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Norm Properties of S-Universal Operators

Joshua O. Muholo[1]*, Job O. Bonyo [2]

Abstract
We investigate the norm properties of a generalized derivation on a norm ideal $\mathscr{J}$ in $\mathscr{B}(H)$, the algebra of bounded linear operators on a Hilbert space H. Specifically, we extend the concept of $S-$universality from the inner derivation to the generalized derivation context, establish the necessary conditions for the attainment of the optimal value of the circumdiameters of numerical ranges and the spectra of two bounded linear operators on H. Moreover, we characterize the antidistance from an operator to its similarity orbit in terms of the circumdiameters, norms, numerical and spectra radii of a pair of S-universal operators.

Keywords: Spectrum, Numerical range, Generalized derivations, Circumdiameter, S-universal operators, Spectra, Numerical ranges.
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[1] *Department of Pure and Applied Mathematics, Maseno University, P.O. Box 120-40105, Maseno, Kenya, ORCID: 0000-0001-8411-6606*
[2] *Department of Pure and Applied Mathematics, Maseno University, P.O. Box 333-40105, Maseno, Kenya, ORCID: 0000-0002-6442-4211*
***Corresponding author**: otienomuholo@gmail.com
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1. Introduction

A derivation δ on an algebra $\mathscr{A}$ is a linear map $\delta : \mathscr{A} \to \mathscr{A}$ such that for all $A, B \in \mathscr{A}$, $\delta(AB) = \delta(A)B + A\delta(B)$. Fix $A, B \in \mathscr{A}$ and define a mapping of $\mathscr{A}$ into $\mathscr{A}$ by $\delta_{A,B}(X) = AX - XB$ for all $X \in \mathscr{A}$. Then $\delta_{A,B}$ is called a generalized derivation on $\mathscr{A}$. In the case that $A = B$, we have an inner derivation $\delta_A := \delta_{A,A}$. That is, $\delta_A(X) = \delta_{A,A}(X) = AX - XA$ for all $X \in \mathscr{A}$. Now, for a fixed $A \in \mathscr{A}$, the mappings R_A and L_A of $\mathscr{A}$ into $\mathscr{A}$ defined by $L_A(X) = AX$ and $R_A(X) = XA$, for all $X \in \mathscr{A}$, are called the left and the right multiplications by an operator A, respectively.
Let H be a complex Hilbert space and let $\mathscr{B}(H)$ be the algebra of all bounded linear operators on H. Stampfli [1] computed the norms of both the inner and generalized derivation on $\mathscr{B}(H)$; in particular, he proved that for fixed $A, B \in \mathscr{B}(H)$,

$$\|\delta_A\| = 2d(A), \tag{1.1}$$

where $d(A) = \inf\{\|A - \lambda I\| : \lambda \in \mathbb{C}\}$, and

$$\|\delta_{A,B}\| = \inf\{\|A - \lambda I\| + \|B - \lambda I\| : \lambda \in \mathbb{C}\}. \tag{1.2}$$

Following [2], a norm ideal $(\mathscr{J}, \|\cdot\|_{\mathscr{J}})$ in $\mathscr{B}(H)$ consists of a proper two-sided ideal $\mathscr{J}$ together with the norm $\|\cdot\|_{\mathscr{J}}$ satisfying the following conditions;

(i) $(\mathscr{J}, \|\cdot\|_{\mathscr{J}})$ is a Banach space;

(ii) $\|AXB\|_{\mathscr{J}} \le \|A\|\|X\|_{\mathscr{J}}\|B\|$ for all $X \in \mathscr{J}$ and all operators $A, B \in \mathscr{B}(H)$.

For a good account of the theory of norm ideals, we refer to [2]. An example of such an ideal is the Schatten p-ideal, $C_p(H)$, $1 \leq p \leq \infty$, see for instance [2]. The space $C_p(H)$ consists of the compact operators X such that $\sum_j S_j^p(X) < \infty$, where $\{S_j(X)\}_j$ denotes the sequence of singular values of X. For $X \in C_p(H)$ where $1 \leq p \leq \infty$, we set $\|X\|_p = (\sum_j S_j^p(X))^{\frac{1}{p}}$, where, by convention, $\|X\|_\infty = S_1(X)$ is the usual operator norm of X. Then $(C_p(H), \|\cdot\|_p)$ is a norm ideal. Moreover, $C_1(H), C_2(H)$ and $C_\infty(H)$ are the trace class, the Hilbert-Schmidt class and the class of compact operators respectively.
For $A, B \in \mathscr{B}(H)$, if $X \in \mathscr{J}$, then

$$\begin{aligned} \|\delta_{A,B}(X)\|_{\mathscr{J}} &= \|AX - XB\|_{\mathscr{J}} \\ &= \|(A-\lambda)X - X(B-\lambda)\|_{\mathscr{J}} \\ &\leq (\|A-\lambda\| + \|B-\lambda\|)\,\|X\|_{\mathscr{J}}. \end{aligned}$$

Taking supremum over all $X \in \mathscr{J}$, we get $\|\delta_{A,B}|\mathscr{J}\| \leq \|A-\lambda\| + \|B-\lambda\|$, and from equation (1.2), it follows that the restriction $\delta_{A,B}|\mathscr{J}$ of $\delta_{A,B}$ to $\mathscr{J}$ is a bounded linear operator on $(\mathscr{J}, \|\cdot\|_{\mathscr{J}})$ and

$$\|\delta_{A,B}|\mathscr{J}\| \leq \|\delta_{A,B}\| \tag{1.3}$$

for each norm ideal $\mathscr{J}$ in $\mathscr{B}(H)$. If $A = B$ in (1.3), then

$$\|\delta_A|\mathscr{J}\| \leq \|\delta_A\| = 2d(A). \tag{1.4}$$

The question as to when the equality is attained in (1.4) was considered by Fialkow [3] who introduced the concept of $S-$universal operators. An operator $A \in \mathscr{B}(H)$ is said to be $S-$universal if $\|\delta_A|\mathscr{J}\| = 2d(A)$. Having introduced the concept of $S-$universal operators, Fialkow in [3] studied the criteria of $S-$universality for a subnormal operator and posed several questions. Barraa and Boumazgour [4] later characterized $S-$universality for arbitrary hyponormal operators thereby answering a question posed by [3] in the affirmative. Motivated by the work [4], the current second author and his co-authors gave a number of results on the properties of these operators in [5, 6]. In the current paper, we extend $S-$universality to the setting of generalized derivations thereby giving a condition for a pair of operators on H to be $S-$universal.
Given an algebra $\mathscr{A}$ with a unit, let $\mathrm{Inv}(\mathscr{A})$ be the set of invertible elements of $\mathscr{A}$, and $A \in \mathrm{Inv}(\mathscr{A})$ be fixed; then the mapping α_A of $\mathscr{A}$ into $\mathscr{A}$ given by $\alpha_A(X) = A^{-1}XA$, for all $X \in \mathscr{A}$, is an automorphism on $\mathscr{A}$ and is called an inner automorphism on $\mathscr{A}$. It is clear that $\alpha_A = I$ if A belongs to the centre of $\mathscr{A}$. In particular, if $\mathscr{A}$ is commutative, then I is the only inner automorphism. We refer to [7, 8] for details on inner automorphisms. Now, for fixed $A, B \in \mathrm{Inv}(\mathscr{A})$, we define a mapping $\alpha_{A,B} : \mathscr{A} \to \mathscr{A}$ by $\alpha_{A,B}(X) = A^{-1}XB$ for all $X \in \mathscr{A}$. We shall call $\alpha_{A,B}$ a generalized inner automorphism on $\mathscr{A}$. It can be easily proved that $\alpha_{A,B}$ is indeed an automorphism on $\mathscr{A}$.
Let $A \in \mathscr{B}(H)$, we denote by $\sigma(A)$, $\sigma_p(A)$, $\sigma_{ap}(A)$, $W(A)$, $r(A)$ and $\omega(A)$; the spectrum, the point spectrum, the approximate point spectrum, the numerical range, the spectral and the numerical radii of A, respectively. We refer to [7, 12] for basic properties of numerical ranges and spectra of bounded linear operators. The numerical range and spectrum of generalized derivations on $\mathscr{B}(H)$ and their restrictions to a norm ideal $\mathscr{J}$ have been determined in literature. See for instance [9] and references therein. It was proved that

$$\sigma(\delta_{A,B}) \subseteq \sigma(A) - \sigma(B) \text{ and } \overline{W}(\delta_{A,B}) \subseteq \overline{W(A)} - \overline{W(B)}, \tag{1.5}$$

while for the restriction on $\mathscr{J}$,

$$\sigma(\delta_{A,B}|\mathscr{J}) = \sigma(A) - \sigma(B) \text{ and } \overline{W}(\delta_{A,B}|\mathscr{J}) = \overline{W(A)} - \overline{W(B)}. \tag{1.6}$$

Let S_1 and S_2 be two nonempty sets. We call the set $\mathrm{diam}_c(S_1, S_2) = \sup\{|\alpha - \beta| : \alpha \in S_1, \beta \in S_2\}$ the circumdiameter of the sets S_1 and S_2. If $S_1 = S_2$, then we simply obtain the usual diameter of S_1, $\mathrm{diam}(S_1) = \sup\{|\alpha - \beta| : \alpha, \beta \in S_1\}$. In this study, we shall consider two circumdiameters $\mathrm{diam}_c(\overline{W}(A;B)) := \mathrm{diam}_c(\overline{W}(A), \overline{W}(B))$ and $\mathrm{diam}_c(\sigma(A;B)) := \mathrm{diam}_c(\sigma(A), \sigma(B))$. It is important to note that when $A = B$, then the circumdiameters $\mathrm{diam}_c(W(A;A))$ and $\mathrm{diam}_c(\sigma(A;A))$ turn out to be the diameters of the numerical range and the spectrum of A, respectively, and whose relationships with the norms of derivations were well studied in [5, 6].

2. Algebraic Properties of Generalized Derivations

In this section, we study various properties of the generalized derivation acting on an algebra $\mathscr{A}$.

Proposition 2.1. *A generalized derivation $\delta_{A,B}$ is linear but fails to be a derivation on an algebra $\mathscr{A}$ while an inner derivation $\delta_{A,A}$ is a derivation on $\mathscr{A}$.*

Proof. First we prove that $\delta_{A,B}$ is linear. Fix $A,B \in \mathscr{A}$ and let $\alpha,\beta \in \mathbb{C}$. Then for all $X,Y \in \mathscr{A}$, we have

$$\begin{aligned}\delta_{A,B}(\alpha X+\beta Y) &= A(\alpha X+\beta Y)-(\alpha X+\beta Y)B\\ &= \alpha(AX-XB)+\beta(AY-YB)\\ &= \alpha\delta_{A,B}(X)+\beta\delta_{A,B}(Y).\end{aligned}$$

Next we show that $\delta_{A,B}$ fails to be a derivation on $\mathscr{A}$. Indeed, for all $X,Y \in \mathscr{A}$, we have;

$$\begin{aligned}\delta_{A,B}(XY) &= A(XY)-(XY)B\\ &= AXY-XYB+XBY-XBY\\ &= (AX-XB)Y+X(BY-YB)\\ &= \delta_{A,B}(X)Y+X\delta_{B,B}(Y).\end{aligned}$$

Since $\delta_{A,B}(X)Y+X\delta_{B,B}(Y)$ is not equal to $\delta_{A,B}(X)Y+X\delta_{A,B}(Y)$, it follows that $\delta_{A,B}$ fails to be a derivation on $\mathscr{A}$. On the other hand, an inner derivation $\delta_{A,A}$ turns out to be a derivation. Indeed, the linearity of δ_A follows from the linearity of $\delta_{A,B}$. Now for a fixed $A \in \mathscr{A}$, we have for all $X,Y \in \mathscr{A}$,

$$\begin{aligned}\delta_A(XY) &= A(XY)-(XY)A.\\ &= (AX-XA)Y+X(AY-YA)\\ &= \delta_A(X)Y+X\delta_A(Y), \quad \text{as desired.}\end{aligned}$$

This completes the proof. □

In the next proposition, we prove that the sum of two generalized derivations is a generalized derivation

Proposition 2.2. *The sum of two generalized derivations on $\mathscr{A}$ is a generalized derivation on $\mathscr{A}$. In particular, for fixed $A,B,C,D \in \mathscr{A}$,*

$$\delta_{A,B}+\delta_{C,D}=\delta_{A+C,B+D}.$$

Proof. For fixed $A,B,C,D \in \mathscr{A}$ and for all $X \in \mathscr{A}$, it follows from the linearity of a generalized derivation that,

$$\begin{aligned}(\delta_{A,B}+\delta_{C,D})(X) &= \delta_{A,B}(X)+\delta_{C,D}(X)\\ &= AX-XB+CX-XD\\ &= (A+C)X-X(B+D)\\ &= \delta_{A+C,B+D}(X).\end{aligned}$$

□

The following is an immediate consequence of proposition 2.2 above.

Corollary 2.3. *For fixed $A,C \in \mathscr{A}$, we have $\delta_A+\delta_C=\delta_{A+C}$.*

Remark 2.4. The question of when the product of two derivations is a derivation has been considered by a number of authors. For instance [11] characterized when the product $\delta_{C,D}\delta_{A,B}$ is a generalized derivation in the cases when $\mathscr{A}$ is the algebra of all bounded operators on a Banach space and when $\mathscr{A}$ is a C^*-algebra.

Proposition 2.5. *Let $\delta_{A,B}$ be a generalized derivation on an algebra $\mathscr{A}$, then for each $n \in \mathbb{N}$,*

$$\delta^n_{A,B}(X)=\sum_{r=0}^{n}(-1)^r\binom{n}{r}A^{n-r}XB^r \tag{2.1}$$

for all $X \in \mathscr{A}$.

Proof. By mathematical induction, let $p(n)$ be the statement that for all $n \in \mathbb{N}$, equation (2.1) holds. Then clearly $p(1)$ is true. Now, suppose that $p(k)$ is true for $k \in \mathbb{N}$. This means that for all $X \in \mathscr{A}$, $\delta_{A,B}^{k}(X) = \sum_{r=0}^{k}(-1)^r\binom{k}{r}A^{k-r}XB^r$. Then, for $p(k+1)$, we have

$$\begin{aligned}
\delta_{A,B}^{k+1}(X) &= \delta_{A,B}(\delta_{A,B}^{k}(X)) \\
&= A\delta_{A,B}^{k}(X) - \delta_{A,B}^{k}(X)B \\
&= A\left(\sum_{r=0}^{k}(-1)^r\binom{k}{r}A^{k-r}XB^r\right) - \left(\sum_{r=0}^{k}(-1)^r\binom{k}{r}A^{k-r}XB^r\right)B \\
&= \sum_{r=0}^{k}(-1)^r\binom{k}{r}A^{k-r+1}XB^r - \sum_{r=0}^{k}(-1)^r\binom{k}{r}A^{k-r}XB^{r+1} \\
&= \sum_{r=0}^{k+1}(-1)^r\binom{k}{r}A^{k-r+1}XB^r - \sum_{r=1}^{k+1}(-1)^{r-1}\binom{k}{r-1}A^{k-r+1}XB^r \\
&= \sum_{r=0}^{k+1}(-1)^r\binom{k}{r}A^{k-r+1}XB^r + \sum_{r=0}^{k+1}(-1)^{r}\binom{k}{r-1}A^{k-r+1}XB^r \\
&= \sum_{r=0}^{k+1}(-1)^r\left(\binom{k}{r}+\binom{k}{r-1}\right)A^{k-r+1}XB^r \\
&= \sum_{r=0}^{k+1}(-1)^r\binom{k+1}{r}A^{k-r+1}XB^r.
\end{aligned}$$

Thus $p(k+1)$ is true. Hence $p(k)$ implies $p(k+1)$ and therefore by the principle of mathematical induction, it follows that $p(n)$ is true for all $n \in \mathbb{N}$. $\square$

For inner automorphisms, if δ is a continuous derivation on a Banach algebra $\mathscr{A}$, then $\exp(\delta)$ is a continuous automorphism on $\mathscr{A}$ and if A is an element of a Banach algebra $\mathscr{A}$ with unit, then $\exp(\delta_A) = \alpha_{\exp A}$, see [7, 8]. In the next result, we extend these relations to the setting of the generalized derivation $\delta_{A,B}$ and generalized inner automorphism $\alpha_{\exp A,\exp B}(X)$.

Proposition 2.6. *Let $\delta_{A,B}$ be a generalized derivation on a Banach algebra $\mathscr{A}$. Then,* $\exp\delta_{A,B}(X) = \exp(A)X\exp(-B) = \alpha_{\exp A,\exp B}(X)$.

Proof. Using equation (2.1), we have

$$\begin{aligned}
\exp\delta_{A,B}(X) &= \sum_{n=0}^{\infty}\frac{1}{n!}\delta_{A,B}^{n}(X) \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{r=0}^{n}(-1)^r\binom{n}{r}A^{n-r}XB^r \\
&= \sum_{n=0}^{\infty}\sum_{r=0}^{n}(-1)^r\frac{1}{n!}\frac{n!}{(n-r)!r!}A^{n-r}XB^r \\
&= \sum_{n=0}^{\infty}\sum_{r=0}^{n}(-1)^r\frac{1}{(n-r)!}\frac{1}{r!}A^{n-r}XB^r \\
&= \sum_{n=0}^{\infty}\sum_{r=0}^{n}\left((-1)^r\frac{1}{(n-r)!}A^{n-r}\right)X\left(\frac{1}{r!}B^r\right) \\
&= \sum_{n=0}^{\infty}\sum_{r=0}^{n}\left(\frac{1}{(n-r)!}(A)^{n-r}\right)X(-1)^r\left(\frac{1}{r!}B^r\right) \\
&= \sum_{n=0}^{\infty}\sum_{r=0}^{n}\left(\frac{1}{(n-r)!}(A)^{n-r}\right)X\left((-1)^r\frac{1}{r!}B^r\right) \\
&= \exp(A)X\exp(-B) \\
&= \alpha_{\exp A,\exp B}(X), \text{ as claimed.}
\end{aligned}$$

$\square$

3. $S-$universality and Generalized Derivations

In this section, we consider $\mathscr{A} = \mathscr{B}(H)$, the algebra of bounded linear operators on H and study the norm properties of generalized derivations restricted to norm ideals $\mathscr{J}$ in $\mathscr{B}(H)$. Most importantly, we extend the concept of $S-$universal operators to the setting of generalized derivations.

Theorem 3.1. *Let $A, B \in \mathscr{B}(H)$ be $S-$universal operators and $\mathscr{J}$ a norm ideal in $\mathscr{B}(H)$. Then $\|\delta_{A,B}\| = \|\delta_{A,B}|\mathscr{J}\|$.*

Proof. For fixed $A, B \in \mathscr{B}(H)$, we have that $\|\delta_A\| = 2d(A)$ and $\|\delta_B\| = 2d(B)$. Since A, B are $S-$universal it follows that $\|\delta_A\| = 2\|A\|$ and $\|\delta_B\| = 2\|B\|$, see [6]. Thus, $d(A) + d(B) = \|A\| + \|B\|$. That is; $\inf_{\lambda\in\mathbb{C}} \|A-\lambda\| + \inf_{\lambda\in\mathbb{C}} \|B-\lambda\| = \|A\| + \|B\|$. This implies that $\inf_{\lambda\in\mathbb{C}}(\|A-\lambda\| + \|B-\lambda\|) = \|A\| + \|B\|$. But $\inf_{\lambda\in\mathbb{C}}(\|A-\lambda\| + \|B-\lambda\|) = \|\delta_{A,B}\|$ so that $\|\delta_{A,B}\| = \|A\| + \|B\|$. This is equivalent to $W_N(A) \cap W_N(-B) \neq \emptyset$ which by [6] further implies that $\|\delta_{A,B}\| = \frac{1}{2}(\|\delta_A\| + \|\delta_B\|) = \frac{1}{2}(\|\delta_A|\mathscr{J}\| + \|\delta_B|\mathscr{J}\|) = \|\delta_{A,B}|\mathscr{J}\|$, where $W_N(A)$ is the normalized maximal numerical range of A. This completes the proof. ☐

The theorem 3.1 above extends the notion of S-universality from the setting of inner derivation to the setting of a generalized derivation. In particular, give the following definition;

Definition 3.2. Let $A, B \in \mathscr{B}(H)$. The pair (A, B) is said to be $S-$universal if $\|\delta_{A,B}|\mathscr{J}\| = \|\delta_{A,B}\|$.

As noted earlier, a special class of norm ideals is the Schatten p-ideal $C_p(H)$.

Theorem 3.3. *Let $A, B \in B(H)$ be $S-$universal, then*

$$\|\delta_{A,B}|C_p\| = \|A\| + \|B\|.$$

Proof. Since A, B are $S-$universal and $C_p(H)$ is a norm ideal in $\mathscr{B}(H)$, it follows that $\|\delta_{A,B}|C_p\| = \|\delta_{A,B}\| = \inf_{\lambda\in\mathbb{C}}(\|A-\lambda\| + \|B-\lambda\|)$. By a compactness argument, $\exists \mu \in \mathbb{C}$ such that $\inf_{\lambda\in\mathbb{C}}(\|A-\lambda\| + \|B-\lambda\|) = \|A-\mu\| + \|B-\mu\|$. We note that $\delta_{A,B}|C_p = \delta_{A-\mu,B-\mu}|C_p = L_{A-\mu}|C_p - R_{B-\mu}|C_p$. Thus $\|L_{A-\mu}|C_p - R_{B-\mu}|C_p\| = \|A-\mu\| + \|B-\mu\|$. On the other hand, since $\|L_{A-\mu}\| = \|A-\mu\|$ and $\|R_{B-\mu}\| = \|B-\mu\|$, it follows that $\|L_{A-\mu}|C_p - R_{B-\mu}|C_p\| = \|L_{A-\mu}|C_p\| + \|R_{B-\mu}|C_p\|$. Without loss of generality, we may assume that $\mu = 0$. Then $\|L_A|C_p - R_B|C_p\| = \|L_A|C_p\| + \|R_B|C_p\| = \|A\| + \|B\|$. This completes the proof. ☐

The following are immediate from Theorem 3.3 above;

Corollary 3.4. *Let $A, B \in \mathscr{B}(H)$ be $S-$universal operators, then $\|\delta_{A,B}|\mathscr{J}\| = \|A\| + \|B\|$*

Proof. Since $C_p(H) \subseteq \mathscr{J}$, it follows by Theorem 3.3 that
$\|\delta_{A,B}|\mathscr{J}\| \geq \|\delta_{A,B}|C_p\| = \|A\| + \|B\|$. The rest of the proof follows from the fact that $\|\delta_{A,B}|\mathscr{J}\| \leq \|A\| + \|B\|$. ☐

Remark 3.5. For $A, B \in \mathscr{B}(H)$, the equation

$$\|A-B\| = \|A\| + \|B\| \tag{3.1}$$

was studied by many authors, see for instance [4, 10] and references therein. In [10], it is shown that if A and B satisfy equation (3.1), then 0 must be in the approximate point spectrum of the operator $\|B\|A + \|A\|B$. Moreover, Lin proved that the converse holds if either A or B is an isometric operator. Another result in this direction as provided in [4] asserts that non-zero A and B in $\mathscr{B}(H)$ satisfy the equation (3.1), if and only if $\|A\|\|B\|$ is in the closure of the numerical range of the operator $-A*B$.

We now give further consequences of Theorem 3.3,

Corollary 3.6. *Let $A, B \in \mathscr{B}(H)$. If the operators A, B are S-universal and L_A, R_B are defined on $C_p(H)$. Then $0 \in \sigma_{ap}(\|A\|R_B + \|B\|L_A)$. Moreover, the converse holds if either A or B is isometric.*

Proof. By Theorem 3.3, we have that for $A, B \in \mathscr{B}(H)$ $S-$universal, $\|L_A|C_p - R_B|C_p\| = \|L_A|C_p\| + \|R_B|C_p\| = \|A\| + \|B\|$. The result now follows from Remark 3.5. ☐

Corollary 3.7. *Let $A, B \in \mathscr{B}(H)$. If the operators A, B are $S-$universal, then $\|L_A|C_p\|\|R_B|C_p \in \overline{W(-L_{A^*}|C_pR_B|C_p)}$*

Another consequence which follows from the fact that $\|A\| \in \sigma(A)$ if and only if $\|A\| \in \overline{W(A)}$ and Corollary 3.7 is the following;

Corollary 3.8. *Let $A,B \in \mathscr{B}(H)$ be $S-$universal, then*

$$\|L_A|C_p\|\|R_B|C_p\| \in \sigma(-L_{A^*}|C_pR_B|C_p).$$

In the next results, we consider the pair of $S-$universal operators $A,B \in \mathscr{B}(H)$ and establish the relationship between the circumdiameter
$\mathrm{diam}_c(\overline{W}(A;B))$ and the norm of a generalized derivation.

Theorem 3.9. *Let $A,B \in \mathscr{B}(H)$ be $S-$universal, then $diam_c(\overline{W}(A;B)) = \|A\| + \|B\|$.*

Proof. If the pair A,B are $S-$universal, then by corollary 3.8,
we have $\|L_A|C_p\|\|R_B|C_p\| \in \sigma(-L_{A^*}|C_pR_B|C_p)$. But $\sigma(-L_{A^*}|C_pR_B|C_p) = -\sigma(A^*)\sigma(B)$, and $\|L_A|C_p\|\|R_B|C_p\| = \|A\|\|B\|$. So there exists $\alpha \in \sigma(A), \beta \in \sigma(B)$ such that $\|A\|\|B\| = -\overline{\alpha}\beta$. Since $|\alpha| \leq \|A\|$ and $|\beta| \leq \|B\|$, there exists $\theta \in \mathbb{R}$ such that $\alpha = \|A\|e^{i\theta}$ and $\beta = -\|B\|e^{i\theta}$. Also since $\sigma(\delta_{A,B}|C_p) = \sigma(A) - \sigma(B)$, it follows that $r(\delta_{A,B}|C_p) = \mathrm{diam}_c(\sigma(A;B)) \geq |\alpha - \beta| = |\|A\|e^{i\theta} + \|B\|e^{i\theta}| = \|A\| + \|B\|$. By the spectral inclusion, it follows that $\mathrm{diam}_c(\overline{W}(A;B)) \geq \mathrm{diam}_c(\sigma(A;B)) \geq \|A\| + \|B\|$. For the reverse inequality, we have

$$\begin{aligned}
\mathrm{diam}_c(\overline{W}(A;B)) &= \sup\{|\alpha - \beta| : \alpha \in \overline{W(A)}, \beta \in \overline{W(B)}\} \\
&\leq \sup\{|\alpha| + |\beta| : \alpha \in \overline{W(A)}, \beta \in \overline{W(B)}\} \\
&\leq \sup\{|\alpha| : \alpha \in \overline{W(A)}\} + \sup\{|\beta| : \beta \in \overline{W(B)}\} \\
&\leq \|A\| + \|B\|, \text{ as desired.}
\end{aligned}$$

□

The following consequences are immediate;

Corollary 3.10. *Let $A,B \in \mathscr{B}(H)$ be $S-$universal, then*

$$diam_c(\overline{W}(A;B)) = \|\delta_{A,B}|\mathscr{J}\|.$$

Proof. Follows from Theorem 3.9 and Corollary 3.4. □

Corollary 3.11. *Let $A,B \in \mathscr{B}(H)$ be $S-$universal operators, then*

$$diam_c(\overline{W}(A;B)) = \|\delta_{A,B}\|.$$

Another consequence of Theorem 3.9 which interestingly coincides with and is a summary of the results obtained in [6] is the following;

Corollary 3.12. *Let $A \in \mathscr{B}(H)$ be $S-$universal operator, $\mathscr{J}$ be a norm ideal in $\mathscr{B}(H)$ and $C_p(H)$ be the Schatten norm ideal in $\mathscr{B}(H)$. Then,*
$diam(\overline{W}(A)) = \|\delta_A\| = \|\delta_A|\mathscr{J}\| = \|\delta_A|C_p\| = 2\|A\|$.

Proof. From Theorem 3.9, Corollaries 3.10 and 3.11 above, we have that for $A = B$,

$$\mathrm{diam}_c(\overline{W}(A;A)) = \|\delta_{A,A}|\mathscr{J}\| = \|\delta_{A,A}|\mathscr{B}(H)\| = 2\|A\|.$$

□

For arbitrary operators $A,B \in \mathscr{B}(H)$, the circumdiameters $\mathrm{diam}_c(\overline{W}(A,B))$ and $\mathrm{diam}_c(\sigma(A;B))$ are related to the sum of numerical and spectral radii of A and B, respectively. In fact for $A,B \in \mathscr{B}(H)$,

$$\begin{aligned}
\mathrm{diam}_c(\overline{W}(A;B)) &= \sup\{|\alpha - \beta| : \alpha \in \overline{W}(A), \beta \in \overline{W}(B)\} \\
&\leq \sup\{|\alpha| + |\beta| : \alpha \in \overline{W}(A), \beta \in \overline{W}(B)\} \\
&\leq \omega(A) + \omega(B). \qquad (3.2)
\end{aligned}$$

Similarly, it can be shown that

$$\mathrm{diam}_c(\sigma(A;B)) \leq r(A) + r(B). \qquad (3.3)$$

Remark 3.13. A natural question then arises: When can equalities be obtained in (3.2) and (3.3)? In the next results, we answer this question in the affirmative in the case that the operators are $S-$universal.

Theorem 3.14. *Let $A,B \in \mathscr{B}(H)$ be $S-$universal, then*

1. $diam_c(\overline{W}(A;B)) = \omega(A)+\omega(B)$
2. $diam_c(\sigma(A;B)) = r(A)+r(B)$.

Proof. It is clear from equation (3.2) that for arbitrary operators $A,B \in \mathscr{B}(H)$, we have $\text{diam}_c(W(A;B)) \leq \omega(A)+\omega(B)$. To prove the reverse inequality, we have for A,B $S-$universal, $\text{diam}_c(\overline{W}(A;B)) = \|\delta_{A,B}\| = \|A\|+\|B\| \geq \omega(A)+\omega(B)$, which proves (1). The proof of (2) is similar. $\square$

The diameters $\text{diam}_c(\overline{W}(A;B))$ and $\text{diam}_c(\sigma(A;B))$ are respectively related to the numerical and spectral radii of a generalized derivation. In fact for a generalized derivation on a norm ideal $\mathscr{J}$, $\text{diam}_c(\overline{W}(A;B))$ and $\text{diam}_c(\sigma(A;B))$ turn out to be exactly the numerical and spectral radii of the generalized derivation respectively, as we give in the following theorem;

Theorem 3.15. *For $A,B \in \mathscr{B}(H)$, we have;*

1. $\omega(\delta_{A,B}) \leq diam_c(\overline{W}(A;B))$
2. $r(\delta_{A,B}) \leq diam_c(\sigma(A;B))$. *Moreover, if $\mathscr{J}$ is a norm ideal in $\mathscr{B}(H)$, we have;*
3. $\omega(\delta_{A,B}|\mathscr{J}) = diam_c(\overline{W}(A;B))$
4. $r(\delta_{A,B}|\mathscr{J}) = diam_c(\sigma(A;B))$.

Proof. As remarked in the introduction, we have that for $A,B \in \mathscr{B}(H)$, $\overline{W}(\delta_{A,B}) \subseteq \overline{W(A)} - \overline{W(B)}$. Let $\lambda \in \overline{W}(\delta_{A,B})$. Then $\exists$ $\alpha \in \overline{W(A)}$ and $\beta \in \overline{W(B)}$ such that $|\lambda| \leq |\alpha-\beta|$. Taking supremum over all $\lambda \in \overline{W}(\delta_{A,B})$, we obtain $\omega(\delta_{A,B}) \leq |\alpha-\beta|$. Now, taking supremum over all $\alpha \in \overline{W(A)}$ and $\beta \in \overline{W(B)}$, we get, $\omega(\delta_{A,B}) \leq \sup\{|\alpha-\beta| : \alpha \in \overline{W(A)}, \beta \in \overline{W(B)}\} = \text{diam}_c(\overline{W}(A,B))$. On the other hand, we have; $\sigma(\delta_{A,B}) \subseteq \sigma(A)-\sigma(B)$. Now, by letting $\lambda \in \sigma(\delta_{A,B})$, it follows that $\exists \lambda_1 \in \sigma(A), \lambda_2 \in \sigma(B)$ such that $|\lambda| \leq |\lambda_1-\lambda_2|$. Taking supremum over all $\lambda \in \sigma(\delta_{A,B})$ and then over all $\lambda_1 \in \sigma(A), \lambda_2 \in \sigma(B)$, we obtain $r(\delta_{A,B}) \leq \text{diam}_c(\sigma(A;B))$. This proves assertions 1 and 2. To prove assertions 3 and 4, we have that the restriction of $\delta_{A,B}$ to a norm ideal $\mathscr{J}$ yields the equalities; $\overline{W}(\delta_{A,B}) = \overline{W(A)} - \overline{W(B)}$ and $\sigma(\delta_{A,B}) = \sigma(A)-\sigma(B)$. Now by similar arguments as above, we obtain the assertions 3 and 4. $\square$

As an immediate consequence, we give the following;

Corollary 3.16. *For $A,B \in \mathscr{B}(H)$, we have*

1. $\omega(\delta_{A,B}) \leq \omega(A)+\omega(B)$
2. $r(\delta_{A,B}) \leq r(A)+r(B)$
   Moreover, if $A,B \in \mathscr{B}(H)$ are $S-$universal, then
3. $\omega(\delta_{A,B}|\mathscr{J}) = \omega(A)+\omega(B)$
4. $r(\delta_{A,B}|\mathscr{J}) = r(A)+r(B)$

Proof. Following Theorems 3.14 and 3.15, we have;

$$\omega(\delta_{A,B}) \leq \text{diam}_c(\overline{W}(A;B)) \leq \omega(A)+\omega(B)$$

and

$$r(\delta_{A,B}) \leq \text{diam}_c(\sigma(A;B)) \leq r(A)+r(B).$$

Now, assume that $A,B \in \mathscr{B}(H)$ are $S-$universal operators. Then, Theorems 3.14 and 3.15 yield

$$\omega(\delta_{A,B}|\mathscr{J}) = \text{diam}_c(\overline{W}(A;B)) = \omega(A)+\omega(B)$$

and

$$r(\delta_{A,B}|\mathscr{J}) = \text{diam}_c(\sigma(A;B)) = r(A)+r(B),$$

as desired. $\square$

4. Normaloid and Spectraloid operators

In this section, we explore other special classes of operators for which we obtain the equalities $\text{diam}_c(\overline{W}(A;B)) = \omega(A) + \omega(B)$ and $\text{diam}_c(\sigma(A;B)) = r(A) + r(B)$ without the operators $A,B \in \mathscr{B}(H)$ being necessarily S-universal. Recall that an operator $A \in \mathscr{B}(H)$ is said to be **normaloid** if $\omega(A) = \|A\|$, while it is said to be **spectraloid** if $r(A) = \omega(A)$. Note that a normaloid operator is a spectraloid operator. We refer to [12] for details on these operators. We give the following result;

Theorem 4.1. *If $A,B \in \mathscr{B}(H)$ are both normaloid operators, then;*

1. *$\text{diam}_c(\sigma(A;B)) = r(A) + r(B)$*
2. *$\text{diam}_c(\overline{W}(A;B)) = \omega(A) + \omega(B)$.*

Proof. By equation (3.3), we have that for arbitrary $A,B \in \mathscr{B}(H)$,
$\text{diam}_c(\sigma(A,B)) \leq r(A) + r(B)$. Now, we suppose that both $A,B \in \mathscr{B}(H)$ are normaloid and prove the reverse inequality. By definition; $\text{diam}_c(\sigma(A,B)) = \sup\{|\alpha - \beta| : \alpha \in \sigma(A), \beta \in \sigma(B)\} \geq |\alpha - \beta|$ for all $\alpha \in \sigma(A), \beta \in \sigma(B)$. For $\alpha \in \sigma(A)$ and $\beta \in \sigma(B)$, we have that $|\alpha| \leq \|A\|$ and $|\beta| \leq \|B\|$. Let $\theta \in \mathbb{R}$ such that $\alpha = \|A\|e^{i\theta}$ and $\beta = -\|B\|e^{i\theta}$. Then since A,B are normaloid, it follows that

$$|\alpha - \beta| = |\|A\|e^{i\theta} + \|B\|e^{i\theta}| = \|A\| + \|B\| = \omega(A) + \omega(B) \geq r(A) + r(B).$$

Therefore $\text{diam}_c(\sigma(A;B)) \geq r(A) + r(B)$ and hence $\text{diam}_c(\sigma(A;B)) = r(A) + r(B)$, as desired. This proves 1. To prove assertion 2, recall from equation (3.2) that $\text{diam}_c(\overline{W}(A;B)) \leq \omega(A) + \omega(B)$ for arbitrary $A,B \in \mathscr{B}(H)$. Now, by the spectral inclusion, the definition of a normaloid operator as well as the proof of assertion 1 above, we have:

$$\text{diam}_c(\overline{W}(A;B)) \geq \text{diam}_c(\sigma(A;B)) = \|A\| + \|B\| = \omega(A) + \omega(B).$$

This completes the proof. $\square$

Theorem 4.2. *Let $A,B \in \mathscr{B}(H)$. Then the following are equivalent:*

1. *Both A and B are normaloid.*
2. *Both A and B are spectraloid.*
3. *$\text{diam}_c(\overline{W}(A;B)) = \omega(A) + \omega(B)$.*
4. *$\text{diam}_c(\sigma(A;B)) = r(A) + r(B)$.*
5. *The pair (A,B) is $S-$universal.*

Proof. $(1) \Rightarrow (2)$: By the fact that a normaloid operator is a spectraloid. Clearly, $(2) \Rightarrow (3)$. From the proof of Theorem 4.1, we have that $\text{diam}_c(\sigma(A;B)) \geq \|A\| + \|B\|$. But we know that $\text{diam}_c(\sigma(A;B)) \leq \|A\| + \|B\|$. Hence $\text{diam}_c(\sigma(A;B)) = \|A\| + \|B\|$. This implies that $\text{diam}_c(\overline{W}(A;B)) = \|A\| + \|B\|$ since it is obvious that $\text{diam}_c(\overline{W}(A;B)) \leq \|A\| + \|B\|$ and $\text{diam}_c(\overline{W}(A;B)) \geq \text{diam}_c(\sigma(A;B)) = \|A\| + \|B\|$. Thus $\text{diam}_c(\sigma(A;B)) = \text{diam}_c(\overline{W}(A;B)) = \omega(A) + \omega(B) \geq r(A) + r(B)$. So $\text{diam}_c(\sigma(A;B)) = r(A) + r(B)$. Hence $(3) \Rightarrow (4)$. $(4) \Rightarrow (5)$: Now, $\text{diam}_c(\sigma(A;B)) = r(A) + r(B) = r(\delta_{A,B}|\mathscr{J})$ which implies that A,B are $S-$universal by Corollary 3.16. $(5) \Rightarrow (1)$: If A,B are $S-$universal, then by Theorem 3.15 and Corollary 3.16, we have $\text{diam}_c(\overline{W}(A;B)) = \omega(\delta_{A,B}|\mathscr{J}) = \omega(A) + \omega(B)$ which is only true for the class of normaloid operators. $\square$

5. Anti-distance and Similarity orbit

A unitary operator on a Hilbert space H is a bounded linear operator $U : H \to H$ that satisfies $U^*U = UU^* = I$, where U^* is the adjoint of U and $I : H \to H$ is the identity operator. Let $B \in \mathscr{B}(H)$. A unitary similarity orbit through B is defined as the set $U_S = \{U^*BU : U \text{ unitary}\}$. The anti-distance from A to the orbit U_S with respect to the norm $\|\cdot\|$ is given by $\sup\{\|A - U^*BU\| : U \text{ unitary}\}$. In [13], T. Ando determined the upper and lower bounds for the anti-distance $\sup\{\|A - U^*BU\|_\infty : U \text{ unitary}\}$, where U runs over the set of unitary matrices.
Just like in the case of a generalized derivation, two operators $A,B \in \mathscr{B}(H)$ must be fixed in order to define the anti-distance from A to the unitary similarity orbit through B, U_S.
Therefore, the question about the relation between the norms of a generalized derivation and the anti-distance is apparent. From the available literature, very little attempt has been made towards addressing questions in this direction. It is clear that

$\sup\{\|A-U^*BU\| : U \ \text{ unitary}\} \leq \|\delta_{A,B}\|$.
In [?], Boumazgour established that for any $A,B \in \mathscr{B}(H)$,

$$\|\delta_{A,B}\| = \sup\{\|A-U^*BU\| : \mathscr{U} \ \text{ unitary}\}. \tag{5.1}$$

Moreover, he proved the following;

1. If $A,B \in \mathscr{B}(H)$, then for $1 \leq p \leq \infty$,

$$\sup\{\|A-U^*BU\| : U \text{ unitary}\} \leq 2^{\frac{1}{p}}\|\delta_{A,B}|C_p\|. \tag{5.2}$$

2. If A,B are hyponormal and cohyponormal operators respectively, (If, in particular both of them are normal) then,

$$\sup\{\|A-U^*BU\| : U \ \text{ unitary}\} \leq \sqrt{2}\text{diam}_c(\sigma(A;B)) \tag{5.3}$$

In this section, we give some of results in the same direction. In the next result, we characterize the anti-distance in terms of the circumdiameters, norms, numerical and spectral radii of pair of S-universal operators.

Theorem 5.1. *For S-universal operators $A,B \in \mathscr{B}(H)$,*

1. $\sup\{\|A-U^*BU\| : U \ \textit{ unitary}\} = \textit{diam}_c(\overline{W}(A;B))$,
2. $\sup\{\|A-U^*BU\| : U \ \textit{ unitary}\} = r(A)+r(B)$,
3. $\sup\{\|A-U^*BU\| : U \ \textit{ unitary}\} = \|A\|+\|B\|$,
4. $\sup\{\|A-U^*BU\| : U \ \textit{ unitary}\} = \omega(A)+\omega(B)$,
5. $\sup\{\|A-U^*BU\| : U \ \textit{ unitary}\} = \omega(\delta_{A,B}|\mathscr{J})$, *and*
6. $\sup\{\|A-U^*BU\| : U \ \textit{ unitary}\} = r(\delta_{A,B}|\mathscr{J})$,

where $\mathscr{J}$ is a norm ideal in $\mathscr{B}(H)$.

Proof. Let $A,B \in \mathscr{B}(H)$ be S-universal, then by equation (5.1) we get;$\sup\{\|A-U^*BU\| : U \ \text{ unitary}\} = \|\delta_{A,B}|B(H)\| = \|A\|+\|B\|$. Clearly, $\text{diam}_c(\overline{W}(A;B)) = \|\delta_{A,B}\|$ for A,B S-universal. This proves assertions 1 and 3. By Theorem 3.14, we have that $\text{diam}_c(\overline{W}(A;B)) = \omega(A)+\omega(B)$ and $\text{diam}_c(\sigma(A;B)) = r(A)+r(B)$ which implies that $\sup\{\|A-U^*BU\| : U \ \text{ unitary}\} = \omega(A)+\omega(B)$ and that $\sup\{\|A-U^*BU\| : U \ \text{ unitary}\} = r(A)+r(B)$ proving the assertion 2 and 4. The proves for assertions 5 and 6 is clear from Corollary 3.16. ☐

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