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An Examination on NP^*- Curves in $\mathbb{E}^3$

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Abstract. The evolute and involute curves, Mannheim curves or Bertrand curves are the famous examples of the associated curve pairs. In the view of such information we have defined NP^* curve pairs where the principal normal vector of the first curve and the vector P^* lying on the normal plane of the second curve are linearly dependent. We have called these curve pairs NP^*- curves. Second curve is named NP^*- partner curve. Also, while the examination of NP^*- curves we obtain some relations for the curvatures and Frenet apparatus of the second curve based on the Frenet apparatus of the first curve.

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1. Introduction and Preliminaries

The evolute and involute curve are the curves whose tangent lines intersect orthogonally, hence the principal normal vector of the first curve and tangent vector of second curve are linearly dependent. So if the principal normal vector of first curve and tangent vector of second curve are linearly dependent, then first curve is evolute, and the second curve is called involute curve. For more detail see in [2] and [3].

Mannheim curve was firstly defined by A. Mannheim in 1878; a curve is called a Mannheim curve if and only if $\kappa/\left(\kappa^2+\tau^2\right)$ is a nonzero constant, κ is the curvature and τ is the torsion. Also, a new definition of these associated curves was given by Liu and Wang ; if the principal normal vector of the first curve and binormal vector of the second curve are linearly dependent, then the first curve is called Mannheim curve, and the second curve is called Mannheim partner curve. As a result they called these new curves as Mannheim partner curves. For more detail see in [4]. Bertrand pair curves are the curves with common principal normal lines. A curve is Bertrand curve, if and only if there exist nonzero real numbers λ and β such that constant $\lambda\kappa+\beta\tau=1$, κ is the curvature and τ is the torsion for any $s\in I$. For more detail see in [5]. Let $\{T,N,B,\kappa,\tau\}$ are collectively Frenet-Serret apparatus of a curve α, Frenet vector fields be T,N,B, of α and the first and second curvatures of the curve α be κ and τ, respectively.

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2. NP^*- Curves

In this section we use a method to produce the new curves based on the other curves with common principal normal vector fields. Before we examined ND^* curve with common principal normal vector of first curve and Darboux vector of the second curve. The Darboux vector field of any arclengthed curve α has symmetrical properties [1]. Let α and α^* be the curves with Frenet-Serret apparatus $\{T, N, B, \kappa, \tau\}$ and $\{T^*, N^*, B^*, \kappa^*, \tau^*\}$, where κ, $\kappa^*$and τ, τ^* are the curvature functions the first and the second curve, respectively.

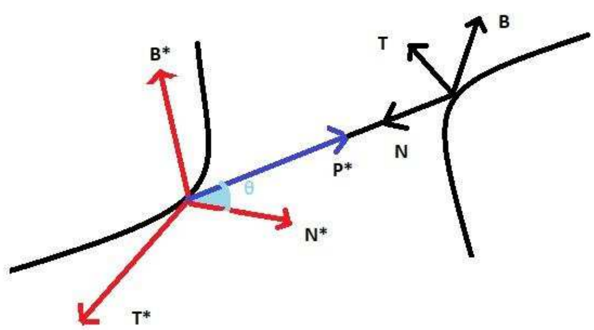

Figure 1. NP^*- partner curve

Definition 2.1. If the principal normal vector N of the first curve and the unit vector field

$$P^*(s) = \frac{aN^*(s) + bB^*(s)}{\sqrt{a^2+b^2}},$$

with non-zero constants a, b of the second curve are linearly dependent, then the first curve is called NP^*- curve, and the second curve is called NP^*- partner curve with the parametrization

$$\alpha^*(s) = \alpha(s) + \lambda(s) P^*(s).$$

As a result we have called these new curves as NP^*- curves. where,P^* lies on the normal plane with $\|P^*\| = 1$. Under the condition $N = P^*$, we get

$$\alpha^*(s) = \alpha(s) + \lambda(s) N(s).$$

Corollary 2.2. *There is the relationship among the non-zero constants a, b of P^*, and the angle $<(N^*, P^*) = \theta$, as in the following way*

$$\cos\theta = \frac{a}{\sqrt{a^2+b^2}},\ \sin\theta = \frac{b}{\sqrt{a^2+b^2}}.$$

Proof. Since

$$\begin{aligned}
\langle N^*, N\rangle &= \langle N^*, P^*\rangle = \left\langle N^*, \frac{aN^*+bB^*}{\sqrt{a^2+b^2}}\right\rangle = \frac{a}{\sqrt{a^2+b^2}},\\
\langle B^*, N\rangle &= \langle B^*, P^*\rangle = \left\langle B^*, \frac{aN^*+bB^*}{\sqrt{a^2+b^2}}\right\rangle = \frac{b}{\sqrt{a^2+b^2}}
\end{aligned}$$

it is trivial. □

Theorem 2.3. *Tangent vector field of NP^*- curve based on the Frenet apparatus of the first curve is*

$$T^* = \frac{(1-\kappa\lambda)\,T + \tau\lambda B}{\sqrt{\delta}}$$

where $\delta = (1-\kappa\lambda)^2 + \tau^2\lambda^2$ *and* $\frac{ds}{ds^*} = \frac{1}{\sqrt{\delta}}$.

Proof. For the curve $\alpha^* = \alpha + \lambda P^* = \alpha + \lambda N$ taking its derivative with respect to it's arclength parameter s^*, we have

$$\begin{aligned}\frac{d\alpha^*}{ds^*} &= \frac{d\alpha^*}{ds}\frac{ds}{ds^*} = \frac{d\,(\alpha+\lambda N)}{ds}\frac{ds}{ds^*}\\ &= ((1-\lambda\kappa)\,T + \lambda' N + \lambda\tau B)\,\frac{ds}{ds^*}.\end{aligned}$$

Also α^* is an arc-lengthed curve with the s^*;$\left\langle \frac{d\alpha^*}{ds^*}, \frac{d\alpha^*}{ds^*}\right\rangle = 1$. Hence if we use the equality $(1-\lambda\kappa)^2 + \lambda'^2 + \tau^2\lambda^2 = \delta$, we have the following result $\frac{ds}{ds^*} = \frac{1}{\sqrt{\delta}}$, and we have tangent vector field as

$$T^* = \frac{(1-\kappa\lambda)\,T + \lambda' N + \tau\lambda B}{\sqrt{(1-\kappa\lambda)^2 + \lambda'^2 + \tau^2\lambda^2}}.$$

Tangent vector T^* of NP^*- *partner curve* is perpendicular its normal plane so tangent vector T^* is perpendicular to $P^* = N$, that is $\langle T^*, N^*\rangle = \langle T^*, B^*\rangle = 0$, so

$$\langle T^*, P^*\rangle = \langle T^*, N\rangle = 0.$$

Hence

$$\begin{aligned}\left\langle \frac{(1-\kappa\lambda)\,T + \lambda' N + \tau\lambda B}{\sqrt{(1-\kappa\lambda)^2 + \lambda'^2 + \tau^2\lambda^2}}, N\right\rangle &= 0,\\ \lambda' &= 0.\end{aligned}$$

Since $\lambda' = 0$, then λ is a non-zero constant, so

$$T^* = \frac{(1-\kappa\lambda)\,T + \tau\lambda B}{\sqrt{(1-\kappa\lambda)^2 + \tau^2\lambda^2}},$$

and $\delta = (1-\kappa\lambda)^2 + \tau^2\lambda^2$, this complete the proof. □

Corollary 2.4. *λ, the distance between NP^*- pair curves is constant.*

Proof. Since $|\alpha^*(s) - \alpha(s)| = |\lambda P^*(s)| = |\lambda|$ and λ is a non-zero constant, it is trivial. □

Theorem 2.5. *First curvature of the curve is NP^*-partner curve based on the Frenet apparatus of the first curve is*

$$\kappa^* = \frac{1}{2\delta^2}\left((-\lambda\kappa' 2\delta - (1-\lambda\kappa)\,\delta')^2 + \left(\kappa - \lambda\left(\kappa^2+\tau^2\right)\right)^2 4\delta^2 + (\lambda\tau' 2\delta - \lambda\tau\delta')^2\right)^{\frac{1}{2}}$$

or $\kappa^* = \frac{\Delta}{2\delta^2}$, *where*

$$\Delta^2 = (-\lambda\kappa' 2\delta - (1-\lambda\kappa)\,\delta')^2 + \left(\kappa - \lambda\left(\kappa^2+\tau^2\right)\right)^2 4\delta^2 + (\lambda\tau' 2\delta - \lambda\tau\delta')^2.$$

Proof. Since $\kappa^* N^* = \frac{dT^*}{ds}\frac{ds}{ds^*}$ and $\frac{ds}{ds^*} = \frac{1}{\sqrt{\delta}}$, it can be calculated as

$$\kappa^* N^* = \left(\frac{[(1-\kappa\lambda)\,T + \lambda\tau B]'\ \sqrt{\delta}}{\delta} - \frac{[(1-\kappa\lambda)\,T + \tau\lambda B]\ \sqrt{\delta}'}{\delta}\right)\frac{ds}{ds^*}$$

hence

$$\kappa^* N^* = \frac{1}{\delta}\left(\begin{array}{c}\left((-2\lambda'\kappa - \lambda\kappa') - (1-\kappa\lambda)\,\frac{\delta'}{2\delta}\right)T\\ +\left(\left(\lambda'' + \kappa - \kappa^2\lambda - \lambda\tau^2\right) - \lambda'\frac{\delta'}{2\delta}\right)N + \left((2\lambda'\tau + \lambda\tau') - \lambda\tau\frac{\delta'}{2\delta}\right)B\end{array}\right).$$

Also $\kappa^{*2} = \langle \kappa^* N^*, \kappa^* N^*\rangle$, so we have

$$\kappa^* = \frac{1}{2\delta^2}\left(\ (-(2\lambda'\kappa + \kappa'\lambda)\,2\delta - (1-\kappa\lambda)\,\delta')^2 + \left(\left(\lambda'' + \kappa - \lambda\left(\kappa^2+\tau^2\right)\right)2\delta - \lambda'\delta'\right)^2 + ((2\lambda'\tau + \lambda\tau')\,2\delta - \lambda\tau\delta')^2\ \right)^{\frac{1}{2}}.$$

For NP^*- *partner curve*, under the condition $\lambda' = 0$ we have the proof. □

Theorem 2.6. *Normal vector field of the curve NP^*–partner curve based on the Frenet apparatus of the first curve is*

$$N^* = \frac{a}{\sqrt{a^2+b^2}}\left(\frac{-2\lambda\kappa'\delta-(1-\kappa\lambda)\delta'}{(\kappa-\lambda(\kappa^2+\tau^2))2\delta}T + N + \frac{2\lambda\tau'\delta-\lambda\tau\delta'}{(\kappa-\lambda(\kappa^2+\tau^2))2\delta}B\right).$$

Proof. Since $\kappa^* N^* = \frac{dT^*}{ds}\frac{ds}{ds^*}$ we have already find out the following result so

$$N^* = \frac{1}{2\delta^2\kappa^*}\Big(\ (-(2\lambda'\kappa+\kappa'\lambda)2\delta-(1-\kappa\lambda)\delta')T + \left(\left(\lambda' + \kappa - \lambda\left(\kappa^2+\tau^2\right)\right)2\delta - \lambda'\delta'\right)N + ((2\lambda'\tau+\lambda\tau')2\delta-\lambda\tau\delta')B \ \Big)$$

for NP^*- curve, under the condition $\lambda' = 0$ and $\kappa^* = \frac{\Delta}{2\delta^2}$ we have normal vector field of the curve NP^*- *partner* curve as in the following way

$$N^* = \frac{1}{\Delta}\Big(\ (-\lambda\kappa' 2\delta-(1-\kappa\lambda)\delta')T + \left(\left(\kappa-\lambda\left(\kappa^2+\tau^2\right)\right)2\delta\right)N + (\lambda\tau' 2\delta-\lambda\tau\delta')B \ \Big),$$

where

$$\Delta = \left[(-\lambda\kappa' 2\delta-(1-\kappa\lambda)\delta')^2 + \left(\kappa-\lambda\left(\kappa^2+\tau^2\right)\right)4\delta^2 + (\lambda\tau' 2\delta-\lambda\tau\delta')^2\right]^{\frac{1}{2}}.$$

Also since

$$\langle N^*, N\rangle = \frac{a}{\sqrt{a^2+b^2}}$$

we get

$$\frac{1}{\Delta}\Big\langle\Big(\ (-\lambda\kappa' 2\delta-(1-\kappa\lambda)\delta')T + \left(\kappa-\lambda\left(\kappa^2+\tau^2\right)\right)2\delta N + (\lambda\tau' 2\delta-\lambda\tau\delta')B \ \Big), N\Big\rangle = \frac{a}{\sqrt{a^2+b^2}}.$$

Hence it is easy to say there is the relationship among the curvatures $\kappa, \tau, \lambda, \Delta$ and a, b of NP^*- partner curve as in

$$\Delta = \frac{\sqrt{a^2+b^2}}{a}\left(\kappa-\lambda\left(\kappa^2+\tau^2\right)\right)2\delta.$$ □

Theorem 2.7. *First curvature of the curve is NP^*- partner curve based on the Frenet apparatus of the first curve and constants a, b is*

$$\kappa^* = \frac{\sqrt{a^2+b^2}}{a}\frac{\left(\kappa-\lambda\left(\kappa^2+\tau^2\right)\right)}{(1-\kappa\lambda)^2+\tau^2\lambda^2}.$$

Proof. It is trivial under the condition $\Delta = \frac{\sqrt{a^2+b^2}}{a}\left(\kappa-\lambda\left(\kappa^2+\tau^2\right)\right)2\delta$ and $\kappa^* = \frac{\Delta}{2\delta^2}$. □

Theorem 2.8. *Binormal vector field of NP^*–partner curve based on the Frenet apparatus of the first curve and constants a, b is*

$$B^* = \frac{a}{\sqrt{a^2+b^2}\sqrt{\delta}}\left(-\lambda\tau T + \frac{\lambda^2(\kappa\tau'-\kappa'\tau)-\lambda\tau'}{\kappa-\lambda(\kappa^2+\tau^2)}N + (1-\lambda\kappa)B\right).$$

Proof. It is trivial under the condition $\Delta = 2\delta\frac{\sqrt{a^2+b^2}}{a}\left(\kappa-\lambda\left(\kappa^2+\tau^2\right)\right)$ and $B^* = T^*\Lambda N^*$, hence

$$\begin{aligned} B^* &= \frac{1}{\sqrt{\delta}\Delta}\begin{vmatrix} T & N & B \\ (1-\kappa\lambda) & 0 & \lambda\tau \\ (-2\lambda\kappa'\delta-(1-\kappa\lambda)\delta') & \left(\left(\kappa-\lambda\left(\kappa^2+\tau^2\right)\right)2\delta\right) & (2\lambda\tau'\delta-\lambda\tau\delta') \end{vmatrix} \\ B^* &= \frac{a}{\sqrt{a^2+b^2}\sqrt{\delta}}\left(\begin{array}{c} -\lambda\tau T + \frac{\lambda\tau(-2\lambda\kappa'\delta-(1-\kappa\lambda)\delta')-(1-\kappa\lambda)(2\lambda\tau'\delta-\lambda\tau\delta')}{2\delta\left(\kappa-\lambda\left(\kappa^2+\tau^2\right)\right)}N \\ +(1-\kappa\lambda)B \end{array}\right). \end{aligned}$$

□

Corollary 2.9. *NP^* − partner curve based can not be Mannheim partner curve*

Proof. Since $\kappa-\lambda\left(\kappa^2+\tau^2\right) \neq 0$,we have $\lambda \neq \frac{\kappa}{(\kappa^2+\tau^2)}$ so $\kappa/\left(\kappa^2+\tau^2\right)$ is not a nonzero constant. □

Theorem 2.10. *The second curvature of the NP^*–partner curve based on the Frenet apparatus of the first curve is*

$$\tau^* = \frac{\tau}{(1-\kappa\lambda)^2+\tau^2\lambda^2}.$$

Proof. Since $\frac{dB^*}{ds^*} = \frac{dB^*}{ds}\frac{ds}{ds^*} = -\tau^* N^*$, we can write $\frac{dB^*}{ds} = -\tau^* N^* \frac{ds^*}{ds}$. Also we have already calculate $\langle B^*, N\rangle = \frac{b}{\sqrt{a^2+b^2}}$, taking its derivative with respect to arclength parameter s^*, we have

$$\begin{aligned}
\left(\left\langle \frac{dB^*}{ds}, N\right\rangle + \left\langle B^*, \frac{dN}{ds}\right\rangle\right)\frac{ds}{ds^*} &= 0\\
\left(\left\langle -\tau^* N^* \frac{ds^*}{ds}, N\right\rangle + \left\langle B^*, \frac{dN}{ds}\right\rangle\right)\frac{ds}{ds^*} &= 0\\
\left(-\sqrt{\delta}\tau^* \langle N^*, N\rangle + \langle B^*, -\kappa T + \tau B\rangle\right)\frac{1}{\sqrt{\delta}} &= 0.
\end{aligned}$$

If we place the product values of the $\langle B^*, T\rangle$, $\langle B^*, B\rangle$ and $\langle N^*, N\rangle$ in

$$\begin{aligned}
\tau^* &= \frac{1}{\sqrt{\delta}}\frac{-\kappa\langle B^*, T\rangle + \tau\langle B^*, B\rangle}{\langle N^*, N\rangle}\\
\tau^* &= \frac{1}{\delta}\tau.
\end{aligned}$$

□

Theorem 2.11. *NP^*-partner curve is defined by $a = 1$ and $b = 1$. $P^* = \dfrac{N^* + B^*}{\sqrt{2}}$ lies on the normal plane, hence Frenet-Serret apparatus are*

$$\begin{aligned}
T^* &= \frac{(1-\kappa\lambda)\,T + \tau\lambda B}{\sqrt{\delta}},\\
N^* &= \frac{1}{\sqrt{2}}\left(\frac{(-2\lambda\kappa'\delta - (1-\kappa\lambda)\,\delta')}{(\kappa - \lambda(\kappa^2+\tau^2))\,2\delta}T + N + \frac{(2\lambda\tau'\delta - \lambda\tau\delta')}{(\kappa - \lambda(\kappa^2+\tau^2))\,2\delta}B\right),\\
B^* &= \frac{-\lambda\tau}{\sqrt{2\delta}}T + \frac{(2\lambda\tau'\delta + \lambda(\tau - \lambda\tau)(1-\lambda\kappa)\,\delta')}{(\kappa - \lambda(\kappa^2+\tau^2))\,2\sqrt{2\delta}\sqrt{\delta}}N + \frac{(1-\kappa\lambda)}{\sqrt{2\delta}}B,\\
\kappa^* &= \sqrt{2}\frac{\left(\kappa - \lambda\left(\kappa^2+\tau^2\right)\right)}{(1-\kappa\lambda)^2 + \tau^2\lambda^2},\\
\tau^* &= \frac{\tau}{(1-\kappa\lambda)^2 + \tau^2\lambda^2}.
\end{aligned}$$

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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