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On The Generation for Numerical Solution of Singularly Perturbed Problem with Right Boundary Layer

Derya Arslan

Department of Mathematics, Faculty of Art and Science, Bitlis Eren University, 13200, Bitlis, Turkey.

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Abstract. In this study, we propose an important numerical method for the numerical solution of singularly perturbed convection-diffusion five points boundary value problem using nonuniform mesh. First, we give the some behaviours of the exact solution and its first derivative. We establish finite difference scheme, which is based on interpolating quadrature rules. Then, we prove the convergence of difference scheme and it is uniformly convergent in ε perturbation parameter. Furthermore, by a numerical experiment, we demonstrate the efficiency of the proposed method.

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1. Introduction

The singularly perturbed convection-diffusion problem with five points boundary value,

$$-\varepsilon u''(x) + a(x)u'(x) + b(x)u(x) = f(x), \quad 0 < x < 1, \tag{1.1}$$

$$u(0) = A, \tag{1.2}$$

$$u(1) = c_1 u(s_1) + c_2 u(s_2) + c_3 u(s_3) + B, \tag{1.3}$$

where $0 < \varepsilon << 1$ is a small perturbation parameter, B and c_i are given constants; $a(x) \geq \alpha > 0$, $b(x)$ and $f(x)$ are continuous functions in $[0, 1]$ and also $0 < s_1 < s_2 < s_3 < 1$ are parameters to be determined. It is easy to see that this problem has a right boundary layer only at $x = 1$.

This kind of problems occur in a number of applications such as chemical-reactor theory, control theory, oceanography, fluid mechanics, quantum mechanics, hydro mechanical problems, meteorology, electrical networks and other physical models [9, 10, 16, 18, 19, 21, 22], and many more. The first time, Samarskii and Bitsadze introduced in nonlocal boundary value problems with using Samarskii-Bitsadze Simple Condition, Samarskii-Bitsadze General Condition, Integral Condition [1, 10]. These problems have been studied by many authors [1–8, 11–17]. Furthermore, finite difference method on various meshes have been used in [2–6, 8, 11–15, 17]. The study of existence and uniqueness of these problems can be seen in [20]. Also, we refer the investigator to the excellent studies by Chegis [16], Bitsadze and Samarskii [10], Amiraliyev [2] and references therein for important theoretical results.

Because of the ε perturbation parameter, standard discretization methods for these singularly perturbed problems create instability. Therefore, we can propose suitable numerical methods such as finite difference method, finite element

Email address: ayredlanu@gmail.com (D. Arslan)

method, etc. We solve the problem of singularly perturbed convection-diffusion with five points condition using finite difference method in this study as well.

This paper is organized as follows: Properties of exact solution are given in Section 2. In Section 3, the finite difference method is presented. The reminder terms are evaluated in Section 4. In Section 5, the results of numerical experiment is presented. These are shown by table and figures.

Throughout the paper, C and C_0 will mean a positive constant independent of ε and the mesh parameter.

2. Properties of The Exact Solution

Here we give useful asymptotic estimates of the exact solution and its first derivative of the problem (1.1) − (1.3), which are needed in the construction of nonuniform mesh.

Lemma 2.1. *Let $a(x)$, $b(x)$ and $f(x)$ be sufficiently smooth on interval $[0,1]$ and*

$$w(1)-[c_1w(s_1)+c_2w(s_2)+c_3w(s_3)]\neq 0,$$

where $w(x)$ is the solution of the following problem:

$$-\varepsilon w''+a(x)w'(x)+b(x)w(x)=0,$$

$$w(0)=0, w(1)=1.$$

Then, the solution of problem (1.1)-(1.3) satisfies the following inequalities:

$$\|u\|_{C[0,1]}\leq C_0, \tag{2.1}$$

where

$$C_0=|v(x)|+|\lambda|\,|w(x)|,$$

and

$$\left|u'(x)\right|\leq C_1\left\{1+\frac{1}{\varepsilon}\left(e^{-\frac{\alpha(1-x)}{\varepsilon}}\right)\right\},\quad 0<x<1. \tag{2.2}$$

Proof. Let us take $u(1)=\lambda$ and $u(x)=v(x)+\lambda w(x)$, where

$$\lambda=\frac{B-v(1)+c_1v(s_1)+c_2v(s_2)+c_3v(s_3)}{w(1)-[c_1w(s_1)+c_2w(s_2)+c_3w(s_3)]},$$

and, the functions $v(x)$ and $w(x)$ is the solution of the following problems:

$$Lv=f(x),$$

$$v(0)=A, v(1)=0,$$

$$Lw=0,$$

$$w(0)=0, w(1)=1.$$

After using the Maximum Principle, we have the inequalities

$$|v(x)|=|v(0)|+|v(1)|+\alpha^{-1}\|f\|_{C[0,1]}\leq C_1, \tag{2.3}$$

and

$$|w(x)|=|w(0)|+|w(1)|\leq 1. \tag{2.4}$$

Finally, from (2.3) and (2.4), we obtain

$$|u(x)|=|v(x)|+|\lambda|\,|w(x)|\leq C_1+1\leq C_0,$$

which proves (2.1).

Now, using $u'(x)=v_1(x)$ and $G(x)=f(x)-b(x)u(x)$ in Equation (1.1), we rewrite (1.1) for proving (2.2) as follows

$$-\varepsilon v_1'(x)+a(x)v_1(x)=G(x),$$

and its solution

$$v_1(x)=u'(x)=u'(0)e^{\frac{1}{\varepsilon}\int_0^x a(\xi)d\xi}-\int_0^x G(\tau)e^{\frac{1}{\varepsilon}\int_\tau^x a(\eta)d\eta}d\tau.$$

After this equation is integrated over $(0, x)$ and some calculations, it is obtained that

$$\left|u'(x)\right| \leq C + \frac{C}{\varepsilon}\left(e^{-\frac{\alpha(1-x)}{\varepsilon}}\right) \leq C\left\{1 + \frac{1}{\varepsilon}\left(e^{-\frac{\alpha(1-x)}{\varepsilon}}\right)\right\}.$$

Thus, the proof of Lemma 2.1 is completed. □

3. Difference Scheme Generating Using Finite Difference Method

In this section, we apply the well-known finite difference method on nonuniform mesh for the problem (1.1)-(1.3). Nonuniform mesh is defined as follows: For a positive integer N, we divide the interval $[0, 1]$ into the two subintervals $[0, 1-\sigma]$ and $[1-\sigma, 1]$. In practice, we usually has $\sigma << 1$. Here σ is transition point which is called as following:

$$\sigma = \min\left\{\frac{1}{2}, -\alpha^{-1}\varepsilon \ln \varepsilon\right\}.$$

We define a set of the mesh points $\bar{\omega}_N = \{x_i\}_{i=0}^{N}$ as

$$x_i = \begin{cases} ih^{(1)}, \ h^{(1)} = \frac{2(1-\sigma)}{N}, \ x_i \in [0, 1-\sigma], \ i = 1, .., \frac{N}{2}; \\ \sigma < \frac{1}{2}, \ 1 - \sigma - \alpha^{-1}\varepsilon \ln\left[1 - (1-\varepsilon)\frac{2i}{N}\right], \ x_i \in [1-\sigma, 1], \ i = \frac{N}{2} + 1, ..., N; \\ \sigma = \frac{1}{2}, \ 1 - \sigma - \alpha^{-1}\varepsilon \ln\left[1 - (1 - e^{\frac{-\alpha}{2\varepsilon}})\frac{2i}{N}\right], \ x_i \in [1-\sigma, 1], \ i = \frac{N}{2} + 1, ..., N. \end{cases}$$

Let us define the following any nonuniform mesh on the interval $[0, 1]$:

$$\omega_N = \{0 < x_1 < x_2 < ... < x_{N-1} < 1\},$$

and

$$\bar{\omega}_N = \omega_N \cup \{x_0 = 0, x_N = 1\}.$$

Before describing our numerical method, we introduce some notations for the mesh functions. We define the following finite difference for any mesh function $g_i = g(x_i)$ given on $\bar{\omega}_N$:

$$g_{\bar{x},i} = \frac{g_i - g_{i-1}}{h_i}, \quad g_{x,i} = \frac{g_{i+1} - g_i}{h_{i+1}},$$

$$g_{\widehat{x},i} = \frac{g_{i+1} - g_i}{\hbar_i}, \quad g_{\bar{x}\widehat{x},i} = \frac{g_{x,i} - g_{\bar{x},i}}{\hbar_i}, \quad \hbar_i = \frac{h_i + h_{i+1}}{2}, \quad h_i = x_i - x_{i-1},$$

$$\|g\|_\infty \equiv \|g\|_{\infty,\bar{\omega}_N} := \max_{0\leqslant i\leqslant N} |g_i|.$$

We shall construct the difference scheme from the following identity,

$$\hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} Lu(x)\varphi_i(x)dx = \hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} f(x)\varphi_i(x)dx, \ i = 1, ..., N-1, \tag{3.1}$$

where the functions $\{\varphi_i(x)\}_{i=1}^{N-1}$ have the from

$$\varphi_i(x) = \begin{cases} \varphi_i^{(1)}(x) = \frac{e^{\frac{a_i(x-x_{i-1})}{\varepsilon}}-1}{e^{\frac{a_i h_i}{\varepsilon}}-1}, & x_{i-1} < x < x_i, \\ \varphi_i^{(2)}(x) = \frac{1-e^{\frac{a_i(x-x_{i+1})}{\varepsilon}}}{1-e^{-\frac{a_i h_{i+1}}{\varepsilon}}}, & x_i < x < x_{i+1}, \\ 0, & x \notin (x_{i-1}, x_{i+1}), \end{cases}$$

We also note that functions $\varphi_i^{(1)}(x)$ and $\varphi_i^{(2)}(x)$ are the solutions of the following problems, respectively

$$\begin{aligned} -\varepsilon\varphi'' + a_i\varphi' + &= 0, \quad x_{i-1} < x < x_i, \\ \varphi(x_{i-1}) &= 0, \quad \varphi(x_i) = 1, \end{aligned}$$

$$\begin{aligned} -\varepsilon\varphi'' + a_i\varphi' &= 0, \quad x_i < x < x_{i+1}, \\ \varphi(x_i) &= 1, \quad \varphi(x_{i+1}) = 0. \end{aligned}$$

The relation (3.1) can be rewritten as

$$\varepsilon\hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} u'(x)\varphi_i'(x)dx + a_i\hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} u'(x)\varphi_i(x)dx + b_i u_i = f_i + R_{a,i} + R_{b,i}, \tag{3.2}$$

where $R_i = f_i + R_{a,i} + R_{b,i}$,

$$R_i = \hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} [a(x_i) - a(x)]\, u'(x)\varphi_i(x)dx + \hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} [b(x_i) - b(x)]\, u(x)\varphi_i(x)dx + \hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} [f(x) - f(x_i)]\, \varphi_i(x)dx. \tag{3.3}$$

Using the interpolating quadrature rules from [2] and as a consequence of (3.2), we propose the following difference scheme for approximating (1.1):

$$-\varepsilon\theta_i u_{\bar{x}\widehat{x},i} + \eta_i u_{\widehat{x},i} + b_i u_i = f_i + R_{a,i} + R_{b,i} = R_i,\, i = \overline{1, N-1}, \tag{3.4}$$

where

$$\theta_i = \frac{\frac{a_i h_i}{\varepsilon}}{1 - e^{-\frac{a_i h_i}{\varepsilon}}}, \tag{3.5}$$

and

$$\eta_i = \frac{-a_i h_i}{h_{i+1}[1 - e^{-\frac{a_i h_i}{\varepsilon}}]} + \frac{a_i}{1 - e^{\frac{a_i h_{i+1}}{\varepsilon}}}. \tag{3.6}$$

Thus, by neglecting R_i in the equation (3.4), we suggest the following difference scheme for approximating (1.1)-(1.3):

$$-\varepsilon\theta_i y_{\bar{x}\widehat{x},i} + \eta_i y_{\widehat{x},i} + b_i y_i = f_i,\ i = \overline{1, N-1}, \tag{3.7}$$

$$y_0 = A, \tag{3.8}$$

$$y_N = c_1 y_{N_1}(x_{N_1}) + c_2 y_{N_2}(x_{N_2}) + c_3 y_{N_3}(x_{N_3}) + B, \tag{3.9}$$

where $x_{N_1}, x_{N_2}, x_{N_3}$ are the mesh points nearest to s_1, s_2, s_3, respectively. And also θ_i and η_i are given by (3.5) and (3.6).

4. Error Estimation

In this section, we obtain the convergence of the method. Let $z_i = y_i - u_i$, $i = 0, 1, ..., N,$. Then the error in the numerical solution satisfies where the truncation error R_i is given by (3.3).

$$-\varepsilon\theta_i z_{\bar{x}\widehat{x},i} + \eta_i z_{\widehat{x},i} + b_i z_i = -R_i,\ \ i = \overline{1, N-1}, \tag{4.1}$$

$$z_0 = 0, \tag{4.2}$$

$$z_N = c_1 z_{N_1}(x_{N_1}) + c_2 z_{N_2}(x_{N_2}) + c_3 z_{N_3}(x_{N_3}). \tag{4.3}$$

Lemma 4.1. *The solution of the problem (4.1)-(4.3) satisfies the following estimates*

$$\|z\|_{\infty,\bar{\omega}_N} \le C\|R\|_{\infty,\omega_N}.$$

holds.

Proof. According to the maximum principle, we have the following inequalities:

$$w(x) = \pm z_i + \alpha^{-1}\|R\|_{\infty,\omega_N}, \tag{4.4}$$

$$w(0) = \pm z_0 + \alpha^{-1}\|R\|_{\infty,\omega_N} \ge 0, \tag{4.5}$$

and

$$w(1) = \pm z_N + \alpha^{-1}\|R\|_{\infty,\omega_N} \ge 0. \tag{4.6}$$

Next, from (4.4)-(4.6), we have

$$\|z_i\| \le \alpha^{-1}\|R\|_{\infty,\omega_N} \le C\|R\|_{\infty,\omega_N},$$

which proves Lemma 4.1. □

Lemma 4.2. *If $a(x), b(x), f(x) \in C^1[0,1]$, then for the truncation error R_i we have*

$$\|R\|_{\infty,\omega_N} \le CN^{-1}.$$

Proof. we can rewrite for the truncation error R_i, such that

$$|R_i| \leq \hbar_i^{-1} \int_{x_{i-1}}^{x_{i+1}} |a(x_i) - a(x)| u'(x)\varphi_i(x)dx + \hbar_i^{-1} \int_{x_{i-1}}^{x_{i+1}} |b(x_i) - b(x)| u(x)\varphi_i(x)dx + \hbar_i^{-1} \int_{x_{i-1}}^{x_{i+1}} |f(x) - f(x_i)| \varphi_i(x)dx. \quad (4.7)$$

Using the mean value theorem for $|a(x_i) - a(x)|$, $|b(x_i) - b(x)|$ and $|f(x) - f(x_i)|$ in (4.7), we get

$$|a(x) - a(x_i)| = \left|a'(\xi)\right| |x - x_i| \leq Ch_i, \ \xi \in [x_i, x],$$

$$|b(x_i) - b(x)| = \left|b'(\xi)\right| |x - x_i| \leq Ch_i, \ \xi \in [x_i, x],$$

$$|f(x) - f(x_i)| = \left|f'(\xi)\right| |x - x_i| \leq Ch_i, \ \xi \in [x_i, x],$$

and also we evaluate (2.2) as

$$\left|u'(x)\right| \leq C\left\{h_i + e^{-\frac{\alpha(1-x_{i+1})}{\varepsilon}}\left(1 - e^{-\frac{\alpha(x_{i+1}-x_{i-1})}{\varepsilon}}\right)\right\} \leq Ch_i.$$

From here with (2.1) and (4.7), we have

$$|R_i| \leq Ch_i. \quad (4.8)$$

Now, we can begin to evaluation for (4.7) on the intervals $[0, 1-\sigma]$ and $[1-\sigma, 1]$, respectively.

In the first case $x_i \in [0, 1-\sigma]$:

$$x_i = \sigma + (i - \frac{N}{2})h^{(1)}, \quad i = 0, ..., \frac{N}{2},$$

where

$$h^{(1)} = \frac{2(1-\sigma)}{N} \leq CN^{-1}. \quad (4.9)$$

It then follows from (4.8) and (4.9), we have

$$|R_i| \leq Ch_i \leq CN^{-1}.$$

In the second case $x_i \in [1-\sigma, 1]$:
For $\sigma < \frac{1}{2}$,

$$x_{i-1} = 1 - \sigma - \alpha^{-1}\varepsilon \ln\left[1 - (1-\varepsilon)\frac{2(i-1)}{N}\right],$$

$$h_i = -\alpha^{-1}\varepsilon \ln\left[1 - (1-\varepsilon)\frac{2i}{N}\right] + \alpha^{-1}\varepsilon \ln\left[1 - (1-\varepsilon)\frac{2(i-1)}{N}\right]. \quad (4.10)$$

Applying the mean value theorem in (4.10), we obtain that

$$h_i = \alpha^{-1}\varepsilon\frac{2(1-\varepsilon)N^{-1}}{1 - 2i_1(1-\varepsilon)N^{-1}} \leq CN^{-1}. \quad (4.11)$$

Thus, from (4.8) and (4.11), we can write

$$|R_i| \leq CN^{-1}, \ \ i = \frac{N}{2} + 1, ..., N.$$

For $\sigma = \frac{1}{2}$,

$$x_{i-1} = 1 - \sigma - \alpha^{-1}\varepsilon \ln\left[1 - (1 - e^{\frac{\alpha}{2\varepsilon}})\frac{2(i-1)}{N}\right],$$

$$h_i = -\alpha^{-1}\varepsilon \ln\left[1 - (1 - e^{\frac{\alpha}{2\varepsilon}})\frac{2i}{N}\right] + \alpha^{-1}\varepsilon \ln\left[1 - (1 - e^{\frac{=\alpha}{2\varepsilon}})\frac{2(i-1)}{N}\right]. \quad (4.12)$$

Applying the mean value theorem in (4.12), we get

$$h_i = \alpha^{-1}\varepsilon\frac{2(1 - e^{\frac{\alpha}{2\varepsilon}})N^{-1}}{1 - 2i_1(1 - e^{\frac{\alpha}{2\varepsilon}})N^{-1}} \leq CN^{-1}. \quad (4.13)$$

Thus, from (4.8) and (4.13), we can write

$$|R_i| \leq CN^{-1}, \ \ i = \frac{N}{2} + 1, ..., N.$$

According to all these situations, we have
$$\|R\|_{\infty,\omega_N} \leq CN^{-1}.$$
□

We can state the convergence result of this study the following Theorem 4.3.

Theorem 4.3. *Let $u(x)$ be the solution of the problem (1.1)-(1.3) and y_i be the solution of the difference scheme (3.7)-(3.9). Then, the following uniform error estimate satisfies*
$$\|y - u\|_{\infty,\bar{\omega}_N} \leq CN^{-1}.$$

5. Numerical Illustration

We apply the scheme (3.7)-(3.9) to calculate the solution of approximation of the following problem:

Example 5.1.
$$-\varepsilon u''(x) + u'(x) + b(x)u(x) = 1, \quad 0 < x < 1,$$
$$u(0) = 0, \quad u(1) = u\left(\frac{1}{4}\right) + 2u\left(\frac{1}{3}\right) + 3u\left(\frac{1}{2}\right) + d.$$
The exact solution is given by
$$u(x) = \frac{\exp\left(-\frac{x}{\sqrt{\varepsilon}}\right) + \exp\left(\frac{x-1}{\sqrt{\varepsilon}}\right)}{1 + \exp\left(-\frac{1}{\sqrt{\varepsilon}}\right)} - \cos^2(\pi x),$$
and
$$b(x) = 0.$$

The corresponding ε uniform convergence rates are computed using the formula
$$P^N = \frac{\ln\left(e^N/e^{2N}\right)}{\ln 2}.$$
The error estimates are denoted by
$$e^N = \max_{\varepsilon} e^N_{\varepsilon}, \quad e^N_{\varepsilon} = \|y - u\|_{\infty,\bar{\omega}_N}.$$

Table 1. The computed maximum pointwise errors e^N and rates of convergence p^N of Example 5.1

ε	$N = 24$	$N = 48$	$N = 96$	$N = 192$	$N = 384$	$N = 768$
2^{-2}	0.0235001	0.0133327	0.0081233	0.0044737	0.0021616	0.0010004
	0.81	0.71	0.86	1.04	1.11	
2^{-3}	0.0272644	0.0142093	0.0072614	0.0036831	0.0018673	0.0009542
	0.94	0.96	0.97	0.97	0.96	
2^{-4}	0.0280771	0.0155298	0.0081457	0.0041684	0.0021081	0.0010602
	0.85	0.93	0.96	0.98	0.99	
2^{-5}	0.0236390	0.0149614	0.0083595	0.0044059	0.0022597	0.0011467
	0.65	0.83	0.92	0.96	0.97	
2^{-6}	0.0160009	0.0121143	0.0077644	0.0043655	0.0023075	0.0011851
	0.40	0.64	0.83	0.91	0.96	
2^{-7}	0.0158096	0.0105741	0.0061431	0.0039676	0.0022392	0.0011856
	0.58	0.78	0.63	0.82	0.91	

The values of ε for which we solve the test problem are $\varepsilon = 2^{-i}, i = 2, 3, 4, 5, 6, 7$. Table 1 verifies first-order the $\varepsilon-$ uniform convergence of the numerical solution on both subintervals and computed rates are essentially in agreement with our theoretical analysis.

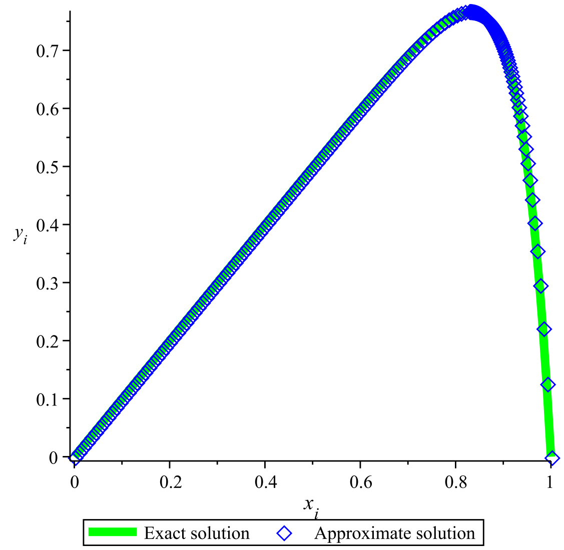

FIGURE 1. Comparison of approximate solution and exact solution of Example 5.1 for N = 192, $\varepsilon = 2^{-4}$

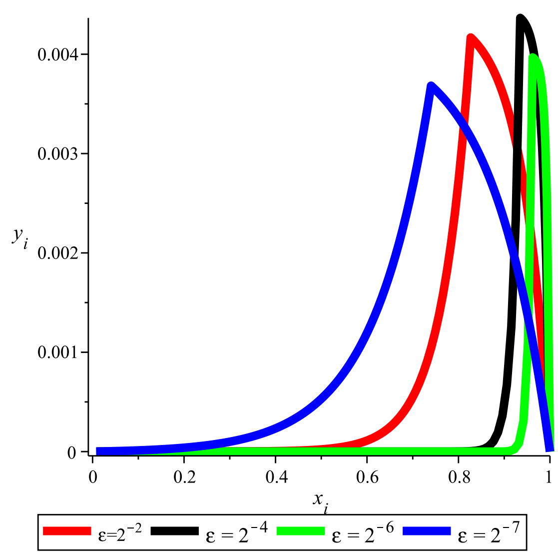

FIGURE 2. Error distribution of Example 5.1 for N = 192, $\varepsilon = 2^{-2}, 2^{-4}, 2^{-6}, 2^{-7}$

The exact solution and approximate solution curves are almost identical as shown in Figure 1. In Figure 2, the errors in boundary layer region are maximum because of the irregularity caused by the sudden and rapid change of the solution in the layer region around $x = 1$ for different ε values.

6. CONCLUSION

In this study, the singularly perturbed convection-diffusion problem with five points boundary condition was examined. To solve this problem, finite difference method on nonuniform mesh was presented. First-order convergence in the discrete maximum norm, independently of the $\varepsilon-$ perturbation parameter was obtained. The errors and rates of convergence are tabulated in Table 1 for the considered test problem in support of the theoretical results. The figures of the exact and the numerical solution of the test problem for different values of $\varepsilon-$ perturbation parameter were plotted in Figure 1. In Figure 2, error distributions of Example 5.1 for N = 192, $\varepsilon = 2^{-2}, 2^{-4}, 2^{-6}, 2^{-7}$ are plotted. We can say that this study can improve academic understanding of the singularly perturbed problems with multipoint boundary condition.

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Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

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