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Some Variations of Janowski Functions Associated with Srivastava-Attiya Operator

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Abstract − In this paper, we consider some new subclasses of analytic functions with bounded boundary and bounded radius rotation associated with Attia-Srivastava operator. The coefficient bounds, integral representations, convolution properties belong to theses classes are investigated.

Keywords − *Srivastava- Attia operator, Janowski functions, subordination, convolution, starlike convex functions*

1. Introduction

Let A be the class of all functions $f(z)$ of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n, \tag{1}$$

which are analytic in the open unit disk U, where

$$U=\{z:z\in\mathbb{C}\quad\text{and}\quad|z|<1\}.$$

For two functions $F(z)$ and $G(z)$ analytic in U , we say that $F(z)$ is subordinate to $G(z)$, denoted by

$$F\prec G\quad\text{ or }\quad F(z)\prec G(z),$$

if there exists an analytic function $w(z)$ with

$$|w(z)|\leq|z|\quad\text{such that}\quad F(z)=G\left(w(z)\right).$$

Furthermore if the function $G(z)$ is univalent in U then we have the following equivalence [1–3]

$$F(z)\prec G(z)\iff F\left(0\right)=G\left(0\right)\text{ and }F\left(U\right)\subset G\left(U\right).$$

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For two analytic functions $f(z)$ given by (1) and $g(z)$

$$g(z) = z + \sum_{n=2}^{\infty} e_n z^n, \quad (z \in U),$$

their Convolution or Hadamard product is given by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n e_n z^n, \quad (z \in U).$$

For arbitrary fixed numbers A, B, α and β satisfying $-1 \leq B < A \leq 1$, $0 \leq \alpha < 1$ and $0 < \beta \leq 1$, let $P_\beta [A, B, \alpha]$ denote the family of functions

$$h(z) = 1 + h_1 z + h_2 z^2 + \cdots,$$

regular in U and such that $h(z)$ is in $P_\beta [A, B, \alpha]$ if and only if

$$h(z) \prec (1-\alpha) \left(\frac{1 + Az}{1 + Bz} \right)^\beta + \alpha. \tag{2}$$

Therefore, $h(z)$ is in $P_\beta [A, B, \alpha]$ if and only if

$$h(z) = \frac{(1-\alpha)(1 + Aw(z))^\beta + \alpha (1 + Bw(z))^\beta}{(1 + Bw(z))^\beta}, \tag{3}$$

for some $w(z)$ with $|w(z)| \leq |z|$. By taking $\beta = 1$, then the class $P_\beta [A, B, \alpha]$, reduces to $P [A, B, \alpha]$, defined by Polatoglu in [4], if we take $\alpha = 0$, $\beta = 1$, then the class $P_\beta [A, B, \alpha]$, reduces to the well known class $P [A, B]$, defined and studied by Janowski in [5] and setting $\alpha = 0$, $\beta = 1$, $A = 1$, $B = -1$, the class $P_\beta [A, B, \alpha]$, reduces to the class P of functions with positive real part. For more details see [6–15].

One can easily verify that $p \in P_\beta [A, B, \alpha]$, if and only if, there exists $g \in P [A, B]$, such that

$$p(z) = (1-\alpha) g(z) + \alpha.$$

The Herglotz representation of the functions of the class $P_\beta [A, B, \alpha]$, is given by

$$h(z) = \alpha + \frac{1-\alpha}{2} \int_0^{2\pi} \left(\frac{1 + Aze^{-i\theta}}{1 + Bze^{-i\theta}} \right)^\beta \mathrm{d}\mu(\theta),$$

where μ is non decreasing function in $[0, 2\pi]$ such that $\int_0^{2\pi} \mathrm{d}\mu(\theta) = 2$.

For $A = 1, B = -1$, the class $P_\beta [A, B, \alpha]$, reduces to the class $P_\beta (\alpha)$, presented by Dziok recently [16, Th.3] and further by setting $\alpha = 0, \beta = 1, A = 1, B = -1$, we obtain the class P of analytic functions with real part greater than zero.

Now we define the subclass $P_{m,\beta} [A, B, \alpha]$, of analytic functions as follows;

Definition 1.1. A function $p(z)$ analytic in U belongs to the class $P_{m,\beta} [A, B, \alpha]$, if and only if

$$p(z) = \alpha + \frac{1-\alpha}{2} \int_0^{2\pi} \left(\frac{1 + Aze^{-i\theta}}{1 + Bze^{-i\theta}} \right)^\beta \mathrm{d}\mu(\theta), \tag{4}$$

where $\mu(\theta)$ is non decreasing function in $[0, 2\pi]$ with

$$\int_0^{2\pi} \mathrm{d}\mu(\theta) = 2 \quad \text{and} \quad \int_0^{2\pi} |\mathrm{d}\mu(\theta)| \leq m,$$

where, $m \geq 2$, $-1 \leq B < A \leq 1$, $0 \leq \alpha < 1$, $0 < \beta \leq 1$.

Now using Horglotz-Stieltjes formula for the functions in the class $P_{m,\beta} [A, B, \alpha]$, given in (4), we obtain

$$p(z) = \left(\frac{m}{4} + \frac{1}{2} \right) p_1(z) - \left(\frac{m}{4} - \frac{1}{2} \right) p_2(z),$$

where $p_1, p_2 \in P_\beta [A, B, \alpha]$ see ([16], Theorem 3).

For $\beta = 1$, the class $P_{m,\beta} [A, B, \alpha] = P_m [A, B, \alpha]$ [33] and for $\alpha = 0, \beta = 1, A = 1, B = -1, P_{m,1} [1, -1, 0] = P_m$ [17].

We consider the function

$$\phi (z; s, b) = \sum_{n=0}^{\infty} \frac{z^n}{(b+n)^s}, \tag{5}$$

where $b \in \mathbb{C} \setminus \mathbb{Z}_0^-$ and $s \in \mathbb{C}$. The function $\phi (z; s, b)$ contain many well known functions as a special case such as Riemann and Hurwitz Zeta functions for more details, see [18, 19].

Using the technique of convolution and the function $\phi (z; s, b)$ Srivastava and Attiya given in [20]. In addition see also ([21, 22]) introduced and studied the linear operator

$$J_{s,b} f : A \to A,$$

defined, in terms of the Hadamard product (or convolution), by

$$J_{s,b}(f)(z) = \phi(z; s, b) * f(z), \quad f \in A, \ (z \in U), \tag{6}$$

where $*$ denotes the convolution and

$$\psi (z; s, b) = (1 + b)^s \left(\phi(z; s, b) - b^{-s}\right) = z + \sum_{n=2}^{\infty} \left(\frac{b+1}{b+n}\right)^s z^n, \quad (z \in U). \tag{7}$$

Therefore, using (6) and (7), we have

$$J_{s,b}(f)(z) = z + \sum_{n=2}^{\infty} \left(\frac{b+1}{b+n}\right)^s a_n z^n, \quad (z \in U).$$

For special values of b and s the operator contain many known operators, see [23, 24].

With the help of the class $P_{m,\beta} [A, B, \alpha]$, along with generalized Srivastava and Attiya operator given in [20] , we now define the following subclass of analytic functions;

Definition 1.2. A function $f \in A$, is in the class $R_{m,\beta}^{s,b} [A, B, \alpha]$, if and only if

$$\frac{z\left(J_{s,b} f(z)\right)'}{J_{s,b} f(z)} \in P_{m,\beta} [A, B, \alpha], \quad (z \in U).$$

Definition 1.3. A function $f \in A$, is in the class $V_{m,\beta}^{s,b} [A, B, \alpha]$, if and only if

$$1 + \frac{z\left(J_{s,b} f(z)\right)''}{\left(J_{s,b} f(z)\right)'} \in P_{m,\beta} [A, B, \alpha], \quad (z \in U).$$

where $m \geq 2$, $b \in \mathbb{C} \setminus \left(\mathbb{Z}_0^- = \{0, -1, -2, \ldots\}\right)$, $s \in \mathbb{C}$, $-1 \leq B < A \leq 1, 0 \leq \alpha < 1$, $0 < \beta \leq 1$. We also note that

$$f(z) \in V_m^{s,b} [A, B, \alpha, \beta] \Leftrightarrow z f(z)' \in R_{m,\beta}^{s,b} [A, B, \alpha]. \tag{8}$$

Remarks:

(i) $R_{m,1}^{0,b} [A, B, 0] = R_m [A, B], V_{m,1}^{0,b} [A, B, 0] = V_m [A, B]$, the well known classes presented and studied in [25] and [26].

(ii) $R_{m,1}^{0,b} [1, -1, 0] = R_m, V_{m,1}^{0,b} [A, B, 0] = V_m$, we have the well known class introduced and studied in [17] and [27].

(iii) $R_{m,1}^{0,b} [2\zeta - 1, -1, 0], V_{m,1}^{0,b} [2\zeta - 1, -1, 0]$, were presented and studied in [28].

To avoid repetition, it is admitted once that $m \geq 2$, $b \in \mathbb{C} \setminus \left(\mathbb{Z}_0^- = \{0, -1, -2, \ldots\}\right)$, $s \in \mathbb{C}$, $-1 \leq B < A \leq 1$, $0 \leq \alpha < 1$, $0 < \beta \leq 1$.

2. Preliminary Lemma

We need the following Lemma which will be used in our main results.

Lemma 2.1. [29] Let $f(z)$ be subordinate to $g(z)$, with

$$f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n, \quad g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n.$$

If $g(z)$ is univalent in U and $g(U)$ is convex, then $|a_n| \leq |b_1|$.

Lemma 2.2. Let $p(z) \in P_{m,\beta}[A, B, \alpha]$, be of the form (1). Then

$$|q_n| \leq \beta (A - B) |1 - \alpha| .$$

The proof is immediate by using Lemma 2.1.

Lemma 2.3. Let $p(z) \in P_{m,\beta}[A, B, \alpha]$, be of the form (1). Then

$$|q_n| \leq \frac{m}{2} \beta (A - B) |1 - \alpha| .$$

The proof is immediate by using Lemma 2.2.

Lemma 2.4. Let $p(z) \in P_{m,\beta}[A, B, \alpha]$, be of the form (1). Then

$$\frac{(1-\alpha)}{4} \left[(m+2) \left(\frac{1-Ar}{1-Br} \right)^{\beta} - (m-2) \left(\frac{1+Ar}{1+Br} \right)^{\beta} \right] + \alpha$$
$$\leq \ \mathfrak{Re}\, p(z) \leq |p(z)| \leq \frac{(1-\alpha)}{4} \left[(m+2) \left(\frac{1+Ar}{1+Br} \right)^{\beta} - (m-2) \left(\frac{1-Ar}{1-Br} \right)^{\beta} \right] + \alpha.$$

This results is sharp.
The proof is immediate by using Lemma 2.3.

Lemma 2.5. [30] Let ψ be convex and let g be starlike in U. Then for F analytic in U with $F(0) = 1$, $\frac{\psi * Fg}{\psi * g}$ is contained in the convex hull of $F(U)$.

3. Main Results

Theorem 3.1. Let $p(z) \in P_{m,\beta}[A, B, \alpha]$, with $m \geq 2$. Then, for $|z| = r < 1$,

$$\left| zp'(z) \right| \leq \frac{(A-B)\beta r \left[(m+2) \frac{(1+Ar)^{\beta-1}}{(1+Br)^{\beta+1}} + (m-2) \frac{(1-Ar)^{\beta-1}}{(1-Br)^{\beta+1}} \right] \mathfrak{Re}\, p(z)}{\left[(m+2) \left(\frac{1+Ar}{1+Br} \right)^{\beta} - (m-2) \left(\frac{1-Ar}{1-Br} \right)^{\beta} \right] + \frac{4\alpha}{1-\alpha}}.$$

Proof. The proof is immediate by using Lemma 2.4. □

Putting $\alpha = 0, \beta = 1$ in Theorem 3.1, we can obtain Corollary 3.2, below which is comparable to the result obtained by Noor and Malik [31].

Corollary 3.2. Let $p(z) \in P_{m,\beta}[A, B, \alpha]$, with $m \geq 2$. Then, for $|z| = r < 1$,

$$\left| zp'(z) \right| \leq \frac{(A-B) r \left\{ m - 4Br + mB^2 r^2 \right\} \mathfrak{Re}\, p(z)}{(1 - Br^2)(2 + mr(A-B) - 2ABr^2)}.$$

Theorem 3.3. Let $f(z) \in R^{s,b}_{m,\beta}[A, B, \alpha]$. Then

$$|a_n| \leq \frac{(b+n)^s \left(\frac{m}{2} \beta (A-B) |1-\alpha| \right)_{n-1}}{(b+1)^s (n-1)!}. \tag{9}$$

This result is sharp.

Proof. Let
$$\frac{z\left(J_{s,b}f(z)\right)'}{J_{s,b}f(z)}=p(z),\qquad (z\in U), \tag{10}$$
where $p(z)\in P_{m,\beta}\left[A,B,\alpha\right]$ and $p(z)=1+\sum\limits_{n=1}^{\infty}q_nz^n$.

Then from the definition we have
$$J_{s,b}f(z)=z+\sum_{n=2}^{\infty}b_nz^n, \tag{11}$$
where
$$b_n=\left(\frac{b+1}{b+n}\right)^s a_n. \tag{12}$$
From (10) and (11), we have
$$\begin{aligned} z+\sum_{n=2}^{\infty}nb_nz^n &= \left(z+\sum_{n=2}^{\infty}b_nz^n\right)\left(1+\sum_{n=1}^{\infty}q_nz^n\right)\\ &= \left(\sum_{n=1}^{\infty}b_nz^n\right)\left(1+\sum_{n=1}^{\infty}q_nz^n\right),\quad b_1=1\\ &= \sum_{n=1}^{\infty}b_nz^n+\left(\sum_{n=1}^{\infty}b_nz^n\right)\left(\sum_{n=1}^{\infty}q_nz^n\right). \end{aligned}$$
By using the Cauchy's product formula [32], for the power series we have
$$z+\sum_{n=2}^{\infty}nb_nz^n=\sum_{n=1}^{\infty}b_nz^n+\sum_{n=1}^{\infty}\left(\sum_{j=1}^{n-1}b_jq_{n-j}\right)z^n.$$
Equating the coefficient of z^n, we have
$$nb_n=b_n+\sum_{j=1}^{n-1}b_jq_{n-j}.$$
By using induction on n, and Lemma 2.3, we obtain
$$b_n=\frac{\left(\frac{m}{2}\beta\left(A-B\right)|1-\alpha|\right)_{n-1}}{(n-1)!}.$$
Using the value of b_n, we obtain (9).

Sharpness is given for the functions $f_1\in A$ such that
$$\begin{aligned} \frac{z\left(J_{s,b}f_1(z)\right)'}{J_{s,b}f_1(z)} &= \left(\frac{m}{2}+\frac{1}{2}\right)\left((1-\alpha)\left(\frac{1+Az}{1+Bz}\right)^{\beta}+\alpha\right)\\ &\quad -\left(\frac{m}{2}-\frac{1}{2}\right)\left((1-\alpha)\left(\frac{1+Az}{1+Bz}\right)^{\beta}+\alpha\right). \end{aligned}$$
This complete the proof of Theorem 3.3. □

Putting $s=0,\beta=1$ in Theorem 3.3, we can obtained the following Corollary.

Corollary 3.4. Let $f(z)\in R_{m,1}^{0,b}\left[A,B,\alpha\right]$. Then
$$|a_n|\leq\frac{\left(\frac{m}{2}\left(A-B\right)|1-\alpha|\right)_{n-1}}{(n-1)!}.$$
This result is sharp.

Putting $s=0,\beta=1,A=1,B=-1$ in Theorem 3.3, we can obtain Corollary 3.5, below which is comparable to the result obtained by Noor [33].

Corollary 3.5. Let $f(z)\in R_{m,1}^{0,b}\left[1,-1,\alpha\right]=R_m\left(\alpha\right)$. Then

$$|a_n|\leq\frac{\left(m\left|1-\alpha\right|\right)_{n-1}}{(n-1)!},\quad\text{for all } n\geq 2.$$

This result is sharp.

Theorem 3.6. Let $f(z)\in V_{m,\beta}^{s,b}\left[A,B,\alpha\right]$. Then

$$|a_n|\leq\frac{(b+n)^s\left(\frac{m}{2}\beta\left(A-B\right)\left|1-\alpha\right|\right)_{n-1}}{(b+1)^s\, n!}. \tag{13}$$

This result is sharp.

Proof. The proof of Theorem 3.6 is similar to that of Theorem 3.3, so the details are omitted. □

For an analytic functions $f(z)$, we consider the operator

$$F(z)=I_c\left(f(z)\right)=\frac{1+c}{z^c}\int_0^z t^{c-1}f\left(t\right)\mathrm{d}t,\quad c>-1. \tag{14}$$

The operator I_c, when $c\in\mathbb{N}$, was introduced by Bernardi [24]. The operator I_1, was studied by Libera [34] and Livingston [35].

Theorem 3.7. If $f(z)$ is of the form of (1) , belongs to $R_{m,\beta}^{s,a}\left[A,B,\alpha\right]$ and $F(z)=z+\sum\limits_{n=2}^{\infty}d_nz^n$, where $F(z)$, is an integral operator given by (14). Then

$$|d_n|\leq\frac{(1+c)\left(b+n\right)^s\left(\frac{m}{2}\beta\left(A-B\right)\left|1-\alpha\right|\right)_{n-1}}{(n+c)\left(b+1\right)^s(n-1)!}.$$

Proof. From (14), we can easily write

$$(1+c)\,f(z)=cF(z)+zF'(z),$$

or equivalently,

$$(1+c)\,z+\sum_{n=2}^{\infty}(1+c)\,a_nz^n=cz+\sum_{n=2}^{\infty}cd_nz^n+z+\sum_{n=2}^{\infty}nd_nz^n.$$

Thus we have,

$$(n+c)\,d_n=(1+c)\,a_n,$$

using the estimate from Theorem 3.3, we have

$$|d_n|\leq\frac{(1+c)\left(b+n\right)^s\left(\frac{m}{2}\beta\left(A-B\right)\left|1-\alpha\right|\right)_{n-1}}{(n+c)\left(b+1\right)^s(n-1)!}.$$

we obtain the required result. □

Putting $s=0,\beta=1,$ in Theorem 3.7, we can obtained the following Corollary.

Corollary 3.8. If $f(z)$ is of the form of (1), belongs to $R_{m,1}^{0,a}\left[1,-1,\alpha\right]$ and $F(z)=z+\sum\limits_{n=2}^{\infty}d_nz^n$, where $F(z)$, is an integral operator given by (14), then

$$|d_n|\leq\frac{(1+c)\left(\frac{m}{2}\left(A-B\right)\left|1-\alpha\right|\right)_{n-1}}{(n+c)\,n!}.$$

Putting $s=0,\beta=1,A=1,B=-1,$ in Theorem 3.7, we can obtained the following Corollary.

Corollary 3.9. If $f(z)$ is of the form of (1), belongs to $R^{s,a}_{m,\beta}[A,B,\alpha]$ and $F(z)=z+\sum\limits_{n=2}^{\infty} d_n z^n$, where $F(z)$, is an integral operator given by (14), then

$$|d_n| \leq \frac{(1+c)\,(m\,|1-\alpha|)_{n-1}}{(n+c)\,n!}.$$

Theorem 3.10. If $f(z)$ is of the form of (1) , belongs to $R^{s,a}_{m,\beta}[A,B,\alpha]$ and $F(z)=z+\sum\limits_{n=2}^{\infty} d_n z^n$, where $F(z)$, is an integral operator given by (14), then

$$|d_n| \leq \frac{(1+c)\,(b+n)^s\left(\frac{m}{2}\beta\,(A-B)\,|1-\alpha|\right)_{n-1}}{(n+c)\,(b+1)^s\,n!}.$$

Proof. The proof of Theorem 3.10 is similar to that of Theorem 3.7 so the details are omitted. □

Theorem 3.11. If $f(z)$ is of the form of (1) , belongs to $R^{s,b}_{2,\beta}[A,B,\alpha]$ if and only if

$$\frac{1}{z}\left\{ f * \left\{ \begin{array}{c} \left(z+\sum\limits_{n=2}^{\infty} nb_n z^n\right)\left(1+B(e^{i\theta})\right)^{\beta}-\left(z+\sum\limits_{n=2}^{\infty} b_n z^n\right) \\ \times\left((1-\alpha)\left(1+A(e^{i\theta})\right)^{\beta}-\alpha\left(1+B(e^{i\theta})\right)^{\beta}\right) \end{array} \right\}\right\} \neq 0, \tag{15}$$

where b_n is given by (12) and $0 \leq \theta < 2\pi$.

Proof. Assume that $f(z) \in R^{s,b}_{2,\beta}[A,B,\alpha]$, then, we have

$$\frac{z\,(J_{s,b}f(z))'}{J_{s,b}f(z)} \prec (1-\alpha)\left(\frac{1+Az}{1+Bz}\right)^{\beta}+\alpha,$$

if and only if

$$\frac{z\,(J_{s,b}f(z))'}{J_{s,b}f(z)} \neq (1-\alpha)\left(\frac{1+A(e^{i\theta})}{1+B(e^{i\theta})}\right)^{\beta}+\alpha, \tag{16}$$

for all $z \in U$, and $0 \leq \theta < 2\pi$. The condition (16) can be written as

$$z\,(J_{s,b}f(z))'\left(1+B(e^{i\theta})\right)^{\beta}-J_{s,b}f(z)\left((1-\alpha)\left(1+A(e^{i\theta})\right)^{\beta}-\alpha\left(1+B(e^{i\theta})\right)^{\beta}\right) \neq 0. \tag{17}$$

On the other hand we know that

$$z\,(J_{s,b}f(z))' = z+\sum_{n=2}^{\infty} nb_n z^n. \tag{18}$$

Combining (5), (6), (18) and (17) we get the convolution property (15) asserted by Theorem 3.11. □

Putting $s=0, \alpha=0, m=2$ and $\beta=1$ in Theorem 3.11, we can obtain Corollary 3.12, below which is comparable to the result obtained by Silverman and Silvia [36].

Corollary 3.12. A function f defined by (1) is in the class $S[A,B]$, if and only if

$$\frac{1}{z}\left\{f(z) * \frac{z-Lz^2}{(1-z)^2}\right\} \neq 0, \quad (z \in U), \tag{19}$$

for all $L=L_\theta=\frac{e^{-i\theta}+A}{A-B}$ and also $L=1$.

Putting $s=0, \alpha=0, m=2, \beta=1, A=1-2\sigma$ and $B=-1$ in Theorem 3.11, we can obtain Corollary 3.13, below which is comparable to the result obtained by Silverman and Silvia [37].

Corollary 3.13. A function f defined by (1) is in the class $S^*(\alpha)$, if and only if

$$\frac{1}{z}\left\{f(z)*\frac{z-Mz^2}{(1-z)^2}\right\}\neq 0, \quad (z\in U), \tag{20}$$

for all $M=M_\theta=\frac{e^{-i\theta}+1-2\sigma}{2(1-\sigma)}$, $(0\leq\sigma<1)$ and also $M=1$.

Theorem 3.14. A function $f(z)\in V_{2,\beta}^{s,b}[A,B,\alpha]$ if and only if

$$\frac{1}{z}\left\{f*\left\{\begin{array}{c}\left(1+\sum\limits_{n=2}^{\infty}n^2b_nz^{n-1}\right)\left(1+B\left(e^{i\theta}\right)\right)^\beta-\left(1+\sum\limits_{n=2}^{\infty}nb_nz^{n-1}\right)\\ \times\left((1-\alpha)\left(1+A(e^{i\theta})\right)^\beta-\alpha\left(1+B(e^{i\theta})\right)^\beta\right)\end{array}\right\}\right\}\neq 0. \tag{21}$$

for all $b_n=\left(\frac{1+b}{n+b}\right)^s a_n$ and $0\leq\theta<2\pi$.

Proof. The proof of Theorem 3.14 is similar to that of Theorem 3.11 so the details are omitted. □

Theorem 3.15. Let $f(z)\in R_{2,\beta}^{s,b}[A,B,\alpha]$. Then

$$f(z)=\left(z\cdot\exp\left((1-\alpha)\int_0^z\frac{\left((1+Aw(t))^\beta-(1+Bw(t))^\beta\right)}{t\,(1+Bw(t))^\beta}\mathrm{d}t\right)\right)*\left(\sum_{n=0}^{\infty}(b+n)^s z^n\right), \tag{22}$$

where $\omega(z)$ is analytic in U, with $\omega(0)=0$ and $|\omega(z)|<1$.

Proof. For $f(z)\in R_{2,\beta}^{s,b}[A,B,\alpha]$, then from definition of subordination we can have

$$\frac{z\left(J_{s,b}f(z)\right)'}{J_{s,b}f(z)}=(1-\alpha)\left(\frac{1+Aw(z)}{1+Bw(z)}\right)^\beta+\alpha, \tag{23}$$

where $w(z)$ analytic in U, with $w(0)=0$ and $|w(z)|<1$.

$$\frac{\left(J_{s,b}f(z)\right)'}{J_{s,b}f(z)}-\frac{1}{z}=\frac{(1-\alpha)\left((1+Aw(z))^\beta-(1+Bw(z))^\beta\right)}{z\,(1+Bw(z))^\beta}, \tag{24}$$

which, upon integration, yield

$$\log\frac{J_{s,b}f(z)}{z}=(1-\alpha)\int_0^z\frac{\left((1+Aw(t))^\beta-(1+Bw(t))^\beta\right)}{t\,(1+Bw(t))^\beta}\mathrm{d}t. \tag{25}$$

From (5) and (6), we obtain

$$f(z)*\left(\sum_{n=0}^{\infty}\frac{z^n}{(b+n)^s}\right)=z\cdot\exp\left((1-\alpha)\int_0^z\frac{\left((1+Aw(t))^\beta-(1+Bw(t))^\beta\right)}{t\,(1+Bw(t))^\beta}\mathrm{d}t\right), \tag{26}$$

and our assertion follows immediately. □

Putting $\alpha=0,\beta=1$ and $m=2$ in Theorem 3.15, we can obtain the following Corollary

Corollary 3.16. Let $f(z)\in R_{2,1}^{s,b}[A,B,0]$. Then

$$f(z)=\left(z\cdot\exp\left((A-B)\int_0^z\frac{w(t)}{t\,(1+Bw(t))}\mathrm{d}t\right)\right)*\left(\sum_{n=0}^{\infty}z^n\right),$$

where $\omega(z)$ is analytic in U, $\omega(0)=0$ and $|\omega(z)|<1$.

Putting $s = 0, \alpha = 0, \beta = 1, A = 1$ and $B = -1$ in Theorem 3.15, we can obtain the following Corollary.

Corollary 3.17. Let $f(z) \in R_{2,1}^{s,b}[1,-1,0]$. Then

$$f(z) = z \cdot \exp\left(2\int_0^z \frac{w(t)}{t\left(1-w(t)\right)}\mathrm{d}t\right)\left(\sum_{n=0}^{\infty} z^n\right),$$

where $\omega(z)$ is analytic in U, $\omega(0) = 0$ and $|\omega(z)| < 1$.

Putting $\alpha = 0, \beta = 1, A = 1, B = -1$ and $m = 2$ in Theorem 3.15, we can obtain the following Corollary.

Corollary 3.18. Let $f(z) \in R_{2,1}^{s,b}[1,-1,0]$. Then

$$f(z) = z \cdot \exp\left(2\int_0^z \frac{w(t)}{t\left(1-w(t)\right)}\mathrm{d}t\right)\left(\sum_{n=0}^{\infty} z^n\right),$$

where $\omega(z)$ is analytic in U, $\omega(0) = 0$ and $|\omega(z)| < 1$.

Theorem 3.19. Let $f(z) \in V_{2,\beta}^{s,b}[A,B,\alpha]$. Then

$$\begin{aligned} f(z) &= \left(\int_0^z \exp\left((1-\alpha)\int_0^\zeta \frac{\left((1+Aw(t))^\beta - (1+Bw(t))^\beta\right)}{t\left(1+Bw(t)\right)^\beta}\mathrm{d}t\right)d\zeta\right) \\ &\quad * \left(\sum_{n=0}^{\infty}(n+b)^s a_n z^n\right), \end{aligned}$$

where $\omega(z)$ is analytic in U, $\omega(0) = 0$ and $|\omega(z)| < 1$.

Proof. The proof of Theorem 3.19 is similar to that of Theorem 3.15 so the details are omitted. □

Theorem 3.20. Let $\psi \in C$ and $f(z) \in R_{2,\beta}^{s,b}[A,B,\alpha]$. Then $\psi * f \in R_{2,\beta}^{s,b}[A,B,\alpha]$.

Proof. Let $F(z) = \psi * f$. Then by using some properties of convolution we have

$$\begin{aligned} \frac{z\left(J_{s,b}F(z)\right)'}{J_{s,b}F(z)} &= \frac{\psi * z\left(J_{s,b}f(z)\right)'}{\psi * J_{s,b}f(z)} \\ &= \frac{\psi * \frac{z\left(J_{s,b}f(z)\right)'}{J_{s,b}f(z)}J_{s,b}f(z)}{\psi * J_{s,b}f(z)} \\ &= \frac{\psi * p(z)J_{s,b}f(z)}{\psi * J_{s,b}f(z)}, \end{aligned}$$

where $p(z) = \frac{z\left(J_{s,b}f(z)\right)'}{J_{s,b}f(z)}$. Since $f(z) \in R_{2,\beta}^{s,b}[A,B,\alpha]$, therefore $\frac{z\left(J_{s,b}f(z)\right)'}{J_{s,b}f(z)} \in P_\beta[A,B,\alpha] \subset P[A,B,\alpha] \subset P$ [38] and hence.$J_{s,b}f(z) \in S^*$. Then by Lemma 2.5, $F(z)$, lies in the convex hull of $p(z)$ and consequently, $F \in R_{2,\beta}^{s,b}[A,B,\alpha]$. □

Theorem 3.21. Let $f \in V_{m,\beta}^{s,b}[A,B,\alpha]$ and $h \in R_{m,\beta}^{s,b}[A,B,\alpha]$. Let $H(z)$ be defined as

$$J_{s,b}H(z) = \int_0^z \left[\left(J_{s,b}f(t)\right)'\right]^{\lambda_1}\left[\frac{J_{s,b}h(t)}{z}\right]^{\lambda_2}\mathrm{d}t, \tag{27}$$

where λ_1 and λ_2 are positive real numbers with $\lambda_1 + \lambda_2 = 1$. Then $H \in V_{m,\beta}^{s,b}[A,B,\alpha]$.

Proof. Suppose $f(z) \in V^{s,b}_{m,\beta}[A,B,\alpha]$, and $h(z) \in R^{s,b}_{m,\beta}[A,B,\alpha]$.

From (27), we have

$$J_{s,b}H(z) = \left[(J_{s,b}f(z))'\right]^{\lambda_1} \left[\frac{J_{s,b}h(z)}{z}\right]^{\lambda_2}. \tag{28}$$

Logarithmic differentiation implies that

$$\begin{aligned} \frac{z\,(J_{s,b}H(z))'}{J_{s,b}H(z)} &= \frac{\left(z\,(J_{s,b}f)'\right)'}{(J_{s,b}f)'} + \frac{z\,(J_{s,b}h)'}{J_{s,b}h} && (29)\\ &= \lambda_1 p_1(z) + \lambda_2 p_2(z), && (30) \end{aligned}$$

for all $p_1, p_2 \in P_{m,\beta}[A,B,\alpha]$. Using the fact that the class $P_{m,\beta}[A,B,\alpha]$, is convex set. Therefore $\lambda_1 p_1(z) + \lambda_2 p_2(z) \in P_{m,\beta}[A,B,\alpha]$. Hence

$$\frac{z\,(J_{s,b}H(z))'}{J_{s,b}H(z)} \in P_{m,\beta}[A,B,\alpha],$$

and consequently $H \in V^{s,b}_{m,\beta}[A,B,\alpha]$. □

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