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Roughness and Fuzziness Associated with Soft Multisets and Their Application to MADM

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Abstract − In this paper, we tried to hybrid soft sets with multisets. We define some basic properties of soft multiset and present some important results. We define some binary relations, equivalence relations and an indiscernibility relation on soft multiset with examples. The concept of an approximation space associated with each parameter in a soft multiset is discussed and an approximation space associated with the soft multiset is defined. We introduce the novel concepts of roughness and fuzziness associated with soft multiset. We use soft multiset in multi-valued information system. Furthermore, we present an algorithm to cope with uncertainties in multi-attribute decision making (MADM) problems by utilizing soft multisets and related concepts. The efficiency of the algorithm is verified by a case study to find the optimal choice of the real-world problems having uncertainties.

Keywords − Soft multiset, roughness and fuzziness of soft multiset, multi-valued information system, multi-attribute decision-making.

1. Introduction

The rapid development of science has led to an urgent need for the development of modern sets theory. Blizard [1] introduced the multiset theory as a generalization of crisp set theory. Keeping in view the uncertainty element Zadeh [2], in 1965, initiated the idea of fuzzy sets where a membership degree is assigned to each member of the universe of discourse. Molodtsov [3] initiated a novel concept of soft set as a new arithmetical tool for handling uncertainties which traditional arithmetical tools cannot manipulate. Soft set theory and multiset theory has many applications in artificial intelligence, multiple-valued logic, multi-process information fusion, social science, economics, medical science, engineering etc. The advancement in the field of soft set theory has been taking place in a rapid pace due to general nature of parametrization expressed by a soft set, in recent years. Similarly, multiset theory, by assuming that for a given set A an element x occurs a finite number of times, has natural applications in artificial intelligence, multiple-valued logic and decision making problems of the real world problems. Abbas et al. [4] established some generalized operations in soft set theory via relaxed conditions on parameters. Ali et al. [5] introduced some new operations on soft sets. Ali [6] presented some interesting results on on soft sets, rough soft sets and fuzzy soft sets. Ali et al. [7] developed representation of graphs based on neighborhoods and soft sets. Feng et al. [8-13] introduced several interesting results on soft relations applied to semigroups, attribute analysis of information systems, soft sets combined with fuzzy sets, fuzzy soft set, rough set and generalized

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intuitionistic fuzzy soft sets and their applications to multi-attribute decision making. Hayat et al. [14] introduced some new results on type-2 soft sets. Further Maji et al. [15,16] studied the soft set theory and applied this theory to resolve decision-making issues. They also initiated the notion of fuzzy soft set. Aktaş and Çağman [17] presented the concept of soft group. Kong et al. [18] used the soft set hypothetic approach in decision-making issues. Babitha and Sunil [19] introduced the some results on soft set relations, equivalence relations and partitions on soft sets, and soft set function. Babitha and John [20] introduced the idea of soft multiset which is the hybrid structure of multiset and soft set. They shown a relationship between soft multisets and multi-valued information system. They also presented an application of soft multiset in decision-making. Alkhazaleh et al. [21] also introduced some results of soft multiset Applications of soft multiset theory in other fields and real life issues are now capturing momentum. Many researchers have contributed their research work in the field of multiset theory and soft set theory (See [1,5,15,17,18,20-29]).

Liu et al. [30] introduced hesitant IF linguistic operators and presented its application to multi-attribute group decision making (MAGDM) problem. Hashmi et al. [31] introduced the notion of m-polar neutrosophic set and m-polar neutrosophic topology and their applications to multi-criteria decision-making (MCDM) in medical diagnosis and clustering analysis. Hashmi and Riaz [32] introduced a novel approach to censuses process by using Pythagorean m-polar fuzzy Dombi's aggregation operators. Naeem et al. [33] introduced Pythagorean fuzzy soft MCGDM methods based on TOPSIS, VIKOR and aggregation operators. Naeem et al. [34] introduced Pythagorean m-polar Fuzzy Sets and TOPSIS method for the Selection of Advertisement Mode. Riaz et al. [35] introduced N-soft topology and its applications to multi-criteria group decision making (MCGDM). Riaz et al. [36,37] introduced soft rough topology with multi-attribute group decision making problems (MAGDM). Riaz and Hashmi [38] introduced the concept of cubic m-polar fuzzy set and presented multi-attribute group decision making (MAGDM) method for agribusiness in the environment of various cubic m-polar fuzzy averaging aggregation operators. Riaz and Hashmi [39] introduced the notion of linear Diophantine fuzzy Set (LDFS) and its Applications towards multi-attribute decision making problems. Linear Diophantine fuzzy Set (LDFS) is superior than IFSs, PFSs and q-ROFSs. Riaz and Hashmi [40] introduced novel concepts of soft rough Pythagorean m-Polar fuzzy sets and Pythagorean m-polar fuzzy soft rough sets with application to decision-making. Riaz and Tehrim [41-43] established the idea of cubic bipolar fuzzy set and cubic bipolar fuzzy ordered weighted geometric aggregation operators with applications to multi-criteria group decision making (MCGDM). They introduced bipolar fuzzy soft mappings with application to bipolar disorders. Roy and Maji [44] presented a new fuzzy soft set theoretic approach to decision making problems. Şenel [45,46] introduced the relation between soft topological space and soft ditopological space and characterization of soft sets by delta-soft operations. Sezgin and Atagün [47] introduced some new operations of soft sets. Sezgin et al. [48] introduced the idea of soft intersection near-rings with applications. Shabir and Ali [29] established some properties of soft ideals and generalized fuzzy ideals in semigroups. Shabir and Naz [49] introduced the concept of soft topological spaces. Tehrim and Tehrim [50] presented a novel extension of TOPSIS to MCGDM with bipolar neutrosophic soft topology. Wei et al. [51] established hesitant triangular fuzzy operators in MADGDM problems. Xueling et al. [52] introduced decision-making methods based on various hybrid soft sets. Xu and Zhang [53] introduced hesitant fuzzy multi-attribute decision-making based on TOPSIS with incomplete weight information. Xu [54] introduced the concept of intuitionistic fuzzy aggregation operators. Xu and Cai, in their book [55], presented the theory and applications of intuitionistic fuzzy information aggregation. Xu, in his book [56], presented hesitant fuzzy sets theory and various types of hesitant fuzzy aggregation operators. Zhan et al. [57-58] presented the concepts of rough soft hemirings, soft rough covering and its applications to multi-criteria group decision-making (MCGDM) problems. Zhang and Xu [59] presented an extension of TOPSIS in multiple criteria decision making with the help of Pythagorean Fuzzy Sets.

This paper is organized as follows: In Section 2, we present some basic concepts of multiset theory. In Section 3, we discuss some results of soft set theory and soft multiset theory. We also present some new operations on soft multisets. In Section 4, we present some binary relations, equivalence relations and an indiscernibility relations on soft multiset with the help of examples. We also present an application of soft multiset in information system. In Section 5, we present an algorithm to cope with multi-attribute decision making (MADM) problems by utilizing soft multisets and related concepts. This algorithm is also summarized by the flow chart. The efficiency of the algorithm is verified by a case study to find the optimal choice to the various real world problems having uncertainties, imprecisions and vagueness.

2. Preliminaries

In this section, we recall some rudiments of multiset theory.

Definition 2.1. [20] "A multiset over Z is just a pair $< Z, f >$, where $f: Z \rightarrow W$ is a function, Z is a crisp set and W is a set of whole numbers.

In order to avoid any confusion we will use square brackets for multisets and braces for sets. Let A be a multiset over crisp set Z with z occurring m times in A. It is denoted by $z \in^m A$. Multi-set A is given by $A =< Z, f >= \left[\frac{k_1}{z_1}, \frac{k_2}{z_2}, \ldots, \frac{k_n}{z_n}\right]$, where z_1 occurring k_1 times, z_2 occurring k_2 times and so on.

Definition 2.2. [60] Let $A =< Z, f >$ and $B =< Z, g >$ be two multisets. Then A is a sub-multiset of B, denoted by $A \subseteq B$ if for all $z \in A$, $f(z) \leq g(z)$.

Definition 2.3. [20] A sub-multiset $A =< Z, f >$ of $B =< Z, g >$ is a whole sub-multiset of B with each element in A having full multiplicity as in B. i.e. $f(z) = g(z)$, for every z in A.

Definition 2.4. [60] Suppose that $A =< Z, f >$ and $B =< Z, g >$ are two multisets. Then their union, denoted by $A \cup B$, is a multiset $C =< Z, h >$, where for all $z \in Z$ such that $h(z) = \max\ (f(z), g(z))$.

Definition 2.5. [60] Suppose that $A =< Z, f >$ and $B =< Z, g >$ are two multisets. Then their intersection, denoted by $A \cap B$, is a multiset $C =< Z, h >$, where for all $z \in Z$ such that $h(z) = \min\ (f(z), g(z))$.

Definition 2.6. [20] Suppose that $A =< Z, f >$ and $B =< Z, g >$ are two multisets. Then their sum denoted by $A \oplus B$, is a multiset $C =< Z, h >$, where for all $z \in Z$ such that $h(z) = f(z) + g(z)$.

Definition 2.7. [60] Suppose that $A =< Z, f >$ and $B =< Z, g >$ are two multisets. Then the removal of multiset B from A, denoted by $A \ominus B$, is a multiset $C =< Z, h >$, where for all $z \in Z$ such that $h(z) = \max\ (f(z) - g(z), 0)$.

Definition 2.8. [20] Let $A =< Z, f >$ be a multiset and $A_1 =< Z, g >$ be a sub-multiset of A. Then the relative compliment of A_1 is given by $A_1^r = A \ominus A_1$.

Definition 2.9. [20] Let $[Z]^n$ denotes the set of all multisets whose elements are in Z such that no element in a multiset appears more than n times. Let $A \in [Z]^n$ be a multiset. The power whole multiset of A denoted by $PW(A)$ is defined as the set of all whole sub-multisets of A. The cardinality of $PW(A)$ is 2^m, where m is the cardinality of the support set (root set) of A".

Example 2.10. Let $M = \left[\frac{2}{g}, \frac{1}{t}, \frac{1}{k}\right]$ be a multiset. Then, by using Definition 2.2, Definition 2.3 and Definition 2.9, the set of all sub-multisets of M is

$$PW(A) = \left\{ S_1 = \left[\frac{0}{g}, \frac{0}{t}, \frac{0}{k}\right], S_2 = \left[\frac{0}{g}, \frac{0}{t}, \frac{1}{k}\right], S_3 = \left[\frac{0}{g}, \frac{1}{t}, \frac{0}{k}\right], S_4 = \left[\frac{0}{g}, \frac{1}{t}, \frac{1}{k}\right], S_5 = \left[\frac{1}{g}, \frac{0}{t}, \frac{0}{k}\right], S_6 = \left[\frac{1}{g}, \frac{0}{t}, \frac{1}{k}\right], \right.$$
$$\left. S_7 = \left[\frac{1}{g}, \frac{1}{t}, \frac{0}{k}\right], S_8 = \left[\frac{1}{g}, \frac{1}{t}, \frac{1}{k}\right], S_9 = \left[\frac{2}{g}, \frac{0}{t}, \frac{0}{k}\right], S_{10} = \left[\frac{2}{g}, \frac{0}{t}, \frac{1}{k}\right], S_{11} = \left[\frac{2}{g}, \frac{1}{t}, \frac{0}{k}\right], S_{12} = \left[\frac{2}{g}, \frac{1}{t}, \frac{1}{k}\right] \right\}$$

and $card(P(M)) = (2 + 1)(1 + 1)(1 + 1) = 12$.

Furthermore, the power whole multiset is given by $PW(M) = \{S_1, S_2, S_3, S_4, S_9, S_{10}, S_{11}, S_{12}\}$ and its cardinality is given by $card(PW(M)) = 2^3 = 8$.

3. Some Results on Soft Multi-Sets

In this section, we present some basic notions of soft set and soft multiset along with related properties. We present binary relation, equivalence relation and indiscernibility relation on soft multiset with examples and study important results. We also present an application of soft multiset in information system.

In the sequel, the multiset H represents the initial universe, E is a set of parameters or attributes, $PW(H)$ is a power whole multiset of H and $A \subseteq E$.

Definition 3.1. [3] Let X be the universal set, E be a set of attributes, $P(X)$ be a power set of X and $A \subseteq E$. A pair (λ, A) is called a soft set over X, where $\lambda: A \to P(X)$ is a set-valued function.

Definition 3.2. [20] A soft multiset σ_A on the universal multiset H is defined by the set of all ordered pairs $\sigma_A = \{(s, \sigma_A(s)): s \in E, \sigma_A(s) \in PW(H)\}$, where $\sigma_A: E \to PW(H)$ such that $\sigma_A(s) = \emptyset$ if $s \notin A$.

Consider a soft multiset σ_A, where $\mathrm{H} = \left[\frac{\mathrm{k_1}}{\mathrm{h_1}}, \frac{\mathrm{k_2}}{\mathrm{h_2}}, \dots, \frac{\mathrm{k_n}}{\mathrm{h_n}}\right]$, $E = \{s_1, s_2, \dots, s_m\}$ and $A = E$. Tabular illustration of a soft multiset is most helpful for storing soft multiset in a computer. Here,

$$h_{ij} = \begin{cases} k_i, & \text{if } h_i \in^{k_i} \sigma_A(s_j) \\ 0, & \text{otherwise} \end{cases}$$

Tabular representation of a soft multiset can be written as:

σ_A	s_1	$s_2 \cdots$	s_m
h_1	h_{11}	$h_{12} \cdots$	h_{1m}
h_2	h_{21}	$h_{22} \cdots$	h_{2m}
$\cdots$	$\cdots$	$\cdots$	$\cdots$
h_n	h_{n1}	$h_{n2} \cdots$	h_{nm}

Hereafter, $SM(H)$ denotes the family of all soft multisets over H with attributes from S. Now, we elaborate the definition of soft multiset by the succeeding example.

Example 3.3. Let $H = \left[\frac{k_1}{h_1}, \frac{k_2}{h_2}, \frac{k_3}{h_3}, \frac{k_4}{h_4}, \frac{k_5}{h_5}, \frac{k_6}{h_6}\right]$ be the universal multiset consist of perfumes under consideration, where $\mathrm{h}_1 = \text{azzaro}$, $h_2 = \text{coco}$, $h_3 = \text{eternity}$, $h_4 = \text{poison}$, $h_5 = \text{rogue}$, $\mathrm{h}_6 = \text{tresor}$ and k_i denotes the multiplicity of perfume $h_i, i = 1,2,\dots,6$.

Let $E = \{\text{expansive}, \text{affordable}, \text{longlastingfragrance}, \text{impressivepackaging}\}$ be the set of all attribues. Let $A = \{\text{expansive}, \text{affordable}, \text{longlastingfragrance}\} \subseteq E$. Then, the soft multiset σ_A or (σ, A) defined below describe the attractiveness of perfumes under consideration,

$$\sigma_A = (\sigma, A) = \left\{\left(\text{expansive}, \left[\frac{k_2}{h_2}, \frac{k_3}{h_3}\right]\right), \left(\text{affordable}, \left[\frac{k_1}{h_1}, \frac{k_4}{h_4}\right]\right), \left(\text{longlastingfragrance}, \left[\frac{k_5}{h_5}, \frac{k_6}{h_6}\right]\right)\right\}$$

Here the approximation set is multiset. In tabular form, the soft multiset σ_A can be represented as:

σ_A	expensive	affordable	long lasting fragrance
h_1	0	k_1	0
h_2	k_2	0	0
h_3	k_3	0	0
h_4	0	k_4	0
h_5	0	0	k_5
h_6	0	0	k_6

Definition 3.4. [20] "Let $\sigma_A \in SM(H)$. If $\sigma_A(s) = \emptyset$ for all $s \in E$, then σ_A is called an empty or null soft multiset, denoted by σ_ϕ.

Definition 3.5. [20] Let $\sigma_A, \sigma_B \in SM(H)$. Then, σ_A is a soft multi subset of σ_B, denoted by $\sigma_A \,\tilde{\sqsubseteq}\, \sigma_B$, if $\sigma_A(s) \subseteq \sigma_B(s)$, for all $s \in E$".

Example 3.6. Let $H = \left[\frac{k_1}{h_1}, \frac{k_2}{h_2}, \frac{k_3}{h_3}, \frac{k_4}{h_4}, \frac{k_5}{h_5}, \frac{k_6}{h_6}\right]$ be a universal multiset. Let $E = \{s_1, s_2, s_3, s_4\}$ be the set of attributes. Let $A = \{s_1, s_2, s_3\}, B = \{s_1, s_2\} \subseteq E$ and $B \subseteq A$. Consider soft multisets σ_A and σ_B defined on H given as, $\sigma_A = \left\{\left(s_1, \left[\frac{k_2}{h_2}, \frac{k_3}{h_3}\right]\right), \left(s_2, \left[\frac{k_1}{h_1}, \frac{k_4}{h_4}\right]\right), \left(s_3, \left[\frac{k_5}{h_5}, \frac{k_6}{h_6}\right]\right)\right\}$, $\sigma_B = \left\{\left(s_1, \left[\frac{k_2}{h_2}\right]\right), \left(s_2, \left[\frac{k_1}{h_1}, \frac{k_4}{h_4}\right]\right)\right\}$. Then, σ_B is soft multi subset of σ_A.

Definition 3.7. [20] "Let $\sigma_A \in SM(H)$. If $\sigma_A(s) = H$ for all $s \in A$, then σ_A is called A-universal soft multiset, denoted by $\sigma_{\tilde{A}}$. If $A = E$, then A-universal soft multiset is called a universal or absolute soft multiset, denoted by $\sigma_{\tilde{E}}$.

Definition 3.8. [20] Let $\sigma_A, \sigma_B \in SM(H)$. Then, σ_A and σ_B are equal soft multisets, denoted by $\sigma_A \cong \sigma_B$, if and only if $\sigma_A(s) = \sigma_B(s)$, for all $s \in E$.

Definition 3.9. [20] Let $\sigma_A, \sigma_B \in SM(H)$. Then, the union $\sigma_A \,\tilde{\cup}\, \sigma_B$ and the intersection $\sigma_A \,\tilde{\cap}\, \sigma_B$ of σ_A and σ_B is defined by the approximate functions $\sigma_{A \tilde{\cup} B}(s) = \sigma_A(s) \cup \sigma_B(s)$ and $\sigma_{A \tilde{\cap} B}(s) = \sigma_A(s) \cap \sigma_B(s)$ respectively, $\forall s \in E$".

Example 3.10. Let $H = \left[\frac{2}{\mathrm{p}_1}, \frac{4}{\mathrm{p}_2}, \frac{6}{\mathrm{p}_3}, \frac{8}{\mathrm{p}_4}, \frac{10}{\mathrm{p}_5}, \frac{12}{\mathrm{p}_6}\right]$ be a universal multiset consist of smart phones under consideration, where $p_1 =$ Samsung Galaxy Note, $p_2 =$ Nokia lumina(930), $p_3 =$ Huawei Nexus 6p, $p_4 =$ iphone 6, $p_5 =$ Motorola V3i, $p_6 =$ Sony Xperia Z5 Premium, and let $E = \{s_1, s_2, s_3, s_4\}$ be the set of attributes defined as $s_1 =$ long battery timing, $s_2 =$ expensive, $s_3 =$ durable glass screen, and $s_4 =$ metallic body. Let $A = \{s_1, s_2, s_3\}$ and $B = \{s_3, s_4\}$ be subsets of E. Then, we we can write two multisets as follows:

$$\sigma_A = \left\{\left(s_1, \left[\frac{4}{p_2}, \frac{6}{p_3}\right]\right), \left(s_2, \left[\frac{2}{p_1}, \frac{8}{p_4}\right]\right), \left(s_3, \left[\frac{10}{p_5}, \frac{12}{p_6}\right]\right)\right\} \text{ and } \sigma_B = \left\{\left(s_3, \left[\frac{10}{p_5}\right]\right), \left(s_4, \left[\frac{2}{p_1}, \frac{4}{p_2}, \frac{6}{p_3}\right]\right)\right\}$$

Then, their intersection is given by

$$\sigma_A \,\tilde{\cap}\, \sigma_B = \left\{(s_1, \emptyset), (s_2, \emptyset), \left(s_3, \left[\frac{10}{p_5}\right]\right), (s_4, \emptyset)\right\}$$

and their union is given by

$$\sigma_A \,\tilde{\cup}\, \sigma_B = \left\{\left(s_1, \left[\frac{4}{p_2}, \frac{6}{p_3}\right]\right), \left(s_2, \left[\frac{2}{p_1}, \frac{8}{p_4}\right]\right), \left(s_3, \left[\frac{10}{p_5}, \frac{12}{p_6}\right]\right), \left(s_4, \left[\frac{2}{p_1}, \frac{4}{p_2}, \frac{6}{p_3}\right]\right)\right\}$$

Definition 3.11. [20] "Let $A = \{s_1, s_2, \ldots, s_n\}$ be a set of parameters. The NOT set of A denoted by $\neg A$ is defined by $\neg A = \{\neg s_1, \neg s_2, \ldots, \neg s_n\}$, where $\neg s_i = \mathrm{not}s_i, \forall i = 1,2, \ldots, n$.

Definition 3.12. [20] Let $\sigma_A \in SM(H)$. The complement of σ_A over a multiset H is denoted by σ_A^{c} and is defined by approximate function $\sigma_A^{\mathrm{c}}(\neg s) = H \ominus \sigma_A(s)$ for all $\neg s \in \neg A$".

Here we point out that the law of excluded middle do not hold with respect to complement for soft multiset given in Definition 3.12 defined by Babitha and John [20]. We give the following counter example to explain it more effectively.

Counter Example 3.13. Suppose that $H = \left[\frac{1}{b_1}, \frac{3}{b_2}, \frac{2}{b_3}, \frac{5}{b_4}, \frac{4}{b_5}\right]$ is a universal multiset of bags under consideration. Let $E = \{\mathrm{red}, \mathrm{brown}, \mathrm{black}, \mathrm{grey}, \mathrm{white}\}$ be the set of attributes and $A = \{\mathrm{red}, \mathrm{brown}, \mathrm{black}, \mathrm{grey}\} \subseteq E$. Let $\sigma_A \in SM(H)$. That is,

$$\sigma_A = \left\{\left(\mathrm{red}, \left[\frac{1}{b_1}, \frac{3}{b_2}\right]\right), \left(\mathrm{brown}, \left[\frac{1}{b_1}, \frac{2}{b_3}\right]\right), \left(\mathrm{black}, \left[\frac{5}{b_4}, \frac{4}{b_5}\right]\right)\right\}$$

The NOT set of A is given by $\neg A = \{\mathrm{notred}, \mathrm{notbrown}, \mathrm{notblack}, \mathrm{notgrey}\}$. Then,

$$\sigma_A^c = \left\{\left(\mathrm{notred}, \left[\frac{2}{b_3}, \frac{5}{b_4}, \frac{4}{b_5}\right]\right), \left(\mathrm{notbrown}, \left[\frac{3}{b_2}, \frac{5}{b_4}, \frac{4}{b_5}\right]\right), \left(\mathrm{notblack}, \left[\frac{1}{b_1}, \frac{3}{b_2}, \frac{2}{b_3}\right]\right)\right\}$$

Therefore,

$$\sigma_A \tilde{\cup} \sigma_A^c = \left\{\left(red, \left[\frac{1}{b_1}, \frac{3}{b_2}\right]\right), \left(\text{brown}, \left[\frac{1}{b_1}, \frac{2}{b_3}\right]\right), \left(\text{black}, \left[\frac{5}{b_4}, \frac{4}{b_5}\right]\right), \left(\text{notred}, \left[\frac{2}{b_3}, \frac{5}{b_4}, \frac{4}{b_5}\right]\right), \left(\text{notbrown}, \left[\frac{3}{b_2}, \frac{5}{b_4}, \frac{4}{b_5}\right]\right), \text{notblack}\left[\frac{1}{b_1}, \frac{3}{b_2}, \frac{2}{b_3}\right]\right\}$$

$\Rightarrow \sigma_A \tilde{\cup} \sigma_A^c \neq \sigma_{\tilde{E}}$

Now to solve this problem, we modify the definition of complement of soft multiset as given below.

Definition 3.14. Let $\sigma_A \in SM(H)$. Then, the complement σ_A^c of σ_A is defined by the approximate function $\sigma_A^c(s) = H \ominus \sigma_A(s)$, for all $s \in E$. Note that $(\sigma_A^c)^c = \sigma_A$ and $\sigma_\phi^c = \sigma_{\tilde{E}}$.

We see that the law of excluded middle and law of contradiction hold with respect to complement for soft multiset given in Definition 3.14.

Proposition 3.15. Let $\sigma_A \in SM(H)$. Then,

i. Law of excluded middle:

$\sigma_A \tilde{\cup} \sigma_A^c = \sigma_{\tilde{E}}$

ii. Law of contradiction:

$\sigma_A \tilde{\cap} \sigma_A^c = \sigma_\phi$

Proof. The proof is straightforward.

Definition 3.16. Let $\sigma_A, \sigma_B \in SM(H)$. Then, the difference $\sigma_A \tilde{\setminus} \sigma_B$ of σ_A and σ_B is defined by the approximate function $\sigma_{A\tilde{\setminus}B}(s) = \sigma_A(s) \ominus \sigma_B(s)$, $\forall s \in E$.

Definition 3.17. Let $\sigma_A, \sigma_B \in SM(H)$. Then, the symmetric difference or disjunctive union $\sigma_A \tilde{\Delta} \sigma_B$ of σ_A and σ_B is defined by $\sigma_A \tilde{\Delta} \sigma_B = (\sigma_A \tilde{\cup} \sigma_B) \tilde{\setminus} (\sigma_A \tilde{\cap} \sigma_B)$ or $\sigma_A \tilde{\Delta} \sigma_B = (\sigma_A \tilde{\cap} \sigma_B^c) \tilde{\cup} (\sigma_A^c \tilde{\cap} \sigma_B) = (\sigma_A \tilde{\setminus} \sigma_B) \tilde{\cup} (\sigma_B \tilde{\setminus} \sigma_A)$.

Example 3.18. Let $H = \left[\frac{\alpha_1}{h_1}, \frac{\alpha_2}{h_2}, \frac{\alpha_3}{h_3}, \frac{\alpha_4}{h_4}, \frac{\alpha_5}{h_5}, \frac{\alpha_6}{h_6}\right]$ be a universal multiset of universities of the world under consideration, and α_i denotes the multiplicity of campuses of university $h_i, i = 1,2,\ldots,6$. The set of facilities which may be provided by these universities is given by $E = \{s_1, s_2, s_3, s_4, s_5\}$ where $s_1 =$ library, $s_2 =$ hostels, $s_3 =$ computer and internet facility, $s_4 =$ international standard course work, and $s_5 =$ best security system. Let $A = \{s_1, s_2, s_3\}$, $B = \{s_3, s_4, s_5\}$ and assume that

$$\sigma_A = \left\{\left(s_1, \left[\frac{\alpha_1}{h_1}, \frac{\alpha_2}{h_2}\right]\right), \left(s_2, \left[\frac{\alpha_3}{h_3}, \frac{\alpha_4}{h_4}\right]\right), \left(s_3, \left[\frac{\alpha_5}{h_5}, \frac{\alpha_6}{h_6}\right]\right)\right\} \text{ and } \sigma_B = \left\{\left(s_3, \left[\frac{\alpha_5}{h_5}, \frac{\alpha_6}{h_6}\right]\right), (s_4, H), \left(s_5, \left[\frac{\alpha_1}{h_1}, \frac{\alpha_2}{h_2}, \frac{\alpha_3}{h_3}, \frac{\alpha_4}{h_4}\right]\right)\right\}$$

Then,

$$\sigma_A^c = \left\{\left(s_1, \left[\frac{\alpha_3}{h_3}, \frac{\alpha_4}{h_4}, \frac{\alpha_5}{h_5}, \frac{\alpha_6}{h_6}\right]\right), \left(s_2, \left[\frac{\alpha_1}{h_1}, \frac{\alpha_2}{h_2}, \frac{\alpha_5}{h_5}, \frac{\alpha_6}{h_6}\right]\right), \left(s_3, \left[\frac{\alpha_1}{h_1}, \frac{\alpha_2}{h_2}, \frac{\alpha_3}{h_3}, \frac{\alpha_4}{h_4}\right]\right), (s_4, H), (s_5, H)\right\}$$

$$\sigma_B^c = \left\{(s_1, H), (s_2, H), \left(s_3, \left[\frac{\alpha_1}{h_1}, \frac{\alpha_2}{h_2}, \frac{\alpha_3}{h_3}, \frac{\alpha_4}{h_4}, \right]\right) (s_4, \emptyset), \left(s_5, \left[\frac{\alpha_5}{h_5}, \frac{\alpha_6}{h_6}\right]\right)\right\}$$

Now, we observe that

$$\sigma_{A\tilde{\setminus}B}(s_1) = \left[\frac{\alpha_1}{h_1}, \frac{\alpha_2}{h_2}\right], \sigma_{A\tilde{\setminus}B}(s_2) = \left[\frac{\alpha_3}{h_3}, \frac{\alpha_4}{h_4}\right], \sigma_{A\tilde{\setminus}B}(s_3) = \emptyset, \sigma_{A\tilde{\setminus}B}(s_4) = \emptyset, \text{and } \sigma_{A\tilde{\setminus}B}(s_5) = \emptyset$$

Thus we have

$$\sigma_A \tilde{\setminus} \sigma_B = \left\{\left(s_1, \left[\frac{\alpha_1}{h_1}, \frac{\alpha_2}{h_2}\right]\right), \left(s_2, \left[\frac{\alpha_3}{h_3}, \frac{\alpha_4}{h_4}\right]\right), (s_3, \emptyset), (s_4, \emptyset), (s_5, \emptyset)\right\}$$

Now, we examine that

$$\sigma_{B\tilde{\setminus}A}(s_1)=\emptyset,\ \sigma_{B\tilde{\setminus}A}(s_2)=\emptyset,\ \sigma_{B\tilde{\setminus}A}(s_3)=\emptyset,\ \sigma_{B\tilde{\setminus}A}(s_4)=H, \text{ and } \sigma_{B\tilde{\setminus}A}(s_5)=\left[\frac{\alpha_1}{h_1},\frac{\alpha_2}{h_2},\frac{\alpha_3}{h_3},\frac{\alpha_4}{h_4}\right]$$

Thus we obtain

$$\sigma_B\tilde{\setminus}\sigma_A=\left\{(s_1,\emptyset),(s_2,\emptyset),(s_3,\emptyset),(s_4,H),\left(s_5,\left[\frac{\alpha_1}{h_1},\frac{\alpha_2}{h_2},\frac{\alpha_3}{h_3},\frac{\alpha_4}{h_4}\right]\right)\right\}$$

Therefore,

$$\sigma_A\tilde{\Delta}\sigma_B=(\sigma_A\tilde{\setminus}\sigma_B)\ \tilde{\cup}\ (\sigma_B\tilde{\setminus}\sigma_A)$$

$$\sigma_A\tilde{\Delta}\sigma_B=\left\{\left(s_1,\left[\frac{\alpha_1}{h_1},\frac{\alpha_2}{h_2}\right]\right),\left(s_2,\left[\frac{\alpha_3}{h_3},\frac{\alpha_4}{h_4},\right]\right)(s_3,\emptyset),(s_4,H),\left(s_5,\left[\frac{\alpha_1}{h_1},\frac{\alpha_2}{h_2},\frac{\alpha_3}{h_3},\frac{\alpha_4}{h_4}\right]\right)\right\}$$

We observe that De-Morgan's laws also hold in soft multiset case.

Proposition 3.19. Let $\sigma_A,\sigma_B\in SM(H)$. Then,

i. $(\sigma_A\ \tilde{\cup}\ \sigma_B)^c=\sigma_A^c\ \tilde{\cap}\ \sigma_B^c$
ii. $(\sigma_A\ \tilde{\cap}\ \sigma_B)^c=\sigma_A^c\ \tilde{\cup}\ \sigma_B^c$

Proof.

i. Let $\sigma_A\ \tilde{\cup}\ \sigma_B=\sigma_D$ where, $\sigma_D(s)=\sigma_A(s)\cup\sigma_B(s)$, $\forall s\in E$. Then,

$$\begin{aligned}\sigma_D^c(s)=(\sigma_A(s)\cup\sigma_B(s))^c&=(\sigma_A(s))^c\cap(\sigma_B(s))^c\\&=\sigma_A^c(s)\cap\sigma_B^c(s)\\&=\sigma_A^c(s)\cap\sigma_B^c(s),\forall s\in E\end{aligned}$$

Thus $\sigma_D^c=\sigma_A^c\ \tilde{\cap}\ \sigma_B^c$, i.e. $(\sigma_A\ \tilde{\cup}\ \sigma_B)^c=\sigma_A^c\ \tilde{\cap}\ \sigma_B^c$.

ii. Let $\sigma_A\ \tilde{\cap}\ \sigma_B=\sigma_D$ where, $\sigma_D(s)=\sigma_A(s)\cap\sigma_B(s)$, $\forall s\in E$. Then,

$$\begin{aligned}\sigma_D^c(s)=(\sigma_A(s)\cap\sigma_B(s))^c&=(\sigma_A(s))^c\cup(\sigma_B(s))^c\\&=\sigma_A^c(s)\cup\sigma_B^c(s)\\&=\sigma_A^c(s)\cup\sigma_B^c(s),\forall s\in E\end{aligned}$$

Thus $\sigma_D^c=\sigma_A^c\ \tilde{\cup}\ \sigma_B^c$, i.e. $(\sigma_A\ \tilde{\cap}\ \sigma_B)^c=\sigma_A^c\ \tilde{\cup}\ \sigma_B^c$.

Proposition 3.20. Let $\sigma_A\in SM(H)$. Then,

i. $\sigma_A\ \tilde{\cap}\ \sigma_{\tilde{E}}=\sigma_A$ and $\sigma_A\ \tilde{\cup}\ \sigma_{\tilde{E}}=\sigma_{\tilde{E}}$
ii. $\sigma_A\ \tilde{\cup}\ \sigma_A=\sigma_A$ and $\sigma_A\ \tilde{\cap}\ \sigma_A=\sigma_A$
iii. $\sigma_A\ \tilde{\cup}\ \sigma_\phi=\sigma_A$ and $\sigma_A\ \tilde{\cap}\ \sigma_\phi=\sigma_\phi$

Proof. The proof is obvious.

We examine that commutative, associative and distributive laws hold in soft multisets.

Proposition 3.21. Let $\sigma_A,\sigma_B,\sigma_C\in SM(H)$. Then,

i. $\sigma_A\ \tilde{\cup}\ \sigma_B=\sigma_B\ \tilde{\cup}\ \sigma_A$
ii. $\sigma_A\ \tilde{\cap}\ \sigma_B=\sigma_B\ \tilde{\cap}\ \sigma_A$
iii. $(\sigma_A\ \tilde{\cup}\ \sigma_B)\ \tilde{\cup}\ \sigma_C=\sigma_A\ \tilde{\cup}\ (\sigma_B\ \tilde{\cup}\ \sigma_C)$
iv. $(\sigma_A\ \tilde{\cap}\ \sigma_B)\ \tilde{\cap}\ \sigma_C=\sigma_A\ \tilde{\cap}\ (\sigma_B\ \tilde{\cap}\ \sigma_C)$
v. $\sigma_A\ \tilde{\cup}\ (\sigma_B\ \tilde{\cap}\ \sigma_C)=(\sigma_A\ \tilde{\cup}\ \sigma_B)\ \tilde{\cap}\ (\sigma_A\ \tilde{\cup}\ \sigma_C)$
vi. $\sigma_A\ \tilde{\cap}\ (\sigma_B\ \tilde{\cup}\ \sigma_C)=(\sigma_A\ \tilde{\cap}\ \sigma_B)\ \tilde{\cup}\ (\sigma_A\ \tilde{\cap}\ \sigma_C)$

Proof. The proof is obvious.

4. Roughness and Fuzziness Associated with Soft Multi-sets

In [6], Ali has defined soft binary relation, soft equivalence relation and soft indiscernibility relation over soft set. He introduced the idea of approximation space associated with each parameter in a soft set. He also proved that for each soft set over a universe U there is a fuzzy soft set over $P(U)$ which induces a soft equivalence relation over $P(U)$. We extend these ideas to the hybrid soft set with multiset.

Definition 4.1. Let H be a universal multiset and σ_A be a soft multiset over $H \times H$. Then σ_A is called a soft multiset binary relation over H. In fact, σ_A is a parameterized family of binary relations on H, i.e for each parameter or attribute s_i, we have a binary relation $\sigma(s_i)$ on H for each parameter $s_i \in A$.

Definition 4.2. Let σ_A be a soft multiset binary relation over H. If $\sigma_A(s_i) \neq \phi$ is an equivalence relation on H for all $s_i \in A$, then σ_A is called a soft multiset equivalence relation over H.

It is well known that each equivalence relation R on a set partitions the set say H into disjoint classes. Similarly each partition of the set H provides us an equivalence relation R. Therefore a soft multiset equivalence relation over H, provides us a parameterized collection of partitions of H. The following example elaborates this concept more effectively.

Example 4.3. Let $H = \left[\frac{\alpha_1}{h_1}, \frac{\alpha_2}{h_2}, \frac{\alpha_3}{h_3}, \frac{\alpha_4}{h_4}, \frac{\alpha_5}{h_5}, \frac{\alpha_6}{h_6}\right]$ and $E = \{s_1, s_2, s_3, s_4, s_5\}$ and $A = \{s_1, s_2, s_3\}$ be a subset of E. Consider a soft multiset given by

$$\sigma_A = \left\{\left(s_1, \left[\frac{\alpha_1}{h_1}, \frac{\alpha_2}{h_2}\right]\right), \left(s_2, \left[\frac{\alpha_3}{h_3}, \frac{\alpha_4}{h_4}\right]\right), \left(s_3, \left[\frac{\alpha_5}{h_5}\right]\right)\right\}$$

In tabular form, the soft multiset σ_A is written in Table 1.

Table 1. Tabular representation of soft multiset σ_A

σ_A	s_1	s_2	s_3
h_1	α_1	0	0
h_2	α_2	0	0
h_3	0	α_3	0
h_4	0	α_4	0
h_5	0	0	α_5
h_6	0	0	0

Now we see from the table that each attribute $s_i; i = 1,2,3$ generates an equivalence relation on H. Therefore, we get a soft multiset equivalence relation σ_A over H. For each of the equivalence relation, we have the following equivalence classes.

The equivalence classes for $\sigma_A(s_1)$ are $\left[\frac{\alpha_1}{h_1}, \frac{\alpha_2}{h_2}\right], \left[\frac{\alpha_3}{h_3}, \frac{\alpha_4}{h_4}, \frac{\alpha_5}{h_5}, \frac{\alpha_6}{h_6}\right]$.

The equivalence classes for $\sigma_A(s_2)$ are $\left[\frac{\alpha_3}{h_3}, \frac{\alpha_4}{h_4}\right], \left[\frac{\alpha_1}{h_1}, \frac{\alpha_2}{h_2}, \frac{\alpha_5}{h_5}, \frac{\alpha_6}{h_6}\right]$.

The equivalence classes for $\sigma_A(s_3)$ are $\left[\frac{\alpha_5}{h_5}\right], \left[\frac{\alpha_1}{h_1}, \frac{\alpha_2}{h_2}, \frac{\alpha_3}{h_3}, \frac{\alpha_4}{h_4}, \frac{\alpha_6}{h_6}\right]$.

We observe that soft multiset σ_A defines an indiscernibility relation. Now we define an indiscernibility relation in the next definition.

Definition 4.4. Let σ_A be a soft multiset. An indiscernibility relation defined by σ_A is attained by the intersection of all the equivalence relations generated by attributes s_i and is denoted by $\mathrm{IND}(\sigma_A)$. i.e

$$\mathrm{IND}(\sigma_A) = \cap_{s_i \in A} \sigma_A(s_i)$$

Example 4.5. Consider Example 4.3 then the partition obtained by the indiscernibility relation $\mathrm{IND}(\sigma_A)$ is

$$\left[\frac{\alpha_1}{h_1},\frac{\alpha_2}{h_2}\right],\left[\frac{\alpha_3}{h_3},\frac{\alpha_4}{h_4}\right],\left[\frac{\alpha_5}{h_5}\right],\left[\frac{\alpha_6}{h_6}\right]$$

It is evident that for each attribute s_i, where $i = 1,2,3$, the soft muli-set $(H, \sigma(s_i))$ give us an approximation spaces in Pawlak's sense [61]. Also (H, σ) is an approximation space.

In the following definition we extend the idea of approximation space for multiset as an extension of approximation space for crisp set given by Chakrabarty et al. in [25].

Definition 4.6. Let H be a multiset called universe and let R be an equivalence relation on H, called indiscernibility relation. The pair (H, R) is called an approximation space.

Definition 4.7. Consider a soft multi set $\mathcal{S} = (\sigma, A)$ over the universe of multiset H and $\mathcal{E}$ be a set of parameters, where $A \subseteq \mathcal{E}$ and σ is a function given as $\sigma: A \to PW(H)$. Then the pair $P = (H, \sigma)$ is called a soft multi approximation space. Following the soft multi approximation space P, we get two approximations to every subset $J \subseteq H$ given by

$$\underline{apr_P}(J) = \left\{\frac{l}{x} \in H: \exists s_i \in A, \left[\frac{l}{x} \in \sigma(s_i) \subseteq J\right]\right\},$$
$$\overline{apr}_P(J) = \left\{\frac{l}{x} \in H: \exists s_i \in A, \left[\frac{l}{x} \in \sigma(s_i) \cap J \neq \emptyset\right]\right\},$$

which we call soft multi P-lower approximation and soft multi P-upper approximation of J. Generally, $\underline{apr_P}(J)$ and $\overline{apr}_P(J)$ are called SMR-approximations of J w.r.t P. If $\underline{apr_P}(J) \neq \overline{apr}_P(J)$ then J is said to be soft multi P-rough set or soft multi rough set (SMR-set) otherwise soft multi P-definable. Also, Soft multi P-positive region set, Soft multi P-negative region set and Soft multi P-boundary region set are defined as follows

$$POS_P(J) = \underline{apr_P}(J)$$
$$NEG_P(J) = -\overline{apr}_P(J)$$
$$BND_P(J) = \overline{apr}_P(J) - \underline{apr_P}(J)$$

Example 4.8. Suppose that $H = \left[\frac{l_1}{x_1},\frac{l_2}{x_2},\frac{l_2}{x_3}\right]$ be universial multiset of dresses under consideration, where l_1, l_2, l_3 are the multiplicity of dress x_1, x_2 and x_3 respectively. Let $E = \{s_1 = \text{modernstyle}, s_2 = \text{reasonableprice}, s_3 = \text{comfortable}, s_4 = \text{durable}, s_5 = \text{digitalpriniting}, s_6 = \text{expensive}\}$ and $A = \{s_1, \ldots, s_5\} \subseteq E$. Let $\mathcal{S} = (\sigma, A)$ be soft multiset over H, where the $\sigma: A \to PW(H)$ mapping describes the attractiveness of dresses under consideration as follows:

$$\sigma(\text{modernstyle}) = \left[\frac{l_1}{x_1}\right]$$
$$\sigma(\text{reasonableprice}) = \left[\frac{l_2}{x_2}\right]$$
$$\sigma(\text{comfortable}) = \left[\frac{l_1}{x_1},\frac{l_2}{x_2}\right]$$
$$\sigma(\text{durable}) = \left[\frac{l_1}{x_1},\frac{l_2}{x_2},\frac{l_3}{x_3}\right]$$
$$\sigma(\text{digital priniting}) = \left[\frac{l_1}{x_1}\right]$$

Thus the soft multiset can be written as

$$\mathcal{S} = (\sigma, A) = \sigma_A = \left\{\left(s_1, \left[\frac{l_1}{x_1}\right]\right), \left(s_2, \left[\frac{l_2}{x_2}\right]\right), \left(s_3, \left[\frac{l_1}{x_1},\frac{l_2}{x_2}\right]\right), (s_4, [H]), \left(s_5, \left[\frac{l_1}{x_1}\right]\right)\right\}$$

The tabular form of soft multi set $\mathcal{S} = (\sigma, A)$ is given in Table 2.

Table 2: Soft multi set (σ, A)

(σ, A)	$\frac{l_1}{x_1}$	$\frac{l_2}{x_2}$	$\frac{l_3}{x_3}$
modernstyle	1	0	0
reasonableprice	0	1	0
comfortable	1	1	0
durable	1	1	1
digital priniting	1	0	0

The soft multiset (σ, A) induces soft multi approximation space $P = (H, \sigma)$.

Equivalence classes for $\sigma(s_1)$ are $\left[\frac{l_1}{x_1}\right]$, $\left[\frac{l_2}{x_2}, \frac{l_3}{x_3}\right]$.

Equivalence classes for $\sigma(s_2)$ are $\left[\frac{l_2}{x_2}\right]$, $\left[\frac{l_1}{x_1}, \frac{l_3}{x_3}\right]$.

Equivalence classes for $\sigma(s_3)$ are $\left[\frac{l_1}{x_1}, \frac{l_2}{x_2}\right]$, $\left[\frac{l_3}{x_3}\right]$.

Equivalence classes for $\sigma(s_4)$ are $\left[\frac{l_1}{x_1}, \frac{l_2}{x_2}, \frac{l_3}{x_3}\right]$, $\emptyset$.

Equivalence classes for $\sigma(s_5)$ are $\left[\frac{l_1}{x_1}\right]$, $\left[\frac{l_2}{x_2}, \frac{l_3}{x_3}\right]$.

If we consider $J = \left[\frac{l_1}{x_1}, \frac{l_3}{x_3}\right] \subseteq H$, we obtain soft multi P-lower approximation and soft multi P-upper approximation respectively given by

$$\underline{apr_P}(J) = \left[\frac{l_1}{x_1}\right]$$
$$\overline{apr}_P(J) = \left[\frac{l_1}{x_1}, \frac{l_2}{x_2}, \frac{l_3}{x_3}\right]$$

Since $\underline{apr_P}(J) \neq \overline{apr}_P(J)$ and $J = \left[\frac{l_1}{x_1}, \frac{l_3}{x_3}\right]$ is a soft multi P-rough set or soft multi rough set (SMR-set).

Here $POS_P(J) = \{\left[\frac{l_1}{x_1}\right]\}$, $NEG_P(J) = \emptyset$ and $BND_P(J) = \left[\frac{l_2}{x_2}, \frac{l_3}{x_3}\right]$.

Note that in the case $\underline{apr_P}(J) = \overline{apr}_P(J)$, then J is said to be a soft multi P-definable set.

Remark:

It is clear from above example that the approximations of soft multi rough set are multi sets. So the operations used in soft multi rough set are multiset operations.

Definition 4.9. Let σ_A be a soft multiset over a multiset H. Then $\sigma_A: E \to PW(H)$ is a mapping. Define a map $D_s: PW(H) \to [0,1]$, for all $s \in E$ such that

$$D_s(U) = \begin{cases} \frac{|\sigma_A(s) \cap U|}{|\sigma_A(s)|}, & \text{if } \sigma_A(s) \neq \phi \\ 0, & \text{if } \sigma_A(s) = \phi \end{cases}$$

where $\forall U \in PW(H)$. Obviously for each $s \in A$, D_s is a fuzzy multiset over $PW(H)$. Hence σ_A generates a fuzzy soft multiset over $PW(H)$.

Proposition 4.10. Let σ_A be a soft multiset over a multiset H. Then $D_s(U) = D_s(V)$, for any $s \in A$ if and only if $|\sigma_A(s) \cap U| = |\sigma_A(s) \cap V|$, where $U, V \in PW(H)$.

From Proposition 4.10, it is easy to see that soft multiset σ_A generates a soft multiset binary relation over $PW(H)$. This soft multiset binary relation is denoted by λ_A and is defined as $(U, V) \in \lambda_A(s)$ if and only if $D_s(U) = D_s(V)$, where $U, V \in PW(H), s \in A$.

Theorem 4.11. The soft multiset binary relation λ_A over $PW(H)$ is a soft multiset equivalence relation and each partition $PW(H)/\lambda_A(s)$ preserves a strict order between its equivalence classes for all $s \in A$.

PROOF. We know that for each $s \in A$, $\lambda_A(s)$ is an equivalence relation by using the definition of soft multiset binary relation λ_A. Hence λ_A over $PW(H)$ is a soft multiset equivalence relation. So $PW(H)/\lambda_A(s)$ is a partition of $PW(H)$, for each $s \in A$. If for any $s \in A$, a class in $PW(H)/\lambda_A(s)$ including some element $U \in PW(H)$ is denoted by $[U]_{\lambda_A(s)}$, then for each $V \in [U]_{\lambda_A(s)}$ we get $D_s(U) = D_s(V)$, by definition. This means that a unique real number belonging to [0,1] can be assigned to each class in $PW(H)/\lambda_A(s)$. Let this number be u, for the class $[U]_{\lambda_A(s)}$ and say it characteristic of $[U]_{\lambda_A(a)}$. Hence there is a strict order between the classes because each class in $PW(H)/\lambda_A(s)$ has a unique characteristic from [0,1]. Therefore this order is defined as $[W]_{\lambda_A(s)} \prec [U]_{\lambda_A(s)}$ if and only if $w < u$, where class $[W]_{\lambda_A(s)}$ has w characteristic belonging to [0,1].

Example 4.12. By using Example 4.8, we cosider $H = \left[\frac{l_1}{x_1}, \frac{l_2}{x_2}, \frac{l_3}{x_3}\right]$ and soft multiset given by

$$\mathcal{S} = (\sigma, A) = \sigma_A = \left\{\left(s_1, \left[\frac{l_1}{x_1}\right]\right), \left(s_2, \left[\frac{l_2}{x_2}\right]\right), \left(s_3, \left[\frac{l_1}{x_1}, \frac{l_2}{x_2}\right]\right), (s_4, [H]), \left(s_5, \left[\frac{l_1}{x_1}\right]\right)\right\}$$

Let $X = \left[\frac{l_1}{x_1}, \frac{l_2}{x_2}\right]$ The power whole sub multisets (sub msets) are

$$\emptyset = \left[\frac{0}{x_1}, \frac{0}{x_2}, \frac{0}{x_3}\right], \left[\frac{l_1}{x_1}\right], \left[\frac{l_2}{x_2}\right], \left[\frac{l_3}{x_3}\right], \left[\frac{l_1}{x_1}, \frac{l_2}{x_2}\right], \left[\frac{l_2}{x_2}, \frac{l_3}{x_3}\right], \text{ and } \left[\frac{l_1}{x_1}, \frac{l_3}{x_3}\right], H = \left[\frac{l_1}{x_1}, \frac{l_2}{x_2}, \frac{l_3}{x_3}\right]$$

As $\left[\frac{l_1}{x_1}\right] \in \sigma(s_1)$ and $D_{s_1}(X) = \frac{|\sigma(s_1) \cap X|}{|\sigma(s_1|} = \frac{\left|\left[\frac{l_1}{x_1}\right]\right|}{\left|\left[\frac{l_1}{x_1}\right]\right|} = \frac{l_1}{l_1} = 1$, so $D_{s_1}\left(\left[\frac{l_1}{x_1}\right]\right) = 1$.

As $\left[\frac{l_2}{x_2}\right] \notin \sigma(s_1)$, so $D_{s_1}\left(\left[\frac{l_2}{x_2}\right]\right) = 0$. Similarly $\left[\frac{l_3}{x_3}\right] \notin \sigma(s_1)$, so $D_{s_1}\left(\left[\frac{l_3}{x_3}\right]\right) = 0$.

Thus the fuzzy multiset λ_1 associated to the parameter s_1 is given by

$$\lambda_1 = \left\{\frac{0}{\emptyset}, \frac{1}{\left[\frac{l_1}{x_1}\right]}, \frac{0}{\left[\frac{l_2}{x_2}\right]}, \frac{0}{\left[\frac{l_3}{x_3}\right]}, \frac{\frac{l_1}{l_1+l_2}}{\left[\frac{l_1}{x_1}, \frac{l_2}{x_2}\right]}, \frac{0}{\left[\frac{l_2}{x_2}, \frac{l_3}{x_3}\right]}, \frac{\frac{l_1}{l_1+l_3}}{\left[\frac{l_1}{x_1}, \frac{l_3}{x_3}\right]}, \frac{\frac{l_1}{l_1+l_2+l_3}}{\left[\frac{l_1}{x_1}, \frac{l_2}{x_2}, \frac{l_3}{x_3}\right]}\right\}$$

Similarly we can find fuzzy multiset λ_i associated to each parameter s_i, $i = 2,3,4,5$.

It is obvious that degree of membership or rough belongingness is a number from the interval [0,1].

Babitha and John [20] presented the idea of multi-valued information system as given by the following definition.

Definition 4.13. [20] "A multi-valued information system is a quadruple $I = (Z, A, f, U)$ where Z is a non-empty finite set of objects, A is a non-empty finite set of parameters, $U = \bigcup_{a \in A} U_a$, where U is the domain set (value set) of attribute a which has multi-value ($|U_a| \geq 3$) and $f: H \times A \to U$ is a total function such that $f(h, a) \in U_a$ for each $(h, a) \in Z \times A$".

Proposition 4.14. [20] If σ_A is a soft multiset over H, then σ_A is a multi-valued information system.

Example 4.15. Let $H = \left\{\frac{n_1}{h_1}, \frac{n_2}{h_2}, \frac{n_3}{h_3}, \frac{n_4}{h_4}, \frac{n_5}{h_5}, \frac{n_6}{h_6}, \frac{n_7}{h_7}, \frac{n_8}{h_8}, \frac{n_9}{h_9}, \frac{n_{10}}{h_{10}}, \frac{n_{11}}{h_{11}}, \frac{n_{12}}{h_{12}}, \frac{n_{13}}{h_{13}}, \frac{n_{14}}{h_{14}}, \frac{n_{15}}{h_{15}}\right\}$ be a multiset of some brands of shoes under consideration, where $h_i; i = 1,2,3,\ldots,15$. Let $A = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9, \alpha_{10}\}$ be the set of attributes defined as $\alpha_1 =$ leather, $\alpha_2 =$ comfortable, $\alpha_3 =$ stylish, $\alpha_4 =$ perfect fine quality, $\alpha_5 =$ relaxable for feet, $\alpha_6 =$ softer sole, $\alpha_7 =$ better grip, $\alpha_8 =$ longer life shoe, $\alpha_9 =$ discount, and $\alpha_{10} =$ cheap. Then, the soft multiset σ_A describes attractiveness of shoes under consideration as given below:

$$\sigma_A = \{(\alpha_1, \sigma_A(\alpha_1)), (\alpha_2, \sigma_A(\alpha_2)), (\alpha_3, \sigma_A(\alpha_3)), (\alpha_4, \sigma_A(\alpha_4)), (\alpha_5, \sigma_A(\alpha_5)), (\alpha_6, \sigma_A(\alpha_6)), (\alpha_7, \sigma_A(\alpha_7)), (\alpha_8, \sigma_A(\alpha_8)), (\alpha_9, \sigma_A(\alpha_9)), (\alpha_{10}, \sigma_A(\alpha_{10}))\}$$

where

$$\sigma_A(\alpha_1)=\left[\frac{n_1}{h_1},\frac{n_2}{h_2},\frac{n_4}{h_4}\right],\sigma_A(\alpha_2)=\left[\frac{n_2}{h_2},\frac{n_4}{h_4},\frac{n_6}{h_6},\frac{n_8}{h_8},\frac{n_{10}}{h_{10}},\frac{n_{12}}{h_{12}},\frac{n_{14}}{h_{14}}\right],\sigma_A(\alpha_3)=\left[\frac{n_3}{h_3},\frac{n_6}{h_6},\frac{n_9}{h_9},\frac{n_{12}}{h_{12}},\frac{n_{15}}{h_{15}}\right],$$

$$\sigma_A(\alpha_4)=\left[\frac{n_4}{h_4},\frac{n_8}{h_8},\frac{n_{12}}{h_{12}}\right],\sigma_A(\alpha_5)=\left[\frac{n_5}{h_5},\frac{n_{10}}{h_{10}},\frac{n_{15}}{h_{15}}\right],\sigma_A(\alpha_6)=\left[\frac{n_6}{h_6},\frac{n_{12}}{h_{12}}\right],\sigma_A(\alpha_7)=\left[\frac{n_7}{h_7},\frac{n_{14}}{h_{14}}\right],$$

$$\sigma_A(\alpha_8)=\left[\frac{n_1}{h_1},\frac{n_8}{h_8}\right],\sigma_A(\alpha_9)=\left[\frac{n_1}{h_1},\frac{n_2}{h_2}\right] \text{ and } \sigma_A(\alpha_{10})=\left[\frac{n_1}{h_1},\frac{n_2}{h_2},\frac{n_3}{h_3}\right]$$

Then the quadruple $I=(Z,A,f,U)$ corresponding to the soft multiset given above is a multi-valued information system. Here $Z=H$ and A is the same set of parameters as in soft multiset and $U_{\alpha_1}=\{n_1,n_2,n_4\}$, $U_{\alpha_2}=\{n_2,n_4,n_6,n_8,n_{10},n_{12},n_{14}\}$, $U_{\alpha_3}=\{n_3,n_6,n_9,n_{12},n_{15}\}$. For the pair (h_1,α_1), we have $f(h_1,\alpha_1)=n_1$. Similarly we obtain the value of other pairs. Now we construct an information table representing soft multiset σ_A given as:

Table 3. Multi-valued information systems

	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}
h_1	n_1	0	0	0	0	0	0	n_1	n_1	n_1
h_2	n_2	n_2	0	0	0	0	0	0	n_2	n_2
h_3	0	0	n_3	0	0	0	0	0	0	n_3
h_4	n_4	n_4	0	n_4	0	0	0	0	0	0
h_5	0	0	0	0	n_5	0	0	0	0	0
h_6	0	n_6	n_6	0	0	n_6	0	0	0	0
h_7	0	0	0	0	0	0	n_7	0	0	0
h_8	0	n_8	0	n_8	0	0	0	n_8	0	0
h_9	0	0	n_9	0	0	0	0	0	0	0
h_{10}	0	n_{10}	0	0	n_{10}	0	0	0	0	0
h_{11}	0	0	0	0	0	0	0	0	0	0
h_{12}	0	n_{12}	n_{12}	n_{12}	0	n_{12}	0	0	0	0
h_{13}	0	0	0	0	0	0	0	0	0	0
h_{14}	0	n_{14}	0	0	0	0	n_{14}	0	0	0
h_{15}	0	0	n_{15}	0	n_{15}	0	0	0	0	0

Thus according to Proposition 4.14, it is seen that soft multisets are multi-valued information systems. However, it is clear that multi-valued information systems are not necessarily soft multisets.

5. A Soft Multi-Set Approach to Multi-Attribute Decision-Making

In this section, we discuss the fuzzy whole sub-multisets of $PW(H)$, affiliated with each attribute of a soft multiset σ_A over H. These fuzzy sub-multisets of $PW(H)$ generate equivalence relations. These fuzzy sub-multisets and equivalence relations perform a vital job in decision making. We extend some results and algorithm which as used for soft set given in [6] to soft multiset. Thus we deduce that a soft multiset over a multiset H induces a fuzzy soft multiset over $PW(H)$ which yields a soft multiset equivalence relation on $PW(H)$. First we present an algorithm to the multi-attribute decision-making (MADM) for the selection of a best fertilizer.

Algorithm:

Step 1. Input suitable parameter set E and universal multiset H.

Step 2. Input a soft multiset σ_A over H.

Step 3. Compute fuzzy multiset D_{s_i} over $PW(H)$ corresponding to parameter s_i, $\forall s_i \in A$.

Step 4. Construct equivalence relation $\lambda_A(s_i)$, $\forall s_i \in A$ such that an equivalence class $C_j(s_i)$ of $\delta(s_i)$ contains those submultisets of $PW(H)$ that have same degree of membership.

Step 5. Arrange equivalence classes $C_j(s_i)$ of $\lambda_A(s_i)$, $\forall s_i \in A$ in order.

Step 6. Choose the equivalence class represented by $h(s_i)$ that has highest order.

Step 7. Compute the disjunctive union $T(s_i)$ such that

$$T(s_i) = \begin{cases} \Delta k(s_i), & \text{if } |h(s_i)| > 1 \\ k(s_i), & \text{if } |h(s_i)| = 1 \end{cases}$$

where $k(s_i) \in h(s_i)$.

Step 8. Find $P(\frac{n}{x}) = \{s_i : \frac{n}{x} \in T(s_i)\}$.

Step 10. Choose that $\frac{n}{x}$ that has $\max\{|P(\frac{n}{x})|\}$.

Numerical Example 5.1. Let us suppose that a farmer wants to buy a particular type of fertilizer for his fields satisfying his demand. Let $U = [20/u, 30/a, 25/c, 35/p]$ be the universal multiset of fertilizers under consideration, where u denotes "Ammonium Phosphate", a denotes "Urea", c denotes "Calcium Nitrate", p denotes "Ammonium Nitrate" and multiplicity of fertilizer denotes the number of sacks of fertilizer that are required to fertilize the fields. Let $E = \{s_1, s_2, s_3, s_4, s_5\}$ be the set of attributes defined as s_1 = rapid plant growth, s_2 = maintain green color in plants, s_3 = improve alkaline soils, s_4 = bacteria killer, s_5 = activator for enzymes in plants. Consider a soft multiset describing the different types of fertilizers under consideration and is given by

$$\sigma_A = \left\{\left(s_1, \left[\frac{20}{u}\right]\right), \left(s_2, \left[\frac{25}{c}, \frac{35}{p}\right]\right), \left(s_3, \left[\frac{30}{a}\right]\right), \left(s_4, \left[\frac{20}{u}, \frac{30}{a}, \frac{25}{c}, \frac{35}{p}\right]\right), \left(s_5, \left[\frac{20}{u}, \frac{25}{c}, \frac{35}{p}\right]\right)\right\}$$

For the attribute s_1, we have a fuzzy submultiset of $PW(H)$ given as:

$$D_{s_1} = \left\{\frac{0}{\left[\frac{0}{\phi}\right]}, \frac{1}{\left[\frac{20}{u}\right]}, \frac{0}{\left[\frac{30}{a}\right]}, \frac{0}{\left[\frac{25}{c}\right]}, \frac{0}{\left[\frac{35}{p}\right]}, \frac{1}{\left[\frac{20}{u}, \frac{30}{a}\right]}, \frac{1}{\left[\frac{20}{u}, \frac{25}{c}\right]}, \frac{1}{\left[\frac{20}{u}, \frac{35}{p}\right]}, \frac{0}{\left[\frac{30}{a}, \frac{25}{c}\right]}, \frac{0}{\left[\frac{30}{a}, \frac{35}{p}\right]}, \frac{0}{\left[\frac{25}{c}, \frac{35}{p}\right]}, \right.$$
$$\left. \frac{1}{\left[\frac{20}{u}, \frac{30}{a}, \frac{25}{c}\right]}, \frac{1}{\left[\frac{20}{u}, \frac{30}{a}, \frac{35}{p}\right]}, \frac{1}{\left[\frac{20}{u}, \frac{25}{c}, \frac{35}{p}\right]}, \frac{0}{\left[\frac{30}{a}, \frac{25}{c}, \frac{35}{p}\right]}, \frac{1}{\left[\frac{20}{u}, \frac{30}{a}, \frac{25}{c}, \frac{35}{p}\right]}\right\}$$

The equivalence classes of the equivalence relation $\lambda_A(s_1)$ generated by D_{s_1} are

$$C_1(s_1) = \left\{\phi, \left[\frac{30}{a}\right], \left[\frac{25}{c}\right], \left[\frac{35}{p}\right], \left[\frac{30}{a}, \frac{25}{c}\right], \left[\frac{30}{a}, \frac{35}{p}\right], \left[\frac{25}{c}, \frac{35}{p}\right], \left[\frac{30}{a}, \frac{25}{c}, \frac{35}{p}\right]\right\}$$

$$C_2(s_1) = \left\{\left[\frac{20}{u}\right], \left[\frac{20}{u}, \frac{30}{a}\right], \left[\frac{20}{u}, \frac{25}{c}\right], \left[\frac{20}{u}, \frac{35}{p}\right], \left[\frac{20}{u}, \frac{30}{a}, \frac{25}{c}\right], \left[\frac{20}{u}, \frac{30}{a}, \frac{35}{p}\right], \left[\frac{20}{u}, \frac{25}{c}, \frac{35}{p}\right], \left[\frac{20}{u}, \frac{30}{a}, \frac{25}{c}, \frac{35}{p}\right]\right\}$$

Order among these classes is given by $C_2(s_1) \succ C_1(s_1)$.

Since $C_2(s_1)$ has highest order, hence $h(s_1) = C_2(s_1)$.

Then, $T(s_1) = \Delta k(s_1) = (\cup\, k(s_1)) \cap (\cap\, k(s_1))^c$, where $k(s_1) \in h(s_1)$.

$T(s_1) = \left[\frac{20}{u}, \frac{30}{a}, \frac{25}{c}, \frac{35}{p}\right] \cap \left[\frac{20}{u}\right]^c$.

$T(s_1) = \left[\frac{30}{a}, \frac{25}{c}, \frac{35}{p}\right]$.

For the attribute s_2, we have a fuzzy submultiset of $PW(H)$ given as:

$$D_{s_2} = \left\{\frac{0}{\phi}, \frac{0}{\left[\frac{20}{u}\right]}, \frac{0}{\left[\frac{30}{a}\right]}, \frac{0.4}{\left[\frac{25}{c}\right]}, \frac{0.6}{\left[\frac{35}{p}\right]}, \frac{0}{\left[\frac{20}{u}, \frac{30}{a}\right]}, \frac{0.4}{\left[\frac{20}{u}, \frac{25}{c}\right]}, \frac{0.6}{\left[\frac{20}{u}, \frac{35}{p}\right]}, \frac{0.4}{\left[\frac{30}{a}, \frac{25}{c}\right]}, \frac{0.6}{\left[\frac{30}{a}, \frac{35}{p}\right]}, \frac{1}{\left[\frac{25}{c}, \frac{35}{p}\right]}, \frac{0.4}{\left[\frac{20}{u}, \frac{30}{a}, \frac{25}{c}\right]}, \frac{0.6}{\left[\frac{20}{u}, \frac{30}{a}, \frac{35}{p}\right]}, \frac{1}{\left[\frac{20}{u}, \frac{25}{c}, \frac{35}{p}\right]}, \frac{1}{\left[\frac{30}{a}, \frac{25}{c}, \frac{35}{p}\right]}, \frac{1}{\left[\frac{20}{u}, \frac{30}{a}, \frac{25}{c}, \frac{35}{p}\right]}\right\}$$

The equivalence classes of the equivalence relation $\lambda_A(s_2)$ generated by D_{s_2} are

$$C_1(s_2) = \left\{\phi, \left[\frac{20}{u}\right], \left[\frac{30}{a}\right], \left[\frac{20}{u}, \frac{30}{a}\right]\right\}, C_2(s_2) = \left\{\left[\frac{25}{c}\right], \left[\frac{20}{u}, \frac{25}{c}\right], \left[\frac{30}{a}, \frac{25}{c}\right], \left[\frac{20}{u}, \frac{30}{a}, \frac{25}{c}\right]\right\},$$

$$C_3(s_2) = \left\{\left[\frac{35}{p}\right], \left[\frac{20}{u}, \frac{35}{p}\right], \left[\frac{30}{a}, \frac{35}{p}\right], \left[\frac{20}{u}, \frac{30}{a}, \frac{35}{p}\right]\right\}, C_4(s_2) = \left\{\left[\frac{25}{c}, \frac{35}{p}\right], \left[\frac{20}{u}, \frac{25}{c}, \frac{35}{p}\right], \left[\frac{30}{a}, \frac{25}{c}, \frac{35}{p}\right], \left[\frac{20}{u}, \frac{30}{a}, \frac{25}{c}, \frac{35}{p}\right]\right\}$$

Order among these classes is given by $C_4(s_2) \succ C_3(s_2) \succ C_2(s_2) \succ C_1(s_2)$.

Since $C_4(s_2)$ has highest order, hence $h(s_2) = C_4(s_2)$.

Then, $T(s_2) = \Delta k(s_2) = (\cup\, k(s_2)) \cap (\cap\, k(s_2))^c$, where $k(s_2) \in h(s_2)$.

$T(s_2) = \left[\frac{20}{u}, \frac{30}{a}, \frac{25}{c}, \frac{35}{p}\right] \cap \left[\frac{25}{c}, \frac{35}{p}\right]^c$.

$T(s_2) = \left[\frac{20}{u}, \frac{30}{a}\right]$.

For the attribute s_3, we have a fuzzy submultiset of $PW(H)$ given as:

$$D_{s_3} = \left\{\frac{0}{\phi}, \frac{0}{\left[\frac{20}{u}\right]}, \frac{1}{\left[\frac{30}{a}\right]}, \frac{0}{\left[\frac{25}{c}\right]}, \frac{0}{\left[\frac{35}{p}\right]}, \frac{1}{\left[\frac{20}{u}, \frac{30}{a}\right]}, \frac{0}{\left[\frac{20}{u}, \frac{25}{c}\right]}, \frac{0}{\left[\frac{20}{u}, \frac{35}{p}\right]}, \frac{1}{\left[\frac{30}{a}, \frac{25}{c}\right]}, \frac{1}{\left[\frac{30}{a}, \frac{35}{p}\right]}, \frac{0}{\left[\frac{25}{c}, \frac{35}{p}\right]}, \frac{1}{\left[\frac{20}{u}, \frac{30}{a}, \frac{25}{c}\right]}, \frac{1}{\left[\frac{20}{u}, \frac{30}{a}, \frac{35}{p}\right]}, \frac{0}{\left[\frac{20}{u}, \frac{25}{c}, \frac{35}{p}\right]}, \frac{1}{\left[\frac{30}{a}, \frac{25}{c}, \frac{35}{p}\right]}, \frac{1}{\left[\frac{20}{u}, \frac{30}{a}, \frac{25}{c}, \frac{35}{p}\right]}\right\}$$

The equivalence classes of the equivalence relation $\lambda_A(s_3)$ generated by D_{s_3} are

$$C_1(s_3) = \left\{\phi, \left[\frac{20}{u}\right], \left[\frac{25}{c}\right], \left[\frac{35}{p}\right], \left[\frac{20}{u}, \frac{25}{c}\right], \left[\frac{20}{u}, \frac{35}{p}\right], \left[\frac{25}{c}, \frac{35}{p}\right], \left[\frac{20}{u}, \frac{25}{c}, \frac{35}{p}\right]\right\},$$

$$C_2(s_3) = \left\{\left[\frac{30}{a}\right], \left[\frac{20}{u}, \frac{30}{a}\right], \left[\frac{30}{a}, \frac{25}{c}\right], \left[\frac{30}{a}, \frac{35}{p}\right], \left[\frac{20}{u}, \frac{30}{a}, \frac{25}{c}\right], \left[\frac{20}{u}, \frac{30}{a}, \frac{35}{p}\right], \left[\frac{30}{a}, \frac{25}{c}, \frac{35}{p}\right], \left[\frac{20}{u}, \frac{30}{a}, \frac{25}{c}, \frac{35}{p}\right]\right\}$$

Order among these classes is given by $C_2(s_3) \succ C_1(s_3)$.

Since $C_2(s_3)$ has highest order, hence $h(s_3) = C_2(s_3)$.

Then, $T(s_3) = \Delta k(s_3) = (\cup\, k(s_3)) \cap (\cap\, k(s_3))^c$, where $k(s_3) \in h(s_3)$.

$T(s_3) = \left[\frac{20}{u}, \frac{30}{a}, \frac{25}{c}, \frac{35}{p}\right] \cap \left[\frac{30}{a}\right]^c$.

$T(s_3) = \left[\frac{20}{u}, \frac{25}{c}, \frac{35}{p}\right]$.

For the attribute s_4, we have a fuzzy submultiset of $PW(H)$ given as:

$$D_{s_4} = \left\{\frac{0}{\phi}, \frac{0.18}{\left[\frac{20}{u}\right]}, \frac{0.27}{\left[\frac{30}{a}\right]}, \frac{0.23}{\left[\frac{25}{c}\right]}, \frac{0.32}{\left[\frac{35}{p}\right]}, \frac{0.45}{\left[\frac{20}{u}, \frac{30}{a}\right]}, \frac{0.41}{\left[\frac{20}{u}, \frac{25}{c}\right]}, \frac{0.5}{\left[\frac{20}{u}, \frac{35}{p}\right]}, \frac{0.5}{\left[\frac{30}{a}, \frac{25}{c}\right]}, \frac{0.59}{\left[\frac{30}{a}, \frac{35}{p}\right]}, \frac{0.55}{\left[\frac{25}{c}, \frac{35}{p}\right]}, \frac{0.68}{\left[\frac{20}{u}, \frac{30}{a}, \frac{25}{c}\right]}, \frac{0.77}{\left[\frac{20}{u}, \frac{30}{a}, \frac{35}{p}\right]}, \frac{0.73}{\left[\frac{20}{u}, \frac{25}{c}, \frac{35}{p}\right]}, \frac{0.82}{\left[\frac{30}{a}, \frac{25}{c}, \frac{35}{p}\right]}, \frac{1}{\left[\frac{20}{u}, \frac{30}{a}, \frac{25}{c}, \frac{35}{p}\right]}\right\}$$

The equivalence classes of the equivalence relation $\lambda_A(s_4)$ generated by D_{s_4} are

$$C_1(s_4) = \{\phi\}, C_2(s_4) = \left\{\left[\frac{20}{u}\right]\right\}, C_3(s_4) = \left\{\left[\frac{30}{a}\right]\right\}, C_4(s_4) = \left\{\left[\frac{25}{c}\right]\right\}, C_5(s_4) = \left\{\left[\frac{35}{p}\right]\right\},$$
$$C_6(s_4) = \left\{\left[\frac{20}{u}, \frac{30}{a}\right]\right\}, C_7(s_4) = \left\{\left[\frac{20}{u}, \frac{25}{c}\right]\right\}, C_8(s_4) = \left\{\left[\frac{20}{u}, \frac{35}{p}\right], \left[\frac{30}{a}, \frac{25}{c}\right]\right\},$$
$$C_9(s_4) = \left\{\left[\frac{30}{a}, \frac{35}{p}\right]\right\}, C_{10}(s_4) = \left\{\left[\frac{25}{c}, \frac{35}{p}\right]\right\}, C_{11}(s_4) = \left\{\left[\frac{20}{u}, \frac{30}{a}, \frac{25}{c}\right]\right\}, C_{12}(s_4) = \left\{\left[\frac{20}{u}, \frac{30}{a}, \frac{35}{p}\right]\right\},$$
$$C_{13}(s_4) = \left\{\left[\frac{20}{u}, \frac{25}{c}, \frac{35}{p}\right]\right\}, C_{14}(s_4) = \left\{\left[\frac{30}{a}, \frac{25}{c}, \frac{35}{p}\right]\right\}, C_{15}(s_4) = \left\{\left[\frac{20}{u}, \frac{30}{a}, \frac{25}{c}, \frac{35}{p}\right]\right\}$$

Order among these classes is given by

$C_{15}(s_4) \succ C_{14}(s_4) \succ C_{12}(s_4) \succ C_{13}(s_4) \succ C_{11}(s_4) \succ C_9(s_4) \succ C_{10}(s_4) \succ C_8(s_4) \succ C_6(s_4) \succ C_7(s_4) \succ C_5(s_4) \succ C_3(s_4) \succ C_4(s_4) \succ C_2(s_4) \succ C_1(s_4)$.

Since $C_{15}(s_4)$ has highest order, hence $h(s_4) = C_{15}(s_4)$.

Then, $T(s_4) = k(s_4) = [\frac{20}{u}, \frac{30}{a}, \frac{25}{c}, \frac{35}{p}]$, since $|h(s_4)| = 1$.

For the attribute s_5, we have a fuzzy submultiset of PW(H) given as:

$$D_{s_5} = \left\{\frac{0}{\phi}, \frac{0.25}{\left[\frac{20}{u}\right]}, \frac{0}{\left[\frac{30}{a}\right]}, \frac{0.3125}{\left[\frac{25}{c}\right]}, \frac{0.4375}{\left[\frac{35}{p}\right]}, \frac{0.25}{\left[\frac{20}{u}, \frac{30}{a}\right]}, \frac{0.5625}{\left[\frac{20}{u}, \frac{25}{c}\right]}, \frac{0.6875}{\left[\frac{20}{u}, \frac{35}{p}\right]}, \frac{0.3125}{\left[\frac{30}{a}, \frac{25}{c}\right]}, \frac{0.4375}{\left[\frac{30}{a}, \frac{35}{p}\right]}, \frac{0.75}{\left[\frac{25}{c}, \frac{35}{p}\right]}, \frac{0.5625}{\left[\frac{20}{u}, \frac{30}{a}, \frac{25}{c}\right]}, \frac{0.6875}{\left[\frac{20}{u}, \frac{30}{a}, \frac{35}{p}\right]}, \frac{1}{\left[\frac{20}{u}, \frac{25}{c}, \frac{35}{p}\right]}, \frac{0.75}{\left[\frac{30}{a}, \frac{25}{c}, \frac{35}{p}\right]}, \frac{1}{\left[\frac{20}{u}, \frac{30}{a}, \frac{25}{c}, \frac{35}{p}\right]}\right\}$$

The equivalence classes of the equivalence relation $\lambda_A(s_5)$ generated by D_{s_5} are

$$C_1(s_5) = \left\{\phi, \left[\frac{30}{a}\right]\right\}, C_2(s_5) = \left\{\left[\frac{20}{u}\right], \left[\frac{20}{u}, \frac{30}{a}\right]\right\}, C_3(s_5) = \left\{\left[\frac{25}{c}\right], \left[\frac{30}{a}, \frac{25}{c}\right]\right\}, C_4(s_5) = \left\{\left[\frac{35}{p}\right], \left[\frac{30}{a}, \frac{35}{p}\right]\right\},$$
$$C_5(s_5) = \left\{\left[\frac{20}{u}, \frac{25}{c}\right], \left[\frac{20}{u}, \frac{30}{a}, \frac{25}{c}\right]\right\}, C_6(s_5) = \left\{\left[\frac{20}{u}, \frac{35}{p}\right], \left[\frac{20}{u}, \frac{30}{a}, \frac{35}{p}\right]\right\},$$
$$C_7(s_5) = \left\{\left[\frac{25}{c}, \frac{35}{p}\right], \left[\frac{30}{a}, \frac{25}{c}, \frac{35}{p}\right]\right\}, C_8(s_5) = \left\{\left[\frac{20}{u}, \frac{25}{c}, \frac{35}{p}\right], \left[\frac{20}{u}, \frac{30}{a}, \frac{25}{c}, \frac{35}{p}\right]\right\}$$

Order among these classes is given by

$C_8(s_5) \succ C_7(s_5) \succ C_6(s_5) \succ C_5(s_5) \succ C_4(s_5) \succ C_3(s_5) \succ C_2(s_5) \succ C_1(s_5)$.

Since $C_8(s_5)$ has highest order, hence $h(s_5) = C_8(s_5)$.

Then, $T(s_5) = \Delta k(s_5) = (\cup\, k(s_5)) \cap (\cap\, k(s_5))^c$, where $k(s_5) \in h(s_5)$.

$T(s_5) = \left[\frac{20}{u}, \frac{30}{a}, \frac{25}{c}, \frac{35}{p}\right] \cap \left[\frac{20}{u}, \frac{25}{c}, \frac{35}{p}\right]^c = \left[\frac{30}{a}\right]$.

Now $P(\frac{20}{u}) = \{s_2, s_3, s_4\} \Rightarrow |P(\frac{20}{u})| = 3$,

$P(\frac{30}{a}) = \{s_1, s_2, s_4, s_5\} \Rightarrow |P(\frac{30}{a})| = 4$,

$P(\frac{25}{c}) = \{s_1, s_3, s_4\} \Rightarrow |P(\frac{25}{c})| = 3$ and

$P(\frac{35}{p}) = \{s_1, s_3, s_4\} \Rightarrow |P(\frac{35}{p})| = 3$.

Therefore, $\max\{|P(\frac{n}{x})|\} = 4, \forall \frac{n}{x} \in H$.

Thus 30 sacks of "Urea" are selected to fertilize the fields because "Urea" fertilizer has maximum qualities.

6. Conclusion

We introduced some fundamental properties of soft multisets and related results. We defined binary relation, equivalence relation and indiscernibility relation on soft multisets with the help of illustrations. We introduced the concept of an approximation space associated with each parameter in a soft multiset and an approximation space associated with the soft multiset. We presented the novel concepts of roughness and fuzziness associated with soft multiset with the help of illustrations. We studied soft multisets in multi-valued information system. Furthermore, We presented an Algorithm to cope with uncertainties in the multi-attribute decision making problems by utilizing soft multiset theory. The proposed Algorithm is also summarized by the flow chart. The effectiveness of the Algorithm has verified by a case study to make the best selection of fertilizer.

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