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Chord Properties of Some Special Curves in Euclidean Space

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Abstract. In this paper, we define some special curves through the chord that combines two different points of the curve on it, and we examine relations of these curves each other. Especially, these curves have been characterized by their unit tangent vector field itself with symmetric and skew symmetric matrix. Moreover, we show that these curves are the geodesics of the isoparametric surfaces such as spheres, right circular cylinders and spherical cylinders.

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1. Introduction and Preliminaries

Curves with constant curvature or $W-$curves are firstly investigated, and called by Felix Klein and Sophus Lie, in the Euclidean space. These curves are the orbits of instantaneonus motions of space, hence they have applications in several areas; such as kinematics, physics and other applied areas. These curves have been studied intensively in the Euclidean space [1–4]. In [8], these curves classified in $3-$dimensional Lorentz-Minkowski space. In [6], the authors moved some ideas in Euclidean space to Lorentzian space such as vector (parametric) equation of curves with constant curvature is given in $n-$dimensional Lorentz-Minkowski space, by solving some linear differential equation systems. We got these curves as an integral curves of the unit tangent vector field of the curve. Besides, in [7] we showed that the geodesics on the boundry of the de Sitter space are the curve with constant curvature in Lorentz-Minkowski space. Through [2], we have

$$\langle \gamma(t) - \gamma(s), T(t) - T(s) \rangle = 0 \tag{1.1}$$

where $\gamma(s)$ is unit speed curve, and T is the unit tangent vector field. Due to (1.1) we write $T(s) = A\gamma(s) + b$ for skew symmetric matrix A. In addition, $\left|\gamma^{(k)}(s)\right|$ is constant for $1 \leq k \leq n$ [3]. Let us give the Frenet frame and Frenet-Serret

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formulae of the unit speed curve by $\{V_1, V_2, ..., V_n\}$ and

$$\begin{pmatrix} V_1' \\ V_2' \\ . \\ . \\ . \\ V_n' \end{pmatrix} = \begin{pmatrix} 0 & -\kappa_1 & 0 & 0 & ... & 0 \\ \kappa_1 & 0 & -\kappa_2 & 0 & ... & 0 \\ 0 & \kappa_2 & 0 & 0 & ... & 0 \\ & ... & & & & \\ 0 & 0 & 0 & ... & 0 & -\kappa_{n-1} \\ 0 & 0 & 0 & ... & \kappa_{n-1} & 0 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ . \\ . \\ . \\ V_n \end{pmatrix} \tag{1.2}$$

respectively. In fact, $V_1 = T, V_2 = N$ and $V_3 = B$ in $3-$dimensional Euclidean space. In (1.2), $i-th$ Frenet curvature is given as $\kappa_i = \left\langle V_i', V_{i+1} \right\rangle$.

Definition 1.1. Let $I \subset \mathbb{R}$ be an open interval and $\gamma : I \to E^n$ be a Frenet curve with rank n, i.e. set of $\{\gamma', \gamma'', ..., \gamma^n\}$ is linear independent when $\left\{\gamma', \gamma'', ..., \gamma^{n+1}\right\}$ is linear dependent. If $\kappa_i(s)$ are constant for all $s \in I$ and $1 \leq i \leq n-1$, then γ is called by $W-$curve with rank n.

Let us state the vector equation of the $W-$curves by following theorem.

Theorem 1.2. *Unit speed $W-$ curve is given by*

$$\gamma(s) = \overrightarrow{\gamma_0} + as\overrightarrow{e_0} + \sum_{i=1}^{k} r_i \left(\cos(a_i s)\,\overrightarrow{e_{2i-1}} + \sin(a_i s)\,\overrightarrow{e_{2i}}\right) \tag{1.3}$$

such that the set of $\left\{\overrightarrow{e_0}, \overrightarrow{e_1}, ..., \overrightarrow{e_{2k}}\right\}$ is orthonormal base of E^{2k+1}. Here $\overrightarrow{\gamma_0}$ is constant vector, $a \in R$ and $a_1 < a_2 < ... < a_k$ positive real numbers satisfy

$$a^2 + \sum_{i=1}^{k} (r_i a_i)^2 = 1.$$

If $a \neq 0$ in (1.3) then γ fully lies in E^{2k+1} space otherwise it lies on hypersphere in E^{2k} space.

In [6] we called 'curves with constant curvature' as $\mathbf{C}-$curves instead of $W-$curves. Because this notation has some advantages such as the the curve satisfy (1.1); the unit tangent vector field of the curve is written by constant skew symmetric matrix, etc.

Definition 1.3. Let $\gamma : I \to E^n$ be a unit speed curve and $T(s)$ be a unit tangent vector field of it. If

$$\langle \gamma(t) - \gamma(s), T(t) - T(s) \rangle = 0$$

for parameter $t \in I$ and arclenght parameter s, then γ called by $\mathbf{C}-$curve in Euclidean space.

Theorem 1.4. *Let $\gamma(s)$ be a unit speed $\mathbf{C}-$curve in n-dimensional Euclidean space. Unit tangent vector field of this curve is given by*

$$T(s) = A\gamma(s) + b \tag{1.4}$$

where A is $n \times n$ constant skew symmetric matrix and $b \in E_1^n$ column vector.

Note that (1.4) shows that $\mathbf{C}-$curves corresponding to integral curves of the unit tangent vector field of the curve itself.
Through the [3], we give the following theorems.

Theorem 1.5. *Let $\gamma(s)$ be a unit speed curve in n-dimensional Euclidean space. The followings are equivalent:*
i) γ is a $\mathbf{C}-$ curve.
ii) $\left|\gamma^{(k)}(s)\right|$ is constant, $1 \leq k \leq n$.
iii) γ is a $W-$ curve.

Theorem 1.6. *Let M be an isoparametric surface in Euclidean space.*

$$\langle Y - X, G(X) + G(Y) \rangle = 0 \tag{1.5}$$

where G is Gauss map and $X, Y \in M$.

(1.5) takes its source from the idea of 'any hypersphere the chord joining any two points on it meets the sphere at the same angle at the two points'.

2. Special Curves as A Geodesics of Isoparametric Surfaces

Definition 2.1. Let $I \subset \mathbb{R}$, $\gamma : I \to E^n$ be a unit speed curve and N is an unit normal vector field of γ. If

$$\langle \gamma(t) - \gamma(s), N(t) + N(s) \rangle = 0 \tag{2.1}$$

where $t \in I$ and s is the arclenght parameter, then we called the curve γ by $\mathbf{N}-$curve.

It is obvious that the circles and right circular helices satisfy equation (2.1) in two and three dimensional Euclidean space respectively. Let us give some results about the chord properties of special curves by following theorems.

Theorem 2.2. *Let γ be a unit speed $\mathbf{C}-$ curve in $n-$ dimensional Euclidean space. In that case, γ is an $\mathbf{N}-$curve.*

Proof. Let γ be a unit speed $\mathbf{C}-$curve. It follows from Theorem 1.4. that

$$T(s) = A\gamma(s) + b. \tag{2.2}$$

If we derivate the (2.2) according to arclenght s, we get

$$T'(s) = A\gamma'(s) = AT(s). \tag{2.3}$$

Moreover we have

$$T'(s) = \kappa(s)N(s). \tag{2.4}$$

Since the γ is $\mathbf{C}-$ curve, $\kappa(s)$ is constant. It follows from (2.3) and (2.4) that

$$N(s) = \frac{1}{\kappa}AT(s) \tag{2.5}$$

where $\kappa(s) = \kappa$. Due to (2.5) we write

$$\langle \gamma(t) - \gamma(s), N(t) + N(s) \rangle = \frac{1}{\kappa}\left\langle A^T\left(\gamma(t) - \gamma(s)\right), T(t) + T(s)\right\rangle.$$

Since A is skew symmetric matrix we get

$$\langle \gamma(t) - \gamma(s), N(t) + N(s) \rangle = 0.$$

So γ is an $\mathbf{N}-$curve. □

Theorem 2.3. *Let γ be a $\mathbf{C}-$curve in $n-$dimensional Euclidean space. The unit normal vector field of this curve is written by*

$$N(s) = B\gamma(s) + c$$

where B is symmetric matrix.

Proof. It follows from (2.2) and (2.5) that

$$N(s) = \frac{1}{\kappa}A^2\gamma(s) + \frac{1}{\kappa}Ab. \tag{2.6}$$

Since A is skew symmetric matrix A^2 is symmetric matrix. So (2.6) becomes

$$N(s) = B\gamma(s) + c$$

where $\frac{1}{\kappa}A^2 = B$ and $\frac{1}{\kappa}Ab = c$. This completes the proof. □

Now we follow the [3] at proof of the following theorem.

Theorem 2.4. *Let γ be a unit speed curve in $3-$dimensional Euclidean space. The followings are equivalent:*
i) γ is an $\mathbf{N}-$curve.
ii) $N(s) = A\gamma(s) + b$.
iii) γ is a $\mathbf{C}-$curve.

where N is the unit normal vector field and A is symmetric matrix.

Proof. Let γ be an $\mathbf{N}-$curve in three dimensional Euclidean space. Let us use the cancellation for $\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$ and its unit normal vector field $N(t) = (N_1(t), N_2(t), N_3(t))$ such that $\gamma_j(t_k) = \gamma_{jk}$ and $N_j(t_k) = N_{jk}$ for $1 \leq j \leq 3$ and $0 \leq k \leq 3$ respectively. Thus we give the following matrices by

$$A_j = \begin{pmatrix} \gamma_{11} - \gamma_{10} & \gamma_{12} - \gamma_{10} & \gamma_{13} - \gamma_{10} \\ \gamma_{21} - \gamma_{20} & \gamma_{22} - \gamma_{20} & \gamma_{23} - \gamma_{20} \\ \gamma_{31} - \gamma_{30} & \gamma_{32} - \gamma_{30} & \gamma_{33} - \gamma_{30} \end{pmatrix}$$

and

$$B_j = \begin{pmatrix} N_{11} - N_{10} & N_{12} - N_{10} & N_{13} - N_{10} \\ N_{21} - N_{20} & N_{22} - N_{20} & N_{23} - N_{20} \\ N_{31} - N_{30} & N_{32} - N_{30} & N_{33} - N_{30} \end{pmatrix}.$$

Therefore we get $N(s) = A\gamma(s) + b$ where $A^T = B_j A_j^{-1}$. We will show the (2×2) matrix A is symmetric. Proof can be shown for (3×3) matrix by similar operations. Let us recall following matrices,

$$A_j = \begin{pmatrix} \gamma_{11} - \gamma_{10} & \gamma_{12} - \gamma_{10} \\ \gamma_{21} - \gamma_{20} & \gamma_{22} - \gamma_{20} \end{pmatrix}$$

and

$$B_j = \begin{pmatrix} N_{11} - N_{10} & N_{12} - N_{10} \\ N_{21} - N_{20} & N_{22} - N_{20} \end{pmatrix}.$$

Obviously we get

$$\det(A_j) = c = (\gamma_{11} - \gamma_{10})(\gamma_{22} - \gamma_{20}) - (\gamma_{12} - \gamma_{10})(\gamma_{21} - \gamma_{20})$$

and

$$A_j^{-1} = \frac{1}{c} \begin{pmatrix} \gamma_{22} - \gamma_{20} & \gamma_{10} - \gamma_{12} \\ \gamma_{20} - \gamma_{21} & \gamma_{11} - \gamma_{10} \end{pmatrix}.$$

Let us define

$$A^T = B_j A_j^{-1} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Then it is written

$$\begin{aligned} a_{12} &= \frac{1}{c}\left((\gamma_{10} - \gamma_{12})N_{11} + (\gamma_{12} - \gamma_{11})N_{10} + (\gamma_{11} - \gamma_{10})N_{12}\right), \\ a_{21} &= \frac{1}{c}\left((\gamma_{22} - \gamma_{20})N_{21} + (\gamma_{21} - \gamma_{22})N_{20} + (\gamma_{20} - \gamma_{21})N_{22}\right). \end{aligned}$$

If we consider$\{t = t_0, t = t_1\}$, $\{t = t_0, t = t_2\}$ and $\{t = t_1, t = t_2\}$ at (2.1) then we get

$$(\gamma_{11} - \gamma_{10})N_{10} + (\gamma_{11} - \gamma_{10})N_{11} + (\gamma_{21} - \gamma_{20})N_{20} + (\gamma_{21} - \gamma_{20})N_{21} = 0, \tag{2.7}$$

$$(\gamma_{12} - \gamma_{10})N_{10} + (\gamma_{22} - \gamma_{10})N_{12} + (\gamma_{22} - \gamma_{20})N_{20} + (\gamma_{22} - \gamma_{20})N_{22} = 0 \tag{2.8}$$

and

$$(\gamma_{12} - \gamma_{11})N_{11} + (\gamma_{12} - \gamma_{11})N_{12} + (\gamma_{22} - \gamma_{21})N_{21} + (\gamma_{22} - \gamma_{21})N_{22} = 0 \tag{2.9}$$

respectively. (2.7), (2.8) and (2.9) gives $a_{12} = a_{21}$. Hence A is a symmetric matrix.
Let $N(s) = A\gamma(s) + b$ and A be a symmetric matrix. It follows from Frenet-Serret formulae that

$$N'(s) = AT(s) = -\kappa(s)T(s) + \tau(s)B(s). \tag{2.10}$$

If we derivate (2.10) by arclenght parameter we get

$$AT'(s) = -\kappa'(s)T(s) - \left(\kappa^2(s) + \tau^2(s)\right)N(s) + \tau'(s)B(s). \tag{2.11}$$

It follows from (2.10) and (2.11) that

$$\kappa'(s) = 0.$$

Thus $\kappa(s)$ is constant. Let us consider $\kappa(s) = \kappa$ and $\tau(s) \neq 0$. If we derivate (2.11) then we write

$$N'''(s) = \kappa\left(\kappa^2 + \tau^2(s)\right)T(s) - 3\tau(s)\tau'(s)N(s) + \left(\tau''(s) - \tau(s)(\kappa^2 + \tau^2(s))\right)B(s). \tag{2.12}$$

Inner product two sides of (2.12) by $N(s)$ gives

$$\langle N'''(s), N(s)\rangle = -3\tau(s)\tau'(s). \tag{2.13}$$

Moreover, by Frenet-Serret formulae we write

$$N'''(s) = -\kappa^2 AT(s) + \kappa\tau(s)AB(s). \tag{2.14}$$

Inner product two sides of (2.14) by $N(s)$ gives

$$\langle N'''(s), N(s)\rangle = \tau(s)\tau'(s). \tag{2.15}$$

It follows from (2.13) and (2.15) that $\tau(s)$ is constant. So this curve is a **C**−curve. □

Theorem 2.5. *Let M be an isoparametric hypersurface in Euclidean space E^n. Geodesics of M are the* **N**−*curves.*

Proof. Let γ be a geodesic curve on M. Hence the unit normal vector field N of the curve and the unit principal normal vector n of the surface are linear dependent. Thus we write $n = \lambda N$ where λ is real. Since M is isoparametric,

$$\langle \gamma(t) - \gamma(s), n(t) + n(s)\rangle = 0. \tag{2.16}$$

It follows from (2.16) and $n = \lambda N$ that

$$\langle \gamma(t) - \gamma(s), N(t) + N(s)\rangle = 0.$$

which is intended. □

In the light of [5], we give an example of **N**−curve that is the geodesic of spherical cylinder in the four dimensional Euclidean space.

Example 2.6. It is well known that the isoparametric surfaces are the hyperespheres, circular cylinders, and the spherical cylinders in the Euclidean space. Let we consider the spherical cylinder of $\mathbb{S}^2 \times \mathbb{R}$ and the geodesic $\gamma(s) = (x(s), y(s), z(s), t(s))$ on it. These geodesics are given by solutions of the following differential equation system:

$$\begin{aligned} \ddot{x} &= -(1 - \dot{t}^2)x, \\ \ddot{y} &= -(1 - \dot{t}^2)y, \\ \ddot{z} &= -(1 - \dot{t}^2)z, \\ \ddot{t} &= 0. \end{aligned} \tag{2.17}$$

If we consider that $t(s) = \lambda_0 s + t_0$ in (2.17) then we have three cases depends on real number λ_0 .
Case 1. If $\lambda_0 = \mp 1$ then

$$\gamma(s) = (x_0, y_0, z_0, t_0 \mp s).$$

This curve is the line of $\mathbb{S}^2 \times \mathbb{R}$ such that $x_0 = x(0), y_0 = y(0), z_0 = z(0), t_0 = t(0)$.
Case 2. If $\lambda_0 = 0$ then γ is the great circle of $\mathbb{S}^2$ which involve by plane through the origin. The vector equation of this curve is given by

$$\gamma(s) = (x_0 \cos s + u_0 \sin s, y_0 \cos s + v_0 \sin s, z_0 \cos s + \omega_0 \sin s, t_0)$$

where $u_0 = \dot{x}(0), v_0 = \dot{y}(0)$ and $\omega_0 = \dot{z}(0)$.
Case 3. If $\lambda_0 \in (-1, 0) \cup (0, 1)$ then the curve is circular helix on $\mathbb{S}^1 \times \mathbb{R}$ such that

$$\gamma(s) = (x_0 \cos(\mu_0 s) + \frac{u_0}{\mu_0} \sin(\mu_0 s), y_0 \cos(\mu_0 s) + \frac{v_0}{\mu_0} \sin(\mu_0 s), z_0 \cos(\mu_0 s) + \frac{\omega_0}{\mu_0} \sin s, \mu_0 s + t_0)$$

where $\mathbb{S}^1$ is the great circle of $\mathbb{S}^2$ and $\mu_0 = \sqrt{1 - \lambda_0^2}$.
Since the $\mathbb{S}^2 \times \mathbb{R}$ is isoparametric, γ is an **N**− curve in all cases above.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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