Turk. J. Math. Comput. Sci.
12(1)(2020) 49–55
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Pseudo-Quasi Conformal Curvature Tensor on Normal Paracontact Metric Space Forms

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Received: 18-03-2020 • Accepted: 13-06-2020

Abstract. In the present paper we have studied the curvature tensor of a normal paracontact metric manifold satisfying the conditions $R(\xi, X)\widetilde{C} = 0$, $\widetilde{C}(\xi, X)S = 0$, $\widetilde{C}(\xi, X)P = 0$, $\widetilde{C}(\xi, X)\widetilde{Z} = 0$ and pseudo quasi conformal flat, where R, P, S, $\widetilde{Z}$ and $\widetilde{C}$ are the Riemannian curvature, projective curvature, Ricci, concircular curvature and pseudo-quasi conformal curvature tensors, respectively.

2010 AMS Classification: 53C15, 53C25.

Keywords: Pseudo quasi conformal curvature tensor, Normal paracontact, Projective, Concircular.

1. Introduction

The study of paracontact geometry was initiated by Kenayuki and Williams [4]. Zamkovoy studied paracontact metric manifolds and their subclasses [14]. Recently, Welyczko studied curvature and torsion of Frenet Legendre curves in 3-dimensional normal almost paracontact metric manifolds [11]. In the recent years, (para) contact metric manifolds and their curvature properties have been studied by many authors [6, 12].

In 2005, Shaikh and Jana introduced a tensor field, called pseudo-quasi-conformal curvature tensor $\widetilde{C}$ on a Riemannian manifold [9]. The curvature tensor includes the projective, quasi-conformal, Weyl conformal and concircular curvature tensor as special cases. Prakasha et. all. studied properties of pseudo-quasi- conformal curvature tensor to P-Sasakian manifold [7]. In [1,2], we have studied the curvature tensors satisfying some conditions on a $C(\alpha)$-manifold and induced cases have been discussed.

2. Preliminaries

A $n-$dimensional differentiable manifold (M, g) is said to be an almost paracontact metric manifold if there exist on M a $(1, 1)$ tensor field ϕ, a contravariant vector ξ and a 1-form η-such that

$$\phi^2 X = X - \eta(X)\xi, \qquad \phi\xi = 0, \qquad \eta(\phi X) = 0, \qquad \eta(\xi) = 1$$

and

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \qquad \eta(X) = g(X, \xi),$$

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for any $X, Y \in \chi(M)$. If the covariant derivative of ϕ satisfies

$$(\nabla_X \phi)Y = -g(X, Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi \tag{2.1}$$

then, M is called a normal paracontact metric manifold, where ∇ is Levi-Civita connection.
From (2.1), we can easily to see that

$$\nabla_X \xi = \phi X$$

for any $X \in \chi(M)$.

Moreover, if such a manifold has constant sectional curvature equal to c, then its the Riemannian curvature tensor is R given by

$$\begin{aligned} R(X, Y)Z &= \frac{c+3}{4}\{g(Y, Z)X - g(X, Z)Y\} \\ &+\frac{c-1}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi \\ &-g(Y, Z)\eta(X)\xi + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z\}, \end{aligned}$$

for any vector fields $X, Y, Z \in \chi(M)$.

Quasi-conformal curvature tensor of a n-dimensional Riemannian manifold defined as

$$\begin{aligned} C^*(X, Y)Z &= aR(X, Y)Z + b[S(Y, Z)X - S(X, Z)Y \\ &+g(Y, Z)QX - g(X, Z)QY] \\ &-\frac{r}{n}[\frac{a}{n-1} + 2b][g(Y, Z)X - g(X, Z)Y] \end{aligned}$$

where, a and b are non-zero constants.

Let M-be n-dimensional Riemannian manifold. Then the concircular curvature tensor $\widetilde{Z}$, the projective tensor P and the pseudo-quasi conformal curvature tensor $\widetilde{C}$ are defined by

$$\widetilde{Z}(X, Y)Z = R(X, Y)Z - \frac{r}{n(n-1)}[g(Y, Z)X - g(X, Z)Y],$$

$$P(X, Y)Z = R(X, Y)Z - \frac{1}{n-1}[S(Y, Z)X - S(X, Z)Y]$$

and

$$\begin{aligned} \widetilde{C}(X, Y)Z &= (p+q)R(X, Y)Z + (q - \frac{d}{n-1})[S(Y, Z)X - S(X, Z)Y] \\ &+q[g(Y, Z)QX - g(X, Z)QY] \\ &-\frac{r[p + 2(n-1)q]}{n(n-1)}[g(Y, Z)X - g(X, Z)Y], \end{aligned} \tag{2.2}$$

where is $X, Y, Z \in \chi(M)$, S is the Ricci tensor, r is the scalar curvature, Q is the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor S, i.e. $g(QX, Y) = S(X, Y)$ and p, q, d are real constants such that $p^2 + q^2 + d^2 > 0$ [9].

In particular, if (1) $p = q = 0$, $d = 1$; (2) $p \neq 0, q \neq 0$, $d = 0$; (3) $p = 1, q = -\frac{1}{n-2}$, $d = 0$; (4) $p = 1, q = d = 0$; then $\widetilde{C}$ reduces to the projective curvature tensor; quasi-conformal tensor; conformal curvature tensor and concircular curvature tensor, respectively.

The Ricci curvature of a normal paracontact space forms is given by

$$S(X,Y) = (\frac{c(n-5)+3n+1}{4})g(X,Y) + (\frac{(c-1)(5-n)}{4})\eta(X)\eta(Y)$$

which is equivalent to

$$QX = (\frac{c(n-5)+3n+1}{4})X + (\frac{(c-1)(5-n)}{4})\eta(X)\xi$$

for any $X, Y \in \chi(M)$.

Corollary 2.1. *A normal paracontact metric space form is always an η-Einstein manifold.*

Also we can easily see

$$S(X,\xi) = (n-1)\eta(X),$$

$$Q\xi = (n-1)\xi \tag{2.3}$$

and

$$r = \frac{n-1}{4}[c(n-5)+3n+5].$$

Furthermore, in an n-dimensional normal paracontact metric space form M, the following relations hold

$$R(\xi,Y)Z = g(Y,Z)\xi - \eta(Z)Y, \tag{2.4}$$

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y,$$

$$R(X,\xi)\xi = X - \eta(X)\xi,$$

$$\eta(R(X,Y)Z) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y),$$

$$P(\xi,Y)Z = g(Y,Z)\xi - \frac{1}{n-1}S(Y,Z)\xi, \tag{2.5}$$

$$P(\xi,Y)\xi = 0, \tag{2.6}$$

$$\widetilde{Z}(\xi,Y)Z = [1-\frac{r}{n(n-1)}][g(Y,Z)\xi - \eta(Z)Y], \tag{2.7}$$

$$\widetilde{Z}(\xi,Y)\xi = [1-\frac{r}{n(n-1)}][\eta(Y)\xi - Y] \tag{2.8}$$

for any $X, Y, Z \in \chi(M)$, where r is scalar curvature of M.

Also from (2.2), we obtain

$$\begin{aligned}\widetilde{C}(\xi,Y)Z &= [p[1-\frac{r}{n(n-1)}] + q[\frac{c(n-5)+7n-3}{4} - \frac{2r}{n}] + d] \\ &\bigotimes [g(Y,Z)\xi - \eta(Z)Y] \\ &-\frac{d}{n-1}[S(Y,Z)\xi - (n-1)\eta(Z)Y]\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}\widetilde{C}(X,Y)\xi &= [p[1-\frac{r}{n(n-1)}] \\ &+q[\frac{c(n-5)+7n-3}{4} - \frac{2r}{n}]][\eta(Y)X - \eta(X)Y].\end{aligned} \tag{2.10}$$

3. Pseudo-Quasi Conformal Curvature Tensor of a Normal Paracontact Metric Space Form

Theorem 3.1. *Let $M(c)$ be $n-$dimensional a normal paracontact metric space form. Then $M(c)$ is pseudo-quasi conformal semi symmetric if and only if either M reduce an Einstein manifold or the pseudo-quasi conformal curvature tensor reduces quasi conformal curvature tensor.*

Proof. Suppose that $n-$dimensional a normal paracontact metric space form $M(c)$ is pseudo-quasi conformal semi symmetric. Then, we have

$$R(X,Y)\widetilde{C}=0. \tag{3.1}$$

(3.1) implies that

$$\begin{aligned}(R(X,Y)\widetilde{C})(U,W,Z) &= R(X,Y)\widetilde{C}(U,W)Z-\widetilde{C}(R(X,Y)U,W)Z\\ &\quad -\widetilde{C}(U,R(X,Y)W)Z-\widetilde{C}(U,W)R(X,Y)Z\\ &= 0\end{aligned} \tag{3.2}$$

for any $X,Y,U,W,Z\in\chi(M)$. Taking $X=\xi$ in (3.2) and using (2.4) we find

$$\begin{aligned}0 &= g(Y,\widetilde{C}(U,W)Z)\xi-\eta(\widetilde{C}(U,W)Z)Y\\ &\quad -g(Y,U)\widetilde{C}(\xi,W)Z+\eta(U)\widetilde{C}(Y,W)Z\\ &\quad -g(Y,W)\widetilde{C}(U,\xi)Z+\eta(W)\widetilde{C}(U,Y)Z\\ &\quad -g(Y,Z)\widetilde{C}(U,W)\xi+\eta(Z)\widetilde{C}(U,W)Y\end{aligned} \tag{3.3}$$

Using (2.9) and (2.10) in (3.3) and putting $Z=U=\xi$, we obtain

$$\frac{d}{n-1}[S(Y,W)-(n-2)g(Y,W)]=0.$$

This tell us that M is reduces an Einstein manifold or $d=0$. If $d=0$, we know that $\widetilde{C}$ reduce C^{*}. The converse is obvious. The proof is complete. □

Theorem 3.2. *Let $M(c)$ be $n-$dimensional a normal paracontact metric space form. Then $\widetilde{C}(\xi,Y)S=0$ if and only if either $M(c)$ reduce an Einstein manifold or the constants p,q,d linearly dependent.*

Proof. Let us suppose that $\widetilde{C}(\xi,Y)S=0$. Then, we have

$$S(\widetilde{C}(X,Y)Z,W)+S(Z,\widetilde{C}(X,Y)W)=0 \tag{3.4}$$

for any $X,Y,Z,W\in\chi(M)$.
In (3.4), we choosing $X=\xi$, we get

$$S(\widetilde{C}(\xi,Y)Z,W)+S(Z,\widetilde{C}(\xi,Y)W)=0. \tag{3.5}$$

Using (2.10) in (3.5), we obtain

$$\begin{aligned}0 &= [p[1-\frac{r}{n(n-1)}]+q[\frac{c(n-6)+7n-6}{4}-\frac{2r}{n}]]\\ &\quad [g(Y,Z)S(\xi,W)-\eta(Z)S(Y,W)+g(Y,W)S(Z,\xi)-\eta(W)S(Y,Z)]\\ &\quad +d[g(Y,Z)S(\xi,W)-\eta(Z)S(Y,W)-\frac{1}{n-1}S(Y,Z)S(\xi,W)\\ &\quad +\frac{n-2}{n-1}\eta(Z)S(Y,W)+g(Y,W)S(Z,\xi)-\eta(W)S(Y,Z)\\ &\quad -\frac{1}{n-1}S(Y,Z)S(\xi,W)+\frac{n-2}{n-1}\eta(W)S(Y,Z)].\end{aligned} \tag{3.6}$$

Using (2.3) in (3.6) and substituting $Z=\xi$, we obtain

$$[p[1-\frac{r}{n(n-1)}]+q[\frac{c(n-6)+7n-6}{4}-\frac{2r}{n}]+d][S(Y,W)-(n-2)g(Y,W)]=0.$$

This shows that, M is an Einstein manifold or p,q,d are linearly dependent. □

Theorem 3.3. *Let $M(c)$ be $n-$dimensional a normal paracontact metric space form. Then $\widetilde{C}(\xi, Y)P = 0$ if and only if either $M(c)$ reduce an Einstein manifold or p and q are linearly dependent.*

Proof. Suppose that $\widetilde{C}(\xi, Y)P = 0$. Then it can be easily seen that

$$\begin{aligned}(\widetilde{C}(\xi, Y)P)(U, W, Z) &= \widetilde{C}(\xi, Y)P(U, W)Z - P(\widetilde{C}(\xi, Y)U, W)Z \\ &\quad -P(U, \widetilde{C}(\xi, Y)W)Z - P(U, W)\widetilde{C}(\xi, Y)Z \\ &= 0 \end{aligned} \tag{3.7}$$

for any $Y, U, W, Z \in \chi(M)$. Using (2.9) in (3.7), we obtain

$$\begin{aligned} 0 &= [p[1 - \frac{r}{n(n-1)}] + q[\frac{c(n-6)+7n-6}{4} - \frac{2r}{n}] + d] \\ &\quad [g(Y, P(U, W)Z)\xi - \eta(P(U, W)Z)Y - g(Y, U)P(\xi, W)Z \\ &\quad -g(Y, Z)P(U, W)\xi + \eta(Z)P(U, W)Y] \\ &\quad +\frac{d}{n-1}[-S(Y, P(U, W)Z)\xi + (n-2)\eta(P(U, W)Z)Y \\ &\quad +S(Y, U)P(\xi, W)Z - (n-2)\eta(U)P(Y, W)Z \\ &\quad +S(Y, W)P(U, \xi)Z - (n-2)\eta(W)P(U, Y)Z \\ &\quad +S(Y, Z)P(U, W)\xi - (n-2)\eta(Z)P(U, W)Y]. \end{aligned} \tag{3.8}$$

In (3.8), using (2.5) and (2.6), we obtain

$$\begin{aligned} 0 &= [p[1 - \frac{r}{n(n-1)}] + q[\frac{c(n-6)+7n-6}{4} - \frac{2r}{n}]] \\ &\quad \bigotimes [\frac{n-2}{n-1}g(Y, W) - \frac{1}{n-1}S(Y, W)]. \end{aligned}$$

This tell us, M is an Einstein manifold or the constants p and q are linearly dependent.
The converse is obvious. This proves our assertion. □

Theorem 3.4. *Let $M(c)$ be $n-$dimensional a normal paracontact metric space form. Then $\widetilde{C}(\xi, Y)\widetilde{Z} = 0$ if and only if $M(c)$ satisfies one of the least following conditions*
i) $M(c)$ is an Einstein Manifold,
ii) $\widetilde{C}$ quasi-conformal curvature tensor reduces the conformal curvature tensor,
iii) The scalar curvature of $M(c)$ is $r = n(n-1)$.

Proof. Suppose that $\widetilde{C}(\xi, Y)\widetilde{Z} = 0$, we have

$$\begin{aligned}(\widetilde{C}(\xi, Y)\widetilde{Z})(U, W, Z) &= \widetilde{C}(\xi, Y)\widetilde{Z}(U, W)Z - \widetilde{Z}(\widetilde{C}(\xi, Y)U, W)Z \\ &\quad -\widetilde{Z}(U, \widetilde{C}(\xi, Y)W)Z - \widetilde{Z}(U, W)\widetilde{C}(\xi, Y)Z \\ &= 0 \end{aligned} \tag{3.9}$$

for any $Y, U, W, Z \in \chi(M)$. Using (2.9) in (3.9), we obtain

$$\begin{aligned} 0 &= [p[1 - \frac{r}{n(n-1)}] + q[\frac{c(n-6)+7n-6}{4} - \frac{2r}{n}] + d] \\ &\quad [g(Y, \widetilde{Z}(U, W)Z)\xi - \eta(\widetilde{Z}(U, W)Z)Y - g(Y, U)\widetilde{Z}(\xi, W)Z \\ &\quad -g(Y, Z)\widetilde{Z}(U, W)\xi + \eta(Z)\widetilde{Z}(U, W)Y] \\ &\quad +\frac{d}{n-1}[-S(Y, \widetilde{Z}(U, W)Z)\xi + (n-2)\eta(\widetilde{Z}(U, W)Z)Y \\ &\quad +S(Y, U)\widetilde{Z}(\xi, W)Z - (n-2)\eta(U)\widetilde{Z}(Y, W)Z \\ &\quad +S(Y, W)\widetilde{Z}(U, \xi)Z - (n-2)\eta(W)\widetilde{Z}(U, Y)Z \\ &\quad +S(Y, Z)\widetilde{Z}(U, W)\xi - (n-2)\eta(Z)\widetilde{Z}(U, W)Y]. \end{aligned} \tag{3.10}$$

In (3.10), using (2.7) and (2.8), we find

$$\begin{aligned} & [1-\frac{r}{n(n-1)}][-dg(W,Z)Y+d(Y,Z)W] \\ + & \frac{d(n-2)}{n-1}g(W,Z)Y+\frac{d}{n-1}S(Y,W)\eta(Z)\xi-\frac{d(n-2)}{n-1}g(Y,Z)\eta(W)\xi \\ + & \frac{d}{n-1}S(Y,Z)\eta(W)\xi-\frac{d}{n-1}S(Y,Z)W-\frac{d(n-2)}{n-1}g(Y,W)\eta(Z)\xi \\ + & [p[1-\frac{r}{n(n-1)}]+q[\frac{c(n-6)+7n-6}{4}-\frac{2r}{n}]+d][g(Y,Z)W-g(W,Z)Y] \\ = & 0. \end{aligned}$$

In last equation putting $Z=\xi$, we obtain

$$[\frac{d}{n-1}[1-\frac{r}{n(n-1)}]][S(Y,W)-(n-2)g(Y,W)]=0$$

The converse is obvious. This proves our assertion. □

Definition 3.5. An $n-$dimensional normal paracontact metric space form $M(c)$ is called pseudo-quasi conformal flat if the condition

$$\widetilde{C}(X,Y)Z=0$$

holds on $M(c)$.

Let the space form $M(c)$ under consideration is pseudo-quasi conformal flat, then we have from Definition 3.1. and relation (2.2) that

$$\begin{aligned} (p+d)R(X,Y)Z &= (q-\frac{d}{n-1})[S(X,Z)Y-S(Y,Z)X] \\ &+q[g(X,Z)QY-g(Y,Z)QX] \\ &+(\frac{r[p+2(n-1)q]}{n(n-1)})[g(Y,Z)X-g(X,Z)Y]. \end{aligned} \tag{3.11}$$

Substituting $Z=\xi$ in (3.11), we have

$$\begin{aligned} (p+d)[\eta(Y)\eta(X)] &= [\frac{[q(n-1)-d](n-2)}{n-1}][\eta(Y)X-\eta(X)Y] \\ &+\eta(X)QY-\eta(Y)QX \\ &+[\frac{r[p+2(n-1)q]}{n(n-1)}][\eta(Y)X-\eta(X)Y]. \end{aligned} \tag{3.12}$$

From (3.12), we conclude that

$$p[1-\frac{r}{n(n-1)}]+q[\frac{c(n-6)+7n-6}{4}-\frac{2r}{n}]+\frac{d}{n-1}=0.$$

This shows that, p,q,d are linearly independent and we are able to state the following theorem:

Theorem 3.6. *An $n-$dimensional ($n\geq 3$) normal paracontact metric manifold is pseudo-quasi conformal flat, provided that*

$$p[1-\frac{r}{n(n-1)}]+q[\frac{c(n-6)+7n-6}{4}-\frac{2r}{n}]+\frac{d}{n-1}=0.$$

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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