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The Two-Type Estimates for The Boundedness of Generalized Fractional Maximal Operator on the Generalized Weighted Local Morrey Spaces

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Abstract. In this paper, we study two-type estimates which are the Spanne and Adams type estimates for the continuity properties of the generalized fractional maximal operator M_ρ on the generalized weighted local Morrey spaces $M^{\{x_0\}}_{p,\varphi}(w^p)$ and generalized weighted Morrey spaces $M_{p,\varphi^{\frac{1}{p}}}(w)$, including weak estimates. We prove the Spanne type boundedness of the generalized fractional maximal operator M_ρ from generalized weighted local Morrey spaces $M^{\{x_0\}}_{p,\varphi_1}(w^p)$ to the weighted weak space $WM^{\{x_0\}}_{q,\varphi_2}(w^q)$ for $1 \le p < q < \infty$ and from $M^{\{x_0\}}_{p,\varphi_1}(w^p)$ to another space $M^{\{x_0\}}_{q,\varphi_2}(w^q)$ for $1 < p < q < \infty$ with $w^q \in A_{1+\frac{q}{p'}}$. We also prove the Adams type boundedness of M_ρ from $M_{p,\varphi^{\frac{1}{p}}}(w)$ to the weighted weak space $WM_{q,\varphi^{\frac{1}{q}}}(w)$ for $1 \le p < q < \infty$ and from $M_{p,\varphi^{\frac{1}{p}}}(w)$ to $M_{q,\varphi^{\frac{1}{q}}}(w)$ for $1 < p < q < \infty$ with $w \in A_{p,q}$. The all weight functions belong to Muckenhoupt-Weeden class $A_{p,q}$. In all cases the conditions for the boundedness of the operator M_ρ are given in terms of supremal-type integral inequalities on the all φ functions and r which do not assume any assumption on monotonicity of $\varphi_1(x,r)$, $\varphi_2(x,r)$ and $\varphi(x,r)$ in r.

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1. Introduction

Morrey spaces $M_{p,\lambda}(\mathbb{R}^n)$ were introduced by Morrey in [24] and defined as follows: For $0 \le \lambda < n$, $1 \le p \le \infty$, $f \in M_{p,\lambda}(\mathbb{R}^n)$ if $f \in L^{loc}_p(\mathbb{R}^n)$ and

$$\|f\|_{M_{p,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))} < \infty$$

holds. These spaces appeared to be useful in the study of local behavior properties of the solutions of second order elliptic PDEs. Morrey spaces found important applications to potential theory [1], elliptic equations with discountinuous coefficients [4], Navier-Stokes equations [23] and Shrödinger equations [33].

On the other hand, on the weighted Lebesgue spaces $L_p(\mathbb{R}^n, w)$, the boundedness of some classical operators were obtained by Muckenhoupt [25], Mukenhoupt and Wheeden [26], and Coifman and Fefferman [6]. Recently, weighted

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Morrey spaces $M_{p,\kappa}(\mathbb{R}^n, w)$ were introduced by Komori and Shirai [19] as follows: For $1 \le p \le \infty, 0 < \kappa < 1$ and w be a weight, $f \in M_{p,\kappa}(\mathbb{R}^n, w)$ if $f \in L_p^{loc}(\mathbb{R}^n, w)$ and

$$\|f\|_{M_{p,\kappa}(\mathbb{R}^n,w)} = \sup_{x\in\mathbb{R}^n, r>0} w(B(x,r))^{-\frac{\kappa}{p}} \|f\|_{L_p(B(x,r),w)} < \infty.$$

They studied the boundedness of the aforementioned classical operators such as Hardy-Littlewood maximal operator, Calderon-Zygmund operator, fractional integral operator in these spaces. These results were extended to several other spaces (see [16] for example). Weighted inequalities for fractional operators have good applications to potential theory and quantum mechanics.

For a fixed $x_0 \in \mathbb{R}^n$ the generalized weighted local Morrey spaces $M_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n, w)$ are obtained by replacing a function $\varphi(x_0, r)$ instead of r^λ in the definition of weighted local Morrey space, which is the space of all functions $f \in L_p^{\rm loc}(\mathbb{R}^n, w)$ with finite norm

$$\|f\|_{M_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n,w)} = \sup_{r>0} \varphi(x_0, r)^{-1}\, w(B(x_0, r))^{-\frac{1}{p}}\, \|f\chi_{B(x_0,r)}\|_{L_p(\mathbb{R}^n,w)}.$$

For a measurable function $\rho : (0,\infty) \to (0,\infty)$ the generalized fractional maximal operator M_ρ and the generalized fractional integral operator I_ρ are defined by

$$M_\rho f(x) = \sup_{t>0} \frac{\rho(t)}{t^n} \int_{B(x,t)} |f(y)|dy,$$

$$I_\rho f(x) = \int_{\mathbb{R}^n} \frac{\rho(|x-y|)}{|x-y|^n} f(y)dy$$

for any suitable function f on $\mathbb{R}^n$. If $\rho(t) \equiv t^\alpha$, then $M_\alpha \equiv M_{t^\alpha}$ is the fractional maximal operator and $I_\alpha \equiv I_{t^\alpha}$ is the Riesz potential. The generalized fractional integral operator I_ρ was initilally investigated in [10]. Nowadays many authors have been culminating important observations about the operators I_ρ and M_ρ especially in connection with Morrey spaces. Nakai [28] proved the boundedness of I_ρ and M_ρ from the generalized Morrey spaces M_{1,φ_1} to the spaces M_{1,φ_2} for suitable functions φ_1 and φ_2. The boundedness of I_ρ and M_ρ from the generalized Morrey spaces M_{p,φ_1} to the spaces M_{q,φ_2} are studied by Eridani et al [7–9], Guliyev et al [17], Gunawan [18], Kucukaslan et al [20, 21, 27], Kucukaslan [22], Nakai [29, 30], Nakamura [31], Sawano et al [34, 35] and Sugano [36].

During the last decades, the theory of boundedness of classical operators of the harmonic analysis in the generalized Morrey spaces $M_{p,\varphi}(\mathbb{R}^n)$ have been well studied by now. But, Spanne and Adams type boundedness of the generalized fractional maximal operator M_ρ in the generalized weighted local Morrey spaces $M_{p,\varphi}^{\{x_0\}}(w^p)$ and generalized weighted Morrey spaces $M_{p,\varphi}(w)$ have not been studied, yet.

Spanne [32] and Adams [1] studied boundedness of the Riesz potential in Morrey spaces. Their results can be summarized as follows.

Theorem A (Spanne, but published by Peetre, [32])**.** *Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $0 < \lambda < n - \alpha p$. Moreover, let $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$ and $\frac{\lambda}{p} = \frac{\mu}{q}$. Then for $p > 1$, the operator I_α is bounded from $M_{p,\lambda}$ to $M_{q,\mu}$ and for $p = 1$, I_α is bounded from $M_{1,\lambda}$ to $WM_{q,\mu}$.*

Theorem B (Adams, [1])**.** *Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $0 < \lambda < n - \alpha p$ and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}$. Then for $p > 1$, the operator I_α is bounded from $M_{p,\lambda}$ to $M_{q,\lambda}$ and for $p = 1$, I_α is bounded from $M_{1,\lambda}$ to $WM_{q,\lambda}$.*

In particular, the following statement containing both Theorem A and Theorem B was proved in [2].

Theorem C ([2])**.** *Let $1 \le p < q < \infty$, $0 < \lambda, \mu < n$ and $0 < \alpha = \frac{n-\lambda}{p} - \frac{n-\mu}{q} < \frac{n}{p}$. Then, for $p > 1$, the operator I_α is bounded from $M_{p,\lambda}$ to $M_{q,\mu}$, and, for $p = 1$, I_α is bounded from $M_{1,\lambda}$ to $WM_{q,\mu}$.*

In [2] it was also proved that, under the assumptions of Theorem C, the operator I_α, for $p > 1$, is bounded from the local Morrey space $M_{p,\lambda}^{\{x_0\}}$ to $M_{q,\mu}^{\{x_0\}}$, and, for $p = 1$ from $M_{1,\lambda}^{\{x_0\}}$ to the weak local Morrey space $WM_{q,\mu}^{\{x_0\}}$. Since, for some $c > 0$, $(M_\alpha f)(x) \le c(I_\alpha(|f|))(x)$, $x \in \mathbb{R}^n$, it follows that in Theorems A, B, C the operator I_α can be replaced by the operator M_α (including also the case $p = q$). For the operator M_α Theorem C was, in fact, earlier proved in [3].

In the following theorems which were proved in [21], we give Spanne and Adams type results for the boundedness of operator M_ρ on the generalized local Morrey spaces $M_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$ and generalized Morrey spaces $M_{p,\varphi}(\mathbb{R}^n)$, respectively.

Theorem D (Spanne type result, [21])**.** *Let $x_0 \in \mathbb{R}^n$, $1 \le p < \infty$, the function ρ satisfy the conditions (3.1)-(3.3) and (3.4). Let also (φ_1, φ_2) satisfy the conditions*

$$\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x_0, s) s^{\frac{n}{p}} \le C\, \varphi_2(x_0, \frac{t}{2})\, t^{\frac{n}{q}},$$

$$\sup_{t>r}\Big(\operatorname*{ess\,inf}_{t<s<\infty}\varphi_1(x_0,s)s^{\frac{n}{p}}\Big)\frac{\rho(t)}{t^{\frac{n}{p}}}\le C\,\varphi_2(x_0,r),$$

where C does not depend on x_0 and r. Then the operator M_ρ is bounded from $M^{\{x_0\}}_{p,\varphi_1}$ to $M^{\{x_0\}}_{q,\varphi_2}$ for $p>1$ and from $M^{\{x_0\}}_{1,\varphi_1}$ to $WM^{\{x_0\}}_{q,\varphi_2}$ for $p=1$.

Theorem E (Adams type result, [21])**.** *Let $1\le p<\infty$, $q>p$, $\rho(t)$ satisfy the conditions (3.1)-(3.3) and (3.4). Let also $\varphi(x,t)$ satisfy the conditions*

$$\sup_{r<t<\infty}\varphi(x,t)\le C\,\varphi(x,r),$$

$$\int_r^\infty\varphi(x,t)^{\frac{1}{p}}\frac{\rho(t)}{t}dt\le C\rho(r)^{-\frac{p}{q-p}},$$

where C does not depend on $x\in\mathbb{R}^n$ and $r>0$. Then the operator M_ρ is bounded from $M_{p,\varphi^{\frac{1}{p}}}$ to $M_{q,\varphi^{\frac{1}{q}}}$ for $p>1$ and from $M_{1,\varphi}$ to $WM_{q,\varphi^{\frac{1}{q}}}$ for $p=1$.

Guliyev [14] proved the Spanne and Adams type boundedness of Riesz potential operator I_α from the spaces $M_{p,\varphi_1}(\mathbb{R}^n)$ to $M_{q,\varphi_2}(\mathbb{R}^n)$ without any assumption on monotonicity of φ_1,φ_2.

In this study, by using the method given by Guliyev in [13] (see also [14]) we prove the Spanne and Adams type estimates for the boundedness of generalized fractional maximal operator M_ρ on the generalized weighted local Morrey spaces $M^{\{x_0\}}_{p,\varphi}(w^p)$ and generalized weighted Morrey spaces $M_{p,\varphi^{\frac{1}{p}}}(w)$, including weak estimates. We prove the Spanne type boundedness of the generalized fractional maximal operator M_ρ from generalized weighted local Morrey spaces $M^{\{x_0\}}_{p,\varphi_1}(w^p)$ to the weighted weak space $WM^{\{x_0\}}_{q,\varphi_2}(w^q)$ for $1\le p<q<\infty$ and from $M^{\{x_0\}}_{p,\varphi_1}(w^p)$ to another space $M^{\{x_0\}}_{q,\varphi_2}(w^q)$ for $1<p<q<\infty$ with $w^q\in A_{1+\frac{q}{p'}}$. We also prove the Adams type boundedness of M_ρ from $M_{p,\varphi^{\frac{1}{p}}}(w)$ to the weighted weak space $WM_{q,\varphi^{\frac{1}{q}}}(w)$ for $1\le p<q<\infty$ and from $M_{p,\varphi^{\frac{1}{p}}}(w)$ to $M_{q,\varphi^{\frac{1}{q}}}(w)$ for $1<p<q<\infty$ with $w\in A_{p,q}$. In all cases the conditions for the boundedness of M_ρ are given in terms of supremal-type integral inequalities on the all φ functions and r which do not assume any assumption on monotonicity of $\varphi_1(x,r)$, $\varphi_2(x,r)$ and $\varphi(x,r)$ in r.

By $A\lesssim B$ we mean that $A\le CB$ with some positive constant C independent of appropriate quantities. If $A\lesssim B$ and $B\lesssim A$, we write $A\approx B$ and say that A and B are equivalent.

2. Preliminaries

Let $x\in\mathbb{R}^n$ and $r>0$, then we denote by $B(x,r)$ the open ball centered at x of radius r, and by ${}^{\complement}B(x,r)$ denote its complement. Let $|B(x,r)|$ be the Lebesgue measure of the ball $B(x,r)$. A weight function is a locally integrable function on $\mathbb{R}^n$ which takes values in $(0,\infty)$ almost everywhere. For a weight w and a measurable set E, we define $w(E)=\int_E w(x)dx$, the characteristic function of E by χ_E. If w is a weight function, for all $f\in L_1^{loc}(\mathbb{R}^n)$ we denote by $L_p^{loc}(w)\equiv L_p^{loc}(\mathbb{R}^n,w)$ the weighted Lebesgue space defined by the norm

$$\|f\chi_{B(x,r)}\|_{L_p(w)}=\left(\int_{B(x,r)}|f(x)|^p w(x)dx\right)^{\frac{1}{p}}<\infty,$$

when $1\le p<\infty$ and by

$$\|f\chi_{B(x,r)}\|_{L_\infty(w)}=ess\sup_{x\in B(y,r)}|f(x)w(x)|<\infty,$$

when $p=\infty$.

We recall that a weight function w belongs to the Muckenhoupt-Wheeden class $A_{p,q}$ (see [25]) for $1<p<q<\infty$, if

$$\sup_B\left(\frac{1}{|B|}\int_B w(x)^q dx\right)^{\frac{1}{q}}\left(\frac{1}{|B|}\int_B w(x)^{-p'}dx\right)^{\frac{1}{p'}}\le C,$$

if $p=1$, w is in the $A_{1,q}$ with $1<q<\infty$ then

$$\sup_B\left(\frac{1}{|B|}\int_B w(x)^q dx\right)^{\frac{1}{q}}\left(ess\sup_{x\in B}\frac{1}{w(x)}\right)\le C,$$

where $C>0$ and the supremum is taken with respect to all balls B.

Lemma 2.1. *[11, 12] If $w \in A_{p,q}$ with $1 < p < q < \infty$, then the following statements are true.*
(i) $w^q \in A_r$ *with* $r = 1 + \frac{q}{p'}$.
(ii) $w^{-p'} \in A_{r'}$ *with* $r' = 1 + \frac{p}{q'}$.
(iii) $w^p \in A_s$ *with* $s = 1 + \frac{p}{q'}$.
(iv) $w^{-q'} \in A_{s'}$ *with* $s' = 1 + \frac{q'}{p}$.

We find it convenient to define the generalized weighted local Morrey spaces in the form as follows.

Definition 2.2. Let $1 \le p < \infty$ and $\varphi(x, r)$ be a positive measurable function on $\mathbb{R}^n \times (0, \infty)$. For any fixed $x_0 \in \mathbb{R}^n$ we denote by $M^{\{x_0\}}_{p,\varphi}(w) \equiv M^{\{x_0\}}_{p,\varphi}(\mathbb{R}^n, w)$ the generalized weighted local Morrey space, the space of all functions $f \in L^{\rm loc}_p(\mathbb{R}^n, w)$ with finite quasinorm

$$\|f\|_{M^{\{x_0\}}_{p,\varphi}(w)} = \|f(x_0 + \cdot)\|_{M_{p,\varphi}(w)}.$$

Also by $WM^{\{x_0\}}_{p,\varphi}(w) \equiv WM^{\{x_0\}}_{p,\varphi}(\mathbb{R}^n, w)$ we denote the weak generalized weighted local Morrey space of all functions $f \in WL^{\rm loc}_p(\mathbb{R}^n, w)$ for which

$$\|f\|_{WM^{\{x_0\}}_{p,\varphi}(w)} = \|f(x_0 + \cdot)\|_{WM_{p,\varphi}(w)} < \infty.$$

According to this definition, we recover the weighted local Morrey space $M^{\{x_0\}}_{p,\lambda}(w)$ and weighted weak local Morrey space $WM^{\{x_0\}}_{p,\lambda}(w)$ under the choice $\varphi(x_0, r) = r^{\frac{\lambda-n}{p}}$:

$$M^{\{x_0\}}_{p,\lambda}(w) = M^{\{x_0\}}_{p,\varphi}(w)\Big|_{\varphi(x_0,r)=r^{\frac{\lambda-n}{p}}}, \qquad WM^{\{x_0\}}_{p,\lambda}(w) = WM^{\{x_0\}}_{p,\varphi}(w)\Big|_{\varphi(x_0,r)=r^{\frac{\lambda-n}{p}}}.$$

Remark 2.3. (i) If $w \equiv 1$, then $M_{p,\varphi}(w) = M_{p,\varphi}$ is the generalized Morrey space.
(ii) If $\varphi(x, r) \equiv w(B(x, r))^{\frac{\kappa-1}{p}}$, then $M_{p,\varphi}(w) = L_{p,\kappa}(w)$ is the weighted Morrey space.
(iii) If $w \equiv 1$ and $\varphi(x, r) = r^{\frac{\lambda-n}{p}}$ with $0 < \lambda < n$ then $M_{p,\varphi}(w) = L_{p,\lambda}(\mathbb{R}^n)$ is the classical Morrey space and $WM_{p,\varphi}(w) = WL_{p,\lambda}(\mathbb{R}^n)$ is the weak Morrey space.
(iv) If $\varphi(x, r) \equiv w(B(x, r))^{-\frac{1}{p}}$, then $M_{p,\varphi}(w) = L_p(w)$ is the weighted Lebesgue space.

We denote by $L_\infty((0, \infty), w)$ the space of all functions $g(t)$, $t > 0$ with finite norm

$$\|g\|_{L_\infty((0,\infty),w)} = \sup_{t>0} w(t)g(t)$$

and $L_\infty(0, \infty) \equiv L_\infty((0, \infty), 1)$. Let $\mathfrak{M}(0, \infty)$ be the set of all Lebesgue-measurable functions on $(0, \infty)$ and $\mathfrak{M}^+(0, \infty)$ its subset consisting of all nonnegative functions on $(0, \infty)$. We denote by $\mathfrak{M}^+(0, \infty; \uparrow)$the cone of all functions in $\mathfrak{M}^+(0, \infty)$ which are non-decreasing on $(0, \infty)$ and

$$\mathbb{A} = \left\{\varphi \in \mathfrak{M}^+(0, \infty; \uparrow) : \lim_{t\to 0+} \varphi(t) = 0\right\}.$$

The following lemma was proved in [17] which we will use while proving our main results.

Lemma 2.4. *Let w_1, w_2 be non-negative measurable functions satisfying $0 < \|w_1\|_{L_\infty(t,\infty)} < \infty$ for any $t > 0$.*
Then the identity operator I is bounded from $L_\infty((0, \infty), w_1)$ to $L_\infty((0, \infty), w_2)$ on the cone $\mathbb{A}$ if and only if

$$\left\| w_2 \left(\|w_1\|^{-1}_{L_\infty(\cdot,\infty)}\right)\right\|_{L_\infty(0,\infty)} < \infty.$$

We will use the following statement on the boundedness of the weighted Hardy operator

$$H_w g(t) := \int_t^\infty g(s)w(s)d\mu(s), \quad 0 < t < \infty,$$

where w is weight and $d\mu(s)$ is a non-negative Borel measure on $(0, \infty)$.

The following lemma was proved in [5].

Lemma 2.5. *Let w_1, w_2 and w be weights on $(0, \infty)$ and $w_1(t)$ be bounded outside a neighborhood of the origin. The inequality*

$$\operatorname*{ess\,sup}_{t>0} w_2(t)H_w g(t) \le C \operatorname*{ess\,sup}_{t>0} w_1(t)g(t) \tag{2.1}$$

holds for some $C > 0$ for all non-negative and non-decreasing g on $(0, \infty)$ if and only if

$$B := \sup_{t>0} w_2(t) \int_t^\infty \frac{w(s)ds}{\operatorname{ess\,sup}_{s<\tau<\infty} w_1(\tau)} < \infty. \tag{2.2}$$

Moreover, the value $C = B$ is the best constant for (2.1).

Remark 2.6. In (2.1) and (2.2) it is assumed that $\frac{1}{\infty} = 0$ and $0 \cdot \infty = 0$.

3. Spanne Type Estimate for The Operator M_ρ in The Spaces $M^{\{x_0\}}_{p,\varphi}(\mathbb{R}^n, w^p)$

We assume that

$$\sup_{1\le t<\infty} \frac{\rho(t)}{t^n} < \infty, \tag{3.1}$$

so that the fractional maximal function $M_\rho f$ is well defined, at least for characteristic functions $1/|x|^{2n}$ of complementary balls:

$$f(x) = \frac{\chi_{\mathbb{R}^n \setminus B(0,1)}(x)}{|x|^{2n}}.$$

In addition, we shall also assume that ρ satisfies the growth condition: there exist constants $C > 0$ and $0 < 2k_1 < k_2 < \infty$ such that

$$\sup_{r<s\le 2r} \frac{\rho(s)}{s^n} \le C \sup_{k_1 r<t<k_2 r} \frac{\rho(t)}{t^n}, \quad r > 0. \tag{3.2}$$

This condition is weaker than the usual doubling condition for the function $\frac{\rho(t)}{t^n}$: there exists a constant $C > 0$ such that

$$\frac{1}{C}\frac{\rho(t)}{t^n} \le \frac{\rho(r)}{r^n} \le C\frac{\rho(t)}{t^n}, \tag{3.3}$$

whenever r and t satisfy $r, t > 0$ and $\frac{1}{2} \le \frac{r}{t} \le 2$. In the sequel for the generalized fractional maximal operator M_ρ we always assume that ρ satisfies the condition (3.2).

The boundedness of the operator I_ρ in the spaces $L_p(\mathbb{R}^n)$ can be found in [8]. Let $\frac{\rho(t)}{t^n}$ be almost decreasing, that is, there exists a constant C such that $\frac{\rho(t)}{t^n} \le C\,\frac{\rho(s)}{s^n}$ for $s < t$. In this case, there is a close and strong relation between the operators M_ρ and I_ρ such that

$$M_\rho f(x) = \sup_{t>0} \frac{\rho(t)}{t^n} \int_{B(x,t)} |f(y)|dy \lesssim \sup_{t>0} \int_{B(x,t)} \frac{\rho(|x-y|)}{|x-y|^n} |f(y)|dy = \int_{\mathbb{R}^n} \frac{\rho(|x-y|)}{|x-y|^n} |f(y)|dy = I_\rho(|f|)(x).$$

The following lemma is valid for the operator M_ρ.

Lemma 3.1. *Let $w^q \in A_{1+\frac{q}{p'}}$, the function ρ satisfies the conditions(3.1)-(3.3), and $f \in L_1^{loc}(\mathbb{R}^n, w)$. Then there exist $C > 0$ for all $r > 0$ such that the inequality*

$$\rho(r) \le C r^{\frac{n}{p}-\frac{n}{q}} \tag{3.4}$$

is sufficient condition for the boundedness of generalized fractional maximal operator M_ρ from $L_p(w^p)$ to $WL_q(w^q)$ for $1 \le p < q < \infty$, and from $L_p(w^p)$ to $L_q(w^q)$ for $1 < p < q < \infty, w^q \in A_{1+\frac{q}{p'}}$, where the constant C does not depend on f.

Proof. The proof follows from by the inequality

$$M_\rho f(x) \lesssim M_{(\frac{n}{p}-\frac{n}{q})} f(x), x \in \mathbb{R}^n$$

and by using Muckenhoupt-Wheeden theorems in ([25], Theorem 2 and Theorem 3, pp. 265) for weak and strong types boundedness of the operator M_ρ, respectively. □

The following lemma is weighted local L_p-estimate for the operator M_ρ.

Lemma 3.2. *Let fixed $x_0 \in \mathbb{R}^n$, and $1 \le p < q < \infty$, $w^q \in A_{1+\frac{q}{p'}}$ and $\rho(t)$ satisfy the conditions (3.1)-(3.3). If the condition (3.4) is fulfill, then the inequality*

$$\|M_\rho f\chi_{B(x_0,r)}\|_{WL_q(w^q)} \lesssim \|f\chi_{B(x_0,2r)}\|_{L_p(w^p)} + (w^q(B(x_0,r)))^{\frac{1}{q}}\left(\sup_{t>r}\frac{\rho(t)}{t^{\frac{n}{p}}}\|f\chi_{B(x_0,t)}\|_{L_p(w^p)}\right) \tag{3.5}$$

and for $p > 1$ the inequality

$$\|M_\rho f\chi_{B(x_0,r)}\|_{L_q(w^q)} \lesssim \|f\chi_{B(x_0,2r)}\|_{L_p(w^p)} + (w^q(B(x_0,r)))^{\frac{1}{q}}\left(\sup_{t>r}\frac{\rho(t)}{t^{\frac{n}{p}}}\|f\chi_{B(x_0,t)}\|_{L_p(w^p)}\right) \tag{3.6}$$

holds for any ball $B(x_0,r)$ and for all $f \in L_p^{loc}(\mathbb{R}^n, w^p)$.

Proof. Let $1 \le p < q < \infty$ and $w^q \in A_{1+\frac{q}{p'}}$. For fixed $x_0 \in \mathbb{R}^n$, set $B \equiv B(x_0,r)$ for the ball centered at x_0 and of radius r. Write $f = f_1 + f_2$ with $f_1 = f\chi_{2B}$ and $f_2 = f\chi_{\complement_{(2B)}}$. Hence, by the Minkowski inequality we have

$$\|M_\rho f\chi_B\|_{WL_q(w^q)} \le \|M_\rho f_1\chi_B\|_{WL_q(w^q)} + \|M_\rho f_2\chi_B\|_{WL_q(w^q)}.$$

Since $f_1 \in L_p(w^p)$, $M_\rho f_1 \in WL_q(w^q)$ and by Lemma 3.1 the operator M_ρ is bounded from $L_p(w^p)$ to $WL_q(w^q)$ and it follows that:

$$\|M_\rho f_1\chi_B\|_{WL_q(w^q)} \le \|M_\rho f_1\|_{WL_q(\mathbb{R}^n,w^q)} \le C\|f\chi_{2B}\|_{L_p(w^p)},$$

where constant $C > 0$ is independent of f.

Let x be an arbitrary point from B. If $B(x,t) \cap {}^{\complement}(2B) \ne \emptyset$, then $t > r$. Indeed, if $y \in B(x,t) \cap {}^{\complement}(2B)$, then $t > |x-y| \ge |x_0-y| - |x_0-x| > 2r - r = r$. On the other hand, $B(x,t) \cap {}^{\complement}(2B) \subset B(x_0,2t)$. Indeed, $y \in B(x,t) \cap {}^{\complement}(2B)$, then we get $|x_0-y| \le |x-y| + |x_0-x| < t + r < 2t$. Hence for all $x \in B = B(x_0,r)$ we have

$$M_\rho f_2(x) = \sup_{t>0}\frac{\rho(t)}{t^n}\int_{B(x,t)\cap{}^{\complement}(2B)}|f(y)|dy \lesssim \sup_{t>r}\frac{\rho(2t)}{(2t)^n}\int_{B(x_0,2t)}|f(y)|dy = \sup_{t>2r}\frac{\rho(t)}{t^n}\int_{B(x_0,t)}|f(y)|dy. \tag{3.7}$$

Thus applying Hölder's inequality and from Lemma 3.1, we get

$$\begin{aligned}\|M_\rho f\|_{WL_q(B,w^q)} &\lesssim \|M_\rho f\|_{L_q(B,w^q)} \le \|M_\rho f_1\|_{L_q(B,w^q)} + \|M_\rho f_2\|_{L_q(B,w^q)} \\ &\le \|f\chi_{2B}\|_{L_p(w^p)} + (w^q(B(x_0,r)))^{\frac{1}{q}}\left(\sup_{t>2r}\frac{\rho(t)}{t^n}\|f\chi_{B(x_0,t)}\|_{L_1(w)}\right) \\ &\lesssim \|f\chi_{2B}\|_{L_p(w^p)} + (w^q(B(x_0,r)))^{\frac{1}{q}}\left(\sup_{t>r}\frac{\rho(t)}{t^{\frac{n}{p}}}\|f\chi_{B(x_0,t)}\|_{L_p(w^p)}\right).\end{aligned} \tag{3.8}$$

Thus we get (3.5).

Now let $1 < p < q < \infty$ and $w^q \in A_{1+\frac{q}{p'}}$. Since $f_1 \in L_p(w^p)$, $M_\rho f_1 \in L_q(w^q)$ and by Lemma 3.1 the operator M_ρ is bounded from $L_p(w^p)$ to $L_q(w^q)$ and it follows that

$$\|M_\rho f_1\chi_B\|_{L_q(w^q)} \le \|M_\rho f_1\|_{L_q(\mathbb{R}^n,w^q)} \le C\|f_1\|_{L_p(\mathbb{R}^n,w^p)} = C\|f\chi_{2B}\|_{L_p(w^p)}.$$

Thus applying Hölder's inequality and by (3.8), we get (3.6). Hence the proof is completed. □

The following theorem is one of the main results of the paper in which we get the Spanne type boundedness of the generalized fractional maximal operator M_ρ in the generalized weighted local Morrey spaces $M_{p,\varphi}^{\{x_0\}}(w^p)$.

Theorem 3.3. *Let $x_0 \in \mathbb{R}^n$, $1 \le p < q < \infty$, $w^q \in A_{1+\frac{q}{p'}}$, and let the function ρ satisfy the conditions (3.1)-(3.3) and (3.4). Let also (φ_1, φ_2) satisfy the conditions*

$$\operatorname*{ess\,inf}_{t<s<\infty}\varphi_1(x_0,s)s^{\frac{n}{p}} \le C\,\varphi_2(x_0,\frac{t}{2})\,t^{\frac{n}{q}}, \tag{3.9}$$

$$\sup_{t>r}\frac{\left(\operatorname*{ess\,inf}_{t<s<\infty}\varphi_1(x_0,s)(w^p(B(x_0,s)))^{\frac{1}{p}}s^{\frac{n}{p}}\right)\rho(t)}{(w^q(B(x_0,t)))^{\frac{1}{q}}t^{\frac{n}{p}}} \le C\,\varphi_2(x_0,r), \tag{3.10}$$

where C does not depend on x_0 and r. Then the operator M_ρ is bounded from $M_{p,\varphi_1}^{\{x_0\}}(w^p)$ to $WM_{q,\varphi_2}^{\{x_0\}}(w^q)$ and for $p > 1$ from $M_{p,\varphi_1}^{\{x_0\}}(w^p)$ to $M_{q,\varphi_2}^{\{x_0\}}(w^q)$. Moreover, for $1 \le p < q < \infty$

$$\|M_\rho f\|_{WM_{q,\varphi_2}^{\{x_0\}}(w^q)} \lesssim \|f\|_{M_{p,\varphi_1}^{\{x_0\}}(w^p)},$$

and for $p > 1$

$$\|M_\rho f\|_{M^{\{x_0\}}_{q,\varphi_2}(w^q)} \lesssim \|f\|_{M^{\{x_0\}}_{p,\varphi_1}(w^p)}.$$

Proof. Let $x_0 \in \mathbb{R}^n$, $1 \le p < q < \infty$, $w^q \in A_{1+\frac{q}{p'}}$, and let the function ρ satisfy the conditions (3.1)-(3.3) and (3.4), and also (φ_1, φ_2) satisfy the conditions (3.9) and (3.10). By Lemmas 2.4, 2.5 and 3.2 we have

$$\begin{aligned}\|M_\rho f\|_{WM^{\{x_0\}}_{q,\varphi_2}(w^q)} &\lesssim \sup_{r>0} \varphi_2(x_0,r)^{-1} (w^q(B(x_0,r)))^{-\frac{1}{q}} \|f\|_{L_p(B(x_0,2r),w^p)} + \sup_{r>0} \varphi_2(x_0,r)^{-1} \sup_{t>r} \frac{\rho(t)}{t^{\frac{n}{p}}} \|f\|_{L_p(B(x_0,t),w^p)} \\ &\approx \sup_{r>0} \varphi_1(x_0,r)^{-1} (w^p(B(x_0,r)))^{-\frac{1}{p}} \|f\|_{L_p(B(x_0,r),w^p)} = \|f\|_{M^{\{x_0\}}_{p,\varphi_1}(w^p)},\end{aligned}$$

and for $1 < p < q < \infty$

$$\begin{aligned}\|M_\rho f\|_{LM^{\{x_0\}}_{q,\varphi_2}} &\lesssim \sup_{r>0} \varphi_2(x_0,r)^{-1} (w^q(B(x_0,r)))^{-\frac{1}{q}} \|f\|_{L_p(B(x_0,2r),w^p)} + \sup_{r>0} \varphi_2(x_0,r)^{-1} \sup_{t>r} \|f\|_{L_p(B(x_0,2t),w^p)} \frac{\rho(t)}{t^{\frac{n}{p}}} \\ &\approx \sup_{r>0} \varphi_1(x_0,r)^{-1} (w^p(B(x_0,r)))^{-\frac{1}{p}} \|f\|_{L_p(B(x_0,r),w^p)} = \|f\|_{M^{\{x_0\}}_{p,\varphi_1}(w^p)}.\end{aligned}$$

Hence the proof is completed. □

Corollary 3.4. *In the case* $w \equiv 1$ *from Theorem 3.3 we get Theorem D, in which we obtain Spanne type result for generalized fractional maximal operator* M_ρ *in the generalized local Morrey spaces* $M^{\{x_0\}}_{p,\varphi}$ *which was proved in [21] (Theorem 3.1, p.81).*

Corollary 3.5. *In the case* $\rho(t) = t^\alpha, w \equiv 1, x \equiv x_0$ *from Theorem 3.3 we get Spanne type result for fractional maximal operator* M_α *on generalized Morrey spaces* $M_{p,\varphi}$ *which was proved in [15].*

Corollary 3.6. *In the case* $\rho(t) = t^\alpha, w \equiv 1$ *and* $\varphi_(x_0, t = t^{\frac{\lambda-n}{p}}$, $0 < \lambda < n$ *from Theorem 3.3 we get Spanne result for fractional maximal operator* M_α *on local Morrey spaces* $M^{\{x_0\}}_{p,\lambda}$ *which is variant of Theorem A proved in [32].*

4. Adams Type Estimate for The Operator M_ρ in The Spaces $M_{p,\varphi}(\mathbb{R}^n, w)$

The following theorem is another main result of the paper, in which we get the Adams type boundedness of the generalized fractional maximal operator M_ρ in the generalized weighted Morrey spaces $M_{p,\varphi}(w)$.

Theorem 4.1. *Let fixed* $x_0 \in \mathbb{R}^n$, $1 \le p < q < \infty$, $w \in A_{p,q}$, $\frac{\rho(t)}{t^n}$ *be almost decreasing, and let* $\rho(t)$ *satisfy the condition* (3.2) *and the inequality*

$$\int_0^{k_2 r} \frac{\rho(s)}{s} ds \le C\rho(r),$$

where k_2 *is given by the condition* (3.2) *and* C *does not depend on* $r > 0$. *Let also* $\varphi(x,t)$ *satisfy the conditions*

$$\sup_{r<t<\infty} w(B(x,t))^{-1} \left(\operatorname*{ess\,inf}_{t<s<\infty} \varphi(x,s)\, w(B(x,s)) \right) \le C\, \varphi(x,r), \tag{4.1}$$

$$\rho(r)\varphi(x,r) + \left(\sup_{t>r} \frac{\varphi(x,t)^{\frac{1}{p}} w(B(x,t))^{\frac{1}{p}} \rho(t)}{t^{\frac{n}{p}}} \right) \le C\varphi(x,r)^{\frac{p}{q}}, \tag{4.2}$$

where C *does not depend on* $x \in \mathbb{R}^n$ *and* $r > 0$. *Then the operator* M_ρ *is bounded from* $M_{p,\varphi^{\frac{1}{p}}}(w)$ *to* $WM_{q,\varphi^{\frac{1}{q}}}(w)$ *and for* $p > 1$ *from* $M_{p,\varphi^{\frac{1}{p}}}(w)$ *to* $M_{q,\varphi^{\frac{1}{q}}}(w)$. *Moreover, for* $1 \le p < q < \infty$

$$\|M_\rho f\|_{WM_{q,\varphi^{\frac{1}{q}}}(w)} \lesssim \|f\|_{M_{p,\varphi^{\frac{1}{p}}}(w)},$$

and for $1 < p < q < \infty$

$$\|M_\rho f\|_{M_{q,\varphi^{\frac{1}{q}}}(w)} \lesssim \|f\|_{M_{p,\varphi^{\frac{1}{p}}}(w)}$$

Proof. Let fixed $x_0 \in \mathbb{R}^n$, $1 \le p < q < \infty$, $w \in A_{p,q}$ and $f \in M_{p,\varphi^{\frac{1}{p}}}(w)$. Write $f = f_1 + f_2$, where $B = B(x,r)$, $f_1 = f\chi_{2B}$ and $f_2 = f\chi_{\complement_{(2B)}}$. Then we have

$$M_\rho f(x) \le M_\rho f_1(x) + M_\rho f_2(x).$$

The inequality

$$M_\rho f_1(y) \lesssim Mf(x)\rho(r). \tag{4.3}$$

was proved in [21].

By applying Hölder's inequality and for $M_\rho f_2(y)$, $y \in B(x,r)$ from (3.7) we have

$$M_\rho f_2(y) \lesssim \sup_{t>2r} \frac{\rho(t)}{t^n} \int_{B(x,t)} |f(z)|dz \lesssim \sup_{t>2r} \frac{\rho(t)}{t^{\frac{n}{p}}} \|f\|_{L_p(B(x,t),w)}. \tag{4.4}$$

Then from condition (4.2) and inequalities (4.3), (4.4) for all $y \in B(x,r)$ we get

$$\begin{aligned} M_\rho f(y) &\lesssim \rho(r)\, Mf(x) + \sup_{t>r} \frac{\rho(t)}{t^{\frac{n}{p}}} \|f\|_{L_p(B(x,t),w)} \\ &\le \rho(r)\, Mf(x) + \|f\|_{M_{p,\varphi^{\frac{1}{p}}}(w)} \left(\sup_{t>r} \frac{\varphi(x,t)^{\frac{1}{p}} w(B(x,t))^{\frac{1}{p}} \rho(t)}{t^{\frac{n}{p}}} \right). \end{aligned} \tag{4.5}$$

Thus, by (4.2) and (4.5) we obtain

$$\begin{aligned} M_\rho f(y) &\lesssim \min\left\{ \varphi(x,t)^{\frac{p}{q}-1} Mf(x), \varphi(x,t)^{\frac{p}{q}} \|f\|_{M_{p,\varphi^{\frac{1}{p}}}(w)} \right\} \\ &\lesssim \sup_{s>0} \left(\min\left\{ s^{\frac{p}{q}-1} Mf(x), s^{\frac{p}{q}} \|f\|_{M_{p,\varphi^{\frac{1}{p}}}(w)} \right\} \right) = (Mf(x))^{\frac{p}{q}} \|f\|^{1-\frac{p}{q}}_{M_{p,\varphi^{\frac{1}{p}}}(w)}, \end{aligned}$$

where we have used that the supremum is achieved when the minimum parts are balanced. Hence for all $y \in B(x,r)$, we have

$$M_\rho f(y) \lesssim (Mf(x))^{\frac{p}{q}} \|f\|^{1-\frac{p}{q}}_{M_{p,\varphi^{\frac{1}{p}}}(w)}.$$

Consequently the statement of the theorem follows in view of the boundedness of the maximal operator M in $M_{p,\varphi^{\frac{1}{p}}}(w)$ provided in [16]. Thus, in virtue of the boundedness of the operator M_ρ from $L_p(w)$ to $L_q(w)$ and condition (4.1). Hence we get

$$\begin{aligned} \|M_\rho f\|_{WM_{q,\varphi^{\frac{1}{q}}}(w)} &= \sup_{x\in\mathbb{R}^n, t>0} \varphi(x,t)^{-\frac{1}{q}} w(B(x,t))^{-\frac{1}{q}} \|M_\rho f\|_{WL_q(B(x,t),w)} \\ &\lesssim \|f\|^{1-\frac{p}{q}}_{M_{p,\varphi^{\frac{1}{p}}}(w)} \left(\sup_{x\in\mathbb{R}^n, t>0} \varphi(x,t)^{-\frac{1}{q}} w(B(x,t))^{-\frac{1}{q}} \|Mf\|^{\frac{p}{q}}_{WL_p(B(x,t),w)} \right) \\ &= \|f\|^{1-\frac{p}{q}}_{M_{p,\varphi^{\frac{1}{p}}}(w)} \left(\sup_{x\in\mathbb{R}^n, t>0} \varphi(x,t)^{-\frac{1}{p}} w(B(x,t))^{-\frac{1}{p}} \|Mf\|_{WL_p(B(x,t),w)} \right)^{\frac{p}{q}} \\ &= \|f\|^{1-\frac{p}{q}}_{M_{p,\varphi^{\frac{1}{p}}}(w)} \|Mf\|^{\frac{p}{q}}_{WM_{p,\varphi^{\frac{1}{p}}}(w)} \\ &\lesssim \|f\|_{M_{p,\varphi^{\frac{1}{p}}}(w)}, \end{aligned}$$

for $1 \le p < q < \infty$, and

$$\begin{aligned}
\|M_\rho f\|_{M_{q,\varphi^{\frac{1}{q}}}(w)} &= \sup_{x\in\mathbb{R}^n,t>0} \varphi(x,t)^{-\frac{1}{q}} w(B(x,t))^{-\frac{1}{q}} \|M_\rho f\|_{L_q(B(x,t),w)} \\
&\lesssim \|f\|^{1-\frac{p}{q}}_{M_{p,\varphi^{\frac{1}{p}}}(w)} \left(\sup_{x\in\mathbb{R}^n,t>0} \varphi(x,t)^{-\frac{1}{q}} w(B(x,t))^{-\frac{1}{q}} \|Mf\|^{\frac{p}{q}}_{L_p(B(x,t),w)} \right) \\
&= \|f\|^{1-\frac{p}{q}}_{M_{p,\varphi^{\frac{1}{p}}}(w)} \left(\sup_{x\in\mathbb{R}^n,t>0} \varphi(x,t)^{-\frac{1}{p}} w(B(x,t))^{-\frac{1}{p}} \|Mf\|_{L_p(B(x,t),w)} \right)^{\frac{p}{q}} \\
&= \|f\|^{1-\frac{p}{q}}_{M_{p,\varphi^{\frac{1}{p}}}(w)} \|Mf\|^{\frac{p}{q}}_{M_{p,\varphi^{\frac{1}{p}}}(w)} \\
&\lesssim \|f\|_{M_{p,\varphi^{\frac{1}{p}}}(w)},
\end{aligned}$$

for $1 < p < q < \infty$. Hence the proof is completed. □

Corollary 4.2. *In the case $w \equiv 1$ from Theorem 4.1 we get Theorem E, in which we obtain Adams type result for generalized fractional maximal operator M_ρ on generalized Morrey spaces $M_{p,\varphi}$ which was proved in [21] (Theorem 4.2, p.82).*

Corollary 4.3. *In the case $\rho(t) = t^\alpha, w \equiv 1, x \equiv x_0$ from Theorem 4.1 we get Adams type result for fractional maximal operator M_α on generalized Morrey spaces $M_{p,\varphi}$ which was proved in [15] (see Theorem 5.7, p.182).*

Corollary 4.4. *In the case $\rho(t) = t^\alpha, w \equiv 1$ and $\varphi_(x_0, t = t^{\frac{\lambda-n}{p}}$, $0 < \lambda < n$ from Theorem 4.1 we get Adams's result for fractional maximal operator M_α on local Morrey spaces $M^{\{x_0\}}_{p,\lambda}$ which is variant of Theorem B proved in [32].*

Remark 4.5. Note that, the condition (3.1) is weaker than the following condition which was given in [17] for generalized fractional integral operator I_ρ:

$$\int_1^\infty \frac{\rho(t)}{t^n}\frac{dt}{t} < \infty. \tag{4.6}$$

For example, the function

$$\rho(t) = \frac{t^n}{\log(e+t)}, \quad t > 0$$

satisfies (3.1), but not (4.6). This example shows that the function ρ satisfies Theorems 3.3 and 4.1, but does not satisfy the assumptions of Theorems 16 and 22 in [17]. In other words, the condition (3.1) which satisfies our main theorems, is better (more general and comprehensive) than the condition (4.8) which satisfies the main theorems were given in [17].

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Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

References

[1] Adams, D.R., *A note on Riesz potentials*, Duke Math., **42(4)**(1975), 765-778. 1

[2] Burenkov, V., Guliyev, V.S., *Necessary and sufficient conditions for the boundedness of the Riesz operator in local Morrey-type spaces*, Potential Anal., **30(3)**(2009), 211-249. 1

[3] Burenkov, V., Guliyev, H.V., Guliyev, V.S., *Necessary and sufficient conditions for boundedness of the fractional maximal operator in the local Morrey-type spaces*, J. Comput. Appl. Math., **208(1)**(2007), 280-301. 1

[4] Cafarelli, L., *Elliptic second order equations*, Rend. Sem. Mat. Fis. Milano, **58**(1998), 253-284 (1990), DOI 10.1007/BF02925245. 1

[5] Carro, M., Pick, L., Soria, J., Stepanov, V.D., *On embeddings between classical Lorentz spaces*, Math. Inequal. Appl., **4(3)**(2001), 397-428. 2

[6] Coifman, R.R., Fefferman, C., *Weighted norm inequalities for maximal functions and singular integrals*, Tamkang J. Math., Studia Math., **51**(1974), 241-250. 1

[7] Eridani, A., *On the boundedness of a generalized fractional integral on generalized Morrey spaces*, Tamkang J. Math., **33(4)**(2002), 335-340. 1

[8] Eridani, A., Gunawan, H., Nakai, E., Sawano, Y., *Characterizations for the generalized fractional integral operators on Morrey spaces*, Math. Inequal. Appl., **17(2)**(2014), 761-777. 1, 3

[9] Eridani, A., Gunawan, H., Nakai, E., *On generalized fractional integral operators*, Sci. Math. Jpn., **60(3)**(2004), 539-550. 1

[10] Gadjiev, A.D., *On generalized potential-type integral operators*, Dedicated to Roman Taberski on the occasion of his 70th birthday. Funct. Approx. Comment. Math., **25**(1997), 37-44. 1

[11] Garcia-Cuerva, J., Rubio de Francia, J.L., Weighted Norm Inequalities and Related Topics, North-Holland Math., 16, Amsterdam, 1985. 2.1

[12] Grafakos, L., Classical and Modern Fourier Analysis, Pearson Education, Inc. Upper Saddle River, New Jersey, 2004. 2.1

[13] Guliyev, V.S., Integral Operators on Function Spaces on The Homogeneous Groups and on Domains in $\mathbb{R}^n$ [in Russian], Diss. Steklov Math. Inst., Moscow, 1994. 1

[14] Guliyev, V.S, *Boundedness of the maximal, potential and singular operators in the generalized Morrey spaces,* J. Inequal. Appl., 2009, Art. ID 503948, 20 pp. 1

[15] Guliyev, V.S., Shukurov, P.S., *On the boundedness of the fractional maximal operator, Riesz potential and their commutators in generalized Morrey spaces*, Advances in Harmonic Analysis and Operator Theory, Series: Operator Theory: Advances and Applications, **229**(2013), 175-199. 3.5, 4.3

[16] Guliyev, V.S., *Generalized weighted Morrey spaces and higher order commutators of sublinear operators*, Eurasian Math. J., **3(3)**(2012), 33-61. 1, 4

[17] Guliyev, V.S., Ismayilova, A.F., Kucukaslan, A., Serbetci, A., *Generalized fractional integral operators on generalized local Morrey spaces*, Journal of Function Spaces, Volume 2015, Article ID 594323, 8 pages. 1, 2, 4.5, 4.5

[18] Gunawan, H., *A note on the generalized fractional integral operators*, J. Indones. Math. Soc., **9(1)**(2003), 39-43. 1

[19] Komori, T.Y., Shirai, S., *Weighted Morrey spaces and a singular integral operator*, Math. Nachr., **282(2)**, (2009), 219-231. 1

[20] Kucukaslan, A., Hasanov, S.G, Aykol, C., *Generalized fractional integral operators on vanishing generalized local Morrey spaces*, Int. J. of Math. Anal., **11(6)**(2017), 277–291. 1

[21] Kucukaslan, A., Guliyev, V.S., Serbetci, A., *Generalized fractional maximal operators on generalized local Morrey spaces*, Commun. Fac. Sci. Univ. Ank. Ser. A1. Math. Stat., **69(1)**(2020), 73-87, DOI: 10.31801/cfsuasmas. 1, 3.4, 4, 4.2

[22] Kucukaslan, A., *Equivalence of norms of the generalized fractional integral operator and the generalized fractional maximal operator on generalized weighted Morrey spaces*, Annals of Funct. Analysis, 2020, DOI: 10.1007/s43034-020-00066-w 1

[23] Mazzucato, A.L., *Besov-Morrey spaces: function space theory and applications to non-linear PDE*, Trans. Amer. Math. Soc., **355(4)**(2003), 1297-1364, DOI 10.1090/S0002-9947-02-03214-2. 1

[24] Morrey, C.B., *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc., **43**(1938), 126-166. 1

[25] Muckenhoupt, B., Wheeden, R., *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc., **165**(1972), 261-274. 1, 2, 3.1

[26] Muckenhoupt, B., *Weighted norm inequalities for fractional integrals*, Trans. Amer. Math. Soc., **192**(1974), 207-226. 1

[27] Mustafayev, R., Kucukaslan, A., *An extension of Muckenhoupt-Wheeden theorem to generalized weighted Morrey spaces*, Georgian Math. Journal, DOI: https://doi.org/10.1515/gmj-2020-2056. 1

[28] Nakai, E., *Hardy-Littlewood maximal operator, singular integral operators and Riesz potentials on generalized Morrey spaces*, Math. Nachr., **166**(1994), 95-103. 1

[29] Nakai, E., On Generalized Fractional Integrals on The Weak Orlicz Spaces, BMO^{φ}, The Morrey Spaces and The Campanato Spaces, Function Spaces, Interpolation Theory and Related Topics (Lund, 2000), 389-401, de Gruyter, Berlin, 2002. 1

[30] Nakai, E., *Generalized fractional integrals on generalized Morrey spaces*, Math. Nachr., **287(2-3)**(2014), 339–351. 1

[31] Nakamura, S., *Generalized weighted Morrey spaces and classical operators*, Math. Nachr., **289**, No. 1718, (2016). DOI:10.1002/mana.201500260. 1

[32] Peetre, J., *On the theory of* $M_{p,\lambda}$, J. Funct. Anal., **4**(1969) 71-87. 1, 3.6, 4.4

[33] Ruiz, A., Vega, L., *Unique continuation for Schrödinger operators with potentials in the Morrey class*, Publ. Math., **35(2)**(1991), 291-298, Conference of Mathematical Analysis (El Escorial, 1989). 1

[34] Sawano, Y., Sugano, S., Tanaka, H., *Generalized fractional integral operators and fractional maximal operators in the framework of Morrey spaces*, Trans. Amer. Math. Soc., **363(12)**(2011), 6481-6503. 1

[35] Sawano, Y., *Boundedness of the Generalized Fractional Integral Operators on Generalized Morrey Spaces over Metric Measure Spaces*, Journal of Analysis and its Applications, **36(2)**(2017) 159-190 DOI: 10.4171/ZAA/1584 1

[36] Sugano, S., *Some inequalities for generalized fractional integral operators on generalized Morrey spaces*, Math. Inequal. Appl., **14(4)**(2011), 849–865. 1