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Solving Mixed Volterra-Fredholm Integro Differential Equations by Using HAM

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Abstract. In this article, we discussed semi-analytical approximated method for solving mixed Volterra-Fredholm integro-differential equations, namely homotopy analysis method. Moreover, we prove the existence and uniqueness results and convergence of the technique. Finally, an example is included to demonstrate the validity and applicability of the proposed technique.

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1. Introduction

In this work, we consider the mixed Volterra-Fredholm integro-differential equation of the second kind as follows:

$$\sum_{j=0}^{k} p_j(x)\Theta^{(j)}(x) = f(x) + \int_a^x \int_\Omega K(x,t)G(t,\Theta^{(l)}(t))dxdt,$$

with the initial conditions

$$\Theta^{(r)}(a) = b_r, \quad r = 0,1,2,\dots,(k-1), \quad a \le x \le b, \quad \Omega = [a,b],$$

where $\Theta^{(j)}(x)$ is the j^{th} derivative of the unknown function $\Theta(x)$ that will be determined, $K(x,t)$ is the kernel of the equation, $f(x)$ and $p_j(x)$ are analytic functions, $G(t,\Theta^{(l)}(t)), l \ge 0$ is nonlinear analytic function of Θ and b_r, $0 \le r \le (k-1)$ are real finite constants.

In recent years there has been a growing interest in the integro-differential equation. The integro-differential equations be an important branch of modern mathematics. It arises frequently in many applied areas which include engineering, electrostatics, mechanics, the theory of elasticity, potential, and mathematical physics [1, 3, 6, 10, 25, 27–29, 31, 34, 36].

Recently, Wazwaz (2001) presented an efficient and numerical procedure for solving boundary value problems for higher-order integro-differential equations. A variety of methods, exact, approximate and purely numerical techniques are available to solve nonlinear integro-differential equations [5, 9, 11–15, 23, 32, 35]. These methods have been of

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great interest to several authors and used to solve many nonlinear problems. Some of these techniques are Adomian decomposition method [4,32], modified Adomian decomposition method [26,35], Variational iteration method [7,37] and many methods for solving integro-differential equations [2,3,6,16–22,30,33].

In this work, our aim is to solve a general form of mixed Volterra-Fredholm integro-differential equations using semi-analytical approximated method, namely, homotopy analysis method. Also, we prove the existence and uniqueness results and convergence of the technique.

2. Nonlinear Mixed Volterra-Fredholm Integro-Differential Equation of Second Kind

We consider the mixed Volterra-Fredholm integro-differential equation of the second kind as follows:

$$\sum_{j=0}^{k} p_j(x)\Theta^{(j)}(x) = f(x) + \int_a^x \int_\Omega K(x,t)G(t,\Theta^{(l)}(t))dxdt. \tag{2.1}$$

We can write Eq.(2.1) as follows:

$$p_k(x)\Theta^{(k)}(x) + \sum_{j=0}^{k-1} p_j(x)\Theta^{(j)}(x) = f(x) + \int_a^x \int_\Omega K(x,t)G(t,\Theta^{(l)}(t))dxdt,$$

$$\Theta^{(k)}(x) = \frac{f(x)}{p_k(x)} + \int_a^x \int_\Omega \frac{K(x,t)G(t,\Theta^{(l)}(t))}{p_k(t)}dxdt - \sum_{j=0}^{k-1}\frac{p_j(x)}{p_k(x)}\Theta^{(j)}(x). \tag{2.2}$$

Let us set L^{-1} is the multiple integration operator as follows:

$$L^{-1}(.) := \int_a^x \int_a^x \cdots \int_a^x \int_a^x (.) \underbrace{dtdt\ldots dtdt}_{k-times}. \tag{2.3}$$

From Eq.(2.2) and Eq.(2.3)

$$\begin{aligned}\Theta(x) &= L^{-1}\{\frac{f(x)}{p_k(x)}\} + \sum_{r=0}^{k-1}\frac{(x-a)^r}{r!}b_r + L^{-1}\{\int_a^x \int_\Omega \frac{K(x,t)G(t,\Theta^{(l)}(t))}{p_k(t)}dxdt\} \\ &- L^{-1}\{\sum_{j=0}^{k-1}\frac{p_j(x)}{p_k(x)}\Theta^{(j)}(x)\}.\end{aligned} \tag{2.4}$$

We can obtain the term $\sum_{r=0}^{k-1}\frac{(x-a)^r}{r!}b_r$ from the initial conditions. From [8], we have

$$L^{-1}\{\int_a^x \int_\Omega \frac{K(x,t)G(t,\Theta^{(l)}(t))}{p_k(t)}dxdt\} = \int_a^x \int_\Omega \frac{(x-t)^k K(x,t)G(t,\Theta^{(l)}(t))}{(k!)\quad p_k(t)}dxdt \tag{2.5}$$

also

$$L^{-1}\{\sum_{j=0}^{k-1}\frac{p_j(x)}{p_k(x)}\Theta^{(j)}(x)\} = \sum_{j=0}^{k-1}\int_a^x \frac{(x-t)^{k-1}p_j(t)}{(k-1)!\quad p_k(t)}\Theta^{(j)}(t)dt \tag{2.6}$$

By substituting Eq.(2.5)and Eq.(2.6) in Eq.(2.4) we obtain

$$\begin{aligned}\Theta(x) &= L^{-1}\{\frac{f(x)}{p_k(x)}\} + \sum_{r=0}^{k-1}\frac{(x-a)^r}{r!}b_r + \int_a^x \int_\Omega \frac{(x-t)^k K(x,t)G(t,\Theta^{(l)}(t))}{(k!)\quad p_k(t)}dxdt \\ &- \sum_{j=0}^{k-1}\int_a^x \frac{(x-t)^{k-1}p_j(t)}{(k-1)!\quad p_k(t)}\Theta^{(j)}(t)dt.\end{aligned}$$

We set,

$$L^{-1}\{\frac{f(x)}{p_k(x)}\} + \sum_{r=0}^{k-1}\frac{(x-a)^r}{r!}b_r = F(x),$$

$$\int_{\Omega} \frac{(x-t)^k K(x,t)}{(k!) \quad p_k(t)} dx = K_1(x,t),$$
$$\frac{(x-t)^{k-1} p_j(x)}{(k-1)! \quad p_k(x)} = K_2(x,t).$$

So, we have one-dimensional nonlinear integro-differential equation as follows:

$$\Theta(x) = F(x) + \int_a^x K_1(x,t) G(t, \Theta^{(l)}(t)) dt - \sum_{j=0}^{k-1} \int_a^x K_2(x,t) \Theta^{(j)}(t) dt. \tag{2.7}$$

3. Homotopy Analysis Method (HAM)

The basic concept behind the HAM is illustrated by using the following nonlinear equation:

$$N[\Theta] = 0,$$

where N is a nonlinear operator, $\Theta(x)$ is unknown function and x is an independent variable. Let $\Theta_0(x)$ denote an initial guess of the exact solution $\Theta(x), \hbar \neq 0$ an auxiliary parameter, $H_1(x) \neq 0$ an auxiliary function, and L an auxiliary linear operator with the property $L[s(x)] = 0$ when $s(x) = 0$. Then using $q \in [0,1]$ as an embedding parameter, we can construct a homotopy when consider, $N[\Theta] = 0$, as follows [24]:

$$\begin{aligned}(1-q)L[\phi(x;q) - \Theta_0(x)] - q\hbar H_1(x) N[\phi(x;q)] \\ = \hat{H}[\phi(x;q); \Theta_0(x), H_1(x), \hbar, q].\end{aligned} \tag{3.1}$$

It should be emphasized that we have great freedom to choose the initial guess $\Theta_0(x)$, the auxiliary linear operator L, the non-zero auxiliary parameter $\hbar$, and the auxiliary function $H_1(x)$. Enforcing the homotopy Eq.(3.1) to be zero, i.e.,

$$\hat{H}_1[\phi(x;q); \Theta_0(x), H_1(x), \hbar, q] = 0,$$

we have the so-called zero-order deformation equation

$$(1-q)L[\phi(x;q) - \Theta_0(x)] = q\hbar H_1(x) N[\phi(x;q)], \tag{3.2}$$

when $q = 0$, the zero-order deformation Eq.(3.2) becomes

$$\phi(x;0) = \Theta_0(x), \tag{3.3}$$

and when $q = 1$, since $\hbar \neq 0$ and $H_1(x) \neq 0$, the zero-order deformation Eq.(3.2) is equivalent to

$$\phi(x;1) = \Theta(x). \tag{3.4}$$

Thus, according to Eqs.(3.3) and (3.4), as the embedding parameter q increases from 0 to 1, $\phi(x;q)$ varies continuously from the initial approximation $\Theta_0(x)$ to the exact solution $\Theta(x)$. Such a kind of continuous variation is called deformation in homotopy [35]. Due to Taylor's theorem, $\phi(x;q)$ can be expanded in a power series of q as follows:

$$\phi(x;q) = \Theta_0(x) + \sum_{m=1}^{\infty} \Theta_m(x) q^m, \tag{3.5}$$

where,

$$\Theta_m(x) = \frac{1}{m!} \frac{\partial^m \phi(x;q)}{\partial q^m}\Big|_{q=0}.$$

Let the initial guess $\Theta_0(x)$, the auxiliary linear parameter L, the nonzero auxiliary parameter $\hbar$ and the auxiliary function $H_1(x)$ be properly chosen so that the power series (3.5) of $\phi(x;q)$ converges at $q = 1$, then, we have under these assumptions the solution series,

$$\Theta(x) = \phi(x;1) = \Theta_0(x) + \sum_{m=1}^{\infty} \Theta_m(x).$$

From Eq.(3.5), we can write Eq.(3.2) as follows:

$$\begin{aligned}(1-q)L[\phi(x;q) - \Theta_0(x)] &= (1-q)L[\sum_{m=1}^{\infty} \Theta_m(x) q^m] \\ &= q\hbar H_1(x) N[\phi(x;q)],\end{aligned}$$

then,

$$L[\sum_{m=1}^{\infty}\Theta_m(x)q^m] - qL[\sum_{m=1}^{\infty}\Theta_m(x)q^m] = q\hbar H_1(x)N[\phi(x;q)]. \tag{3.6}$$

By differentiating Eq.(3.6) m times with respect to q, we obtain,

$$\begin{aligned}\{L[\sum_{m=1}^{\infty}\Theta_m(x)q^m] - qL[\sum_{m=1}^{\infty}\Theta_m(x)q^m]\}^{(m)} &= q\hbar H_1(x)N[\phi(x;q)]^{(m)}\\ &= m!L[\Theta_m(x) - \Theta_{m-1}(x)]\\ &= \hbar H_1(x)m\frac{\partial^{m-1}N[\phi(x;q)]}{\partial q^{m-1}}|_{q=0}.\end{aligned}$$

Therefore,

$$L[\Theta_m(x) - \chi_m\Theta_{m-1}(x)] = \hbar H_1(x)\mathfrak{R}_m(\overrightarrow{\Theta_{m-1}}(x)), \tag{3.7}$$

where,

$$\mathfrak{R}_m(\overrightarrow{\Theta_{m-1}}(x)) = \frac{1}{(m-1)!}\frac{\partial^{m-1}N[\varphi(x;q)]}{\partial q^{m-1}}|_{q=0}, \tag{3.8}$$

and

$$\chi_m = \begin{cases}0 & m \le 1,\\ 1 & m > 1.\end{cases}$$

Note that the high-order deformation Eq.(3.7) is governing the linear operator L, and the term $\mathfrak{R}_m(\overrightarrow{\Theta_{m-1}}(x))$ can be expressed simply by Eq.(3.8) for any nonlinear operator N.
The homotopy analysis method is applied to solve Volterra-Fredholm integro-differential equation, we have

$$N[\Theta(x)] = \Theta(x) - F(x) - \int_a^x K_1(x,t)G(t,\Theta^{(l)}(t))dt + \sum_{j=0}^{k-1}\int_a^x K_2(x,t)D^{(j)}(\Theta(t))dt.$$

so,

$$\mathfrak{R}_m(\Theta_{m-1}(x)) = \Theta_{m-1}(x) - \int_a^x K_1(x,t)G(t,\Theta_{m-1}^{(l)}(t))dt + \sum_{j=0}^{k-1}\int_a^x K_2(x,t)D^{(j)}(\Theta_{m-1}(t))dt. \tag{3.9}$$

Substituting Eq.(3.9) into Eq.(3.7)

$$\begin{aligned}L[\Theta_m(x) - \chi_m\Theta_{m-1}(x)] &= hH_1(x)[\Theta_{m-1}(x) - \int_a^x K_1(x,t)G(t,\Theta_{m-1}^{(l)}(t))dt\\ &\quad + \sum_{j=0}^{k-1}\int_a^x K_2(x,t)D^{(j)}(\Theta_{m-1}(t))dt].\end{aligned} \tag{3.10}$$

we take an initial guess $\Theta_0(x) = F(x)$, an auxiliary linear operator $L\Theta = \Theta$, a nonzero auxiliary parameter $h = -1$, and auxiliary function $H_1(x) = 1$. This is substituted into Eq.(3.10) to give the recurrence relation

$$\begin{aligned}\Theta_0(x) &= F(x)\\ \Theta_n(x) &= \int_a^x K_1(x,t)G(t,\Theta_{n-1}^{(l)}(t))dt - \sum_{j=0}^{k-1}\int_a^x K_2(x,t)D^{(j)}(\Theta_{n-1}(t))dt, \quad n \ge 1.\end{aligned}$$

4. Existence, Uniqueness and Convergence Results

In this section the existence and uniqueness of the obtained solution and convergence of the method are proved. Consider the Eq.(2.7), we assume $F(x)$ is bounded for all x in Ω and

$$\left|K_1(x,z)\right| \leq M_1, \quad \left|K_2(x,z)\right| \leq M_{1j}, \quad j = 0, 1, \ldots, k-1, \quad \forall x, z \in J.$$

Also, we suppose the nonlinear terms $G(\Theta(x))$ and $D^j(\Theta(x))$ are Lipschitz continuous with

$$\left|G(x,\Theta(x)) - G(x,\Theta^*(x))\right| \leq d\left|\Theta(x) - \Theta^*(x)\right|$$
$$\left|D^j(\Theta(x)) - D^j(\Theta^*(x))\right| \leq C_j\left|\Theta(x) - \Theta^*(x)\right|, \quad j = 0, 1, \ldots, k-1.$$

If we set,

$$\gamma = (b-a)(dM_1 + kCM), \quad C = max\left|C_j\right|, \quad M = max\left|M_{1_j}\right|.$$

Then the following theorems can be proved by using the above assumptions.

Theorem 4.1. *Assume that the above assumptions are hold, and $0 < \gamma < 1$. Then Eq.(2.7) has a unique solution.*

Proof. Let Θ and Θ^* be two different solutions of Eq.(2.7) then

$$\begin{aligned}
\left|\Theta(x) - \Theta^*(x)\right| &= \Big| \int_a^x K_1(x,t)G(t,\Theta^{(l)}(t))dt - \sum_{j=0}^{k-1}\int_a^x K_2(x,t)D^j(\Theta(t))dt \\
&- \int_a^x K_1(x,t)G(t,\Theta(l)^*(t))dt + \sum_{j=0}^{k-1}\int_a^x K_2(x,t)D^j(\Theta^*(t))dt \Big| \\
&\leq \Big| \int_a^x K_1(x,t)\Big[G(t,\Theta^{(l)}(t)) - G(t,\Theta(l)^*(t))\Big]dt \\
&- \sum_{j=0}^{k-1}\int_a^x K_2(x,t)\Big[D^j(\Theta(t)) - D^j(\Theta^*(t))\Big]dt \Big| \\
&\leq \int_a^x \left|K_1(x,t)\right| \left|G(t,\Theta^{(l)}(t)) - G(t,\Theta(l)^*(t))\right| dt \\
&+ \sum_{j=0}^{k-1}\int_a^x \left|K_2(x,t)\right| \left|D^j(\Theta(t)) - D^j(\Theta^*(t))\right| dt \\
&\leq M_1 d\left|\Theta(x) - \Theta^*(x)\right|(b-a) + kMC\left|\Theta(x) - \Theta^*(x)\right|(b-a) \\
&\leq (b-a)(M_1 d + kMC)\left|\Theta(x) - \Theta^*(x)\right| \\
&= \gamma\left|\Theta(x) - \Theta^*(x)\right|.
\end{aligned}$$

So,

$$\left|\Theta(x) - \Theta^*(x)\right| \leq \gamma\left|\Theta(x) - \Theta^*(x)\right|,$$

from which we get $(1-\gamma)\left|\Theta - \Theta^*\right| \leq 0$. Since $0 < \gamma < 1$, so $\left|\Theta - \Theta^*\right| = 0$. Therefore, $\Theta = \Theta^*$, and this completes the proof.

Theorem 4.2. *If the series solution $\Theta(x) = \sum_{m=0}^{\infty}\Theta_m(x)$ obtained by the m-order deformation is convergent, then it converges to the exact solution of the Volterra-Fredholm integro-differential equation (2.7).*

Proof. We assume

$$\Theta(x) = \sum_{m=0}^{\infty}\Theta_m(x), \;\; \hat{G}(\Theta(x)) = \sum_{m=0}^{\infty}G(\Theta_m(x)), \;\; \hat{D}^j(\Theta(x)) = \sum_{m=0}^{\infty}D^j(\Theta_m(x)),$$

where,

$$\lim_{m\to\infty} \Theta_m(x) = 0.$$

We can write,

$$\sum_{m=1}^{n}[\Theta_m(x) - \chi_m\Theta_{m-1}(x)] = \Theta_1(x) + (\Theta_2(x) - \Theta_1(x)) + (\Theta_3(x) - \Theta_2(x)) + \cdots + (\Theta_n(x) - \Theta_{n-1}(x)) = \Theta_n(x). \tag{4.1}$$

Hence, from Eq.(4.1)

$$\lim_{n\to\infty} \Theta_n(x) = 0. \tag{4.2}$$

So, using Eq.(4.2) and the definition of the linear operator L, we have

$$\sum_{m=1}^{\infty} L\Big[\Theta_m(x) - \chi_m\Theta_{m-1}(x)\Big] = L\sum_{m=1}^{\infty}\Big[\Theta_m(x) - \chi_m\Theta_{m-1}(x)\Big] = 0.$$

Therefore from Eq.(4.2), we can obtain that,

$$\sum_{m=1}^{\infty} L[\Theta_m(x) - \chi_m\Theta_{m-1}(x)] = hH(x)\sum_{m=1}^{\infty}\Re_{m-1}(\Theta_{m-1}(x)) = 0.$$

Since $h \neq 0$ and $H(x, y) \neq 0$, we have

$$\sum_{m=1}^{\infty}\Re_{m-1}(\Theta_{m-1}(x)) = 0. \tag{4.3}$$

By substituting $\Re_{m-1}(\Theta_{m-1}(x))$ into the relation (3.9) and simplifying it, we have

$$\begin{aligned}\sum_{m=1}^{\infty}\Re_{m-1}(\Theta_{m-1}(x)) &= \sum_{m=1}^{\infty}\Big[\Theta_{m-1}(x) - \int_a^x K_1(x,t)G(t,\Theta_{m-1}^{(l)}(t))dt \\ &+ \sum_{j=0}^{k-1}\int_a^x K_2(x,t)D^j(\Theta_{m-1}(t))dt - (1-\chi_m)F(x)\Big] \\ &= \Theta(x) - F(x) - \int_a^x K_1(x,t)\Big[\sum_{m=1}^{\infty}G(t,\Theta_{m-1}^{(l)}(t))\Big]dt \\ &+ \sum_{j=0}^{k-1}\int_a^x K_2(x,t)\Big[\sum_{m=1}^{\infty}D^j(\Theta_{m-1}(t))\Big]dt\end{aligned} \tag{4.4}$$

From Eq.(4.3) and Eq.(4.4), we have

$$\Theta(x) = F(x) + \int_a^x K_1(x,t)\hat{G}(t,\Theta^{(l)}(t))dt - \sum_{j=0}^{k-1}\int_a^x K_2(x,t)\hat{D}^j(\Theta(t))dt.$$

Then, $\Theta(x)$ must be the exact solution of Eq.(2.7).

5. Numerical Example

In this section, we present the semi-analytical technique based on HAM to solve Volterra-Fredholm integro-differential equations:

Example 5.1. Consider the Volterra-Fredholm integro-differential equation as follow:

$$\Theta'''(x) + \Theta(x)\sin x^2 = x^2\sin x^2 - \frac{1}{3}x^3 + \int_0^x\int_0^1 xt\Theta'(t)dxdt,$$

with the initial conditions

$$\Theta''(0) = \Theta'(0) = \Theta(0) = 0.$$

The exact solution is $\Theta(x) = x^2, \quad \epsilon = 10^{-2}$

Table 1. Numerical Results of the Example 1.

x	Exact	$HAM_{n=3}$	$HAM_{n=4}$	Er($HAM_{n=3}$)	Er($HAM_{n=4}$)
0.1	0.01	0.022336	0.011246	32.336×10^{-3}	1.246×10^{-3}
0.2	0.04	0.007327	0.033736	32.673×10^{-3}	6.264×10^{-3}
0.4	0.16	0.125735	0.145964	34.265×10^{-3}	14.036×10^{-3}
0.6	0.36	0.324434	0.346395	35.566×10^{-3}	13.605×10^{-3}
0.8	0.64	0.602669	0.633758	37.331×10^{-3}	6.242×10^{-3}

6. Conclusion

In this work, the HAM has been successfully employed to obtain the approximate solutions of a mixed Volterra-Fredholm integro-differential equation. Moreover, we proved the existence and uniqueness results and convergence of the technique. The results show that this method is very efficient, convenient and can be adapted to fit a larger class of problems. The comparison reveals that although the numerical results of this method is similar approximately with exact solutions.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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