TWMS J. App. Eng. Math. V.9, N.4, 2019, pp. 764-772

# MIX-POINT PROPERTY IN QUASI-PSEUDOMETRIC SPACES

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Abstract. In this article, we give new results in the startpoint theory for quasi-pseudometric spaces. The results we present provide us with the existence of startpoint (endpoint, fixed point) for multi-valued maps defined on a bicomplete quasi-pseudometric space. We characterise the existence of startpoint and endpoint by the so-called *mix-point property*. The present results extend known ones in the area.

Keywords: quasi-pseudometric; bi-completeness; startpoint(endpoint); approximate startpoint(endpoint); approximate mix-point property, fixed point.

AMS Subject Classification: 47H09, 47H10.

## 1. Introduction and Preliminaries

The theory of startpoint, first introduced in [1], came to extend the idea of fixed points for multi-valued mappings defined on quasi-pseudometric spaces. A series of three papers, see [1, 2, 3] has given a more or less detailed introduction to the subject. the aim of the present article is to continue this study by introducing the idea of *mix-point property*, which is used to characterise the existence of startpoints.

**Definition 1.1.** *Let $X$ be a non empty set. A function $d : X \times X \to [0, \infty)$ is called a* ***quasi-pseudometric*** *on $X$ if:*

i) $d(x, x) = 0 \quad \forall\, x \in X$,

ii) $d(x, z) \leq d(x, y) + d(y, z) \quad \forall\, x, y, z \in X$.

*Moreover, if $d(x, y) = 0 = d(y, x) \implies x = y$, then $d$ is said to be a $T_0$-**quasi-metric**. The latter condition is referred to as the $T_0$-condition.*

**Remark 1.1.**

- *Let $d$ be a quasi-pseudometric on $X$, then the function $d^{-1}$ defined by $d^{-1}(x, y) = d(y, x)$ whenever $x, y \in X$ is also a quasi-pseudometric on $X$, called the* ***conjugate*** *of $d$. In the literature, $d^{-1}$ is also denoted $d^t$ or $\bar{d}$.*
- *It is easy to verify that the function $d^s$ defined by $d^s := d \vee d^{-1}$, i.e. $d^s(x, y) = \max\{d(x, y), d(y, x)\}$ defines a metric on $X$ whenever $d$ is a $T_0$-quasi-metric on $X$.*

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§ Manuscript received: August 17, 2017; accepted: February 10, 2018.

**Definition 1.2.** [1] *A $T_0$-quasi-metric space $(X, d)$ is called **bicomplete** provided that the metric $d^s$ on $X$ is complete.*

Let $(X, d)$ be a quasi-pseudometric space. We set $\mathscr{P}_0(X) := 2^X \setminus \{\emptyset\}$ where $2^X$ denotes the power set of $X$. For $x \in X$ and $A \in \mathscr{P}_0(X)$, we define:

$$d(x, A) = \inf\{d(x, a), a \in A\}, \quad d(A, x) = \inf\{d(a, x), a \in A\}.$$

We also define the map $H : \mathscr{P}_0(X) \times \mathscr{P}_0(X) \to [0, \infty]$ by

$$H(A, B) = \max\left\{\sup_{a\in A} d(a, B),\ \sup_{b\in B} d(A, b)\right\} \text{ whenever } A, B, \in \mathscr{P}_0(X).$$

Then $H$ is an extended[1] quasi-pseudometric on $\mathscr{P}_0(X)$.

## 2. Some first results

We briefly recall the idea of a startpoint, as initially intended in [1].

**Definition 2.1.** *(Compare* [1]*) Let $(X, d)$ be a $T_0$-quasi-metric space.*
*Let $F : X \to 2^X$ be a set-valued map. An element $x \in X$ is said to be*
- (i) *a fixed point of $F$ if $x \in Fx$,*
- (ii) *a startpoint of $F$ if $H(\{x\}, Fx) = 0$,*
- (iii) *an endpoint of $F$ if $H(Fx, \{x\}) = 0$.*

We recall below the main theorem in the startpoint theory that appeared in [1].

**Theorem 2.1.** [1, Theorem 29] *Let $(X, d)$ be a bicomplete quasi-pseudometric space. Let $F : X \to CB(X)$ be a set-valued map that satisfies*

$$H(Fx, Fy) \le \psi(d(x, y)), \quad \text{for each } x, y \in X, \tag{1}$$

*where $\psi : [0, \infty) \to [0, \infty)$ is upper semicontinuous, $\psi(t) < t$ for each $t > 0$ and $\liminf_{t\to\infty}(t - \psi(t)) > 0$. Then there exists a unique $x_0 \in X$ which is both a startpoint and an endpoint of $F$ if and only if $F$ has the approximate mix-point property.*

We introduce the following definitions:

**Definition 2.2.** *Let $(X, d)$ be a quasi-pseudometric space, $J : X \to X$ be a single valued mapping and $F : X \to 2^X$ be a multi-valued mapping. We say that the mappings $J$ and $F$ have the* **approximate startpoint property** *(resp.* **approximate endpoint property** *), if*

$$\inf_{x\in X} \sup_{y\in Fx} d(Jx, y) = 0 \quad (\textit{resp.} \inf_{x\in X} \sup_{y\in Fx} d(y, Jx) = 0).$$

**Definition 2.3.** *Let $(X, d)$ be a $T_0$-quasi-pseudometric space, $J : X \to X$ be a single valued mapping. We say that $J$ and the set-valued map $F : X \to 2^X$ have the* **approximate mix-point property** *if*

$$\inf_{x\in X} \sup_{y\in Fx} d^s(Jx, y) = 0.$$

**Definition 2.4.** *(Compare* [1]*) Let $(X, d)$ be a quasi-pseudometric space, $J : X \to X$ be a single valued mapping. Let $F : X \to 2^X$ be a set-valued map. An element $x \in X$ is said to be*
- (i) *a $J$-fixed point of $F$ if $Jx \in Fx$,*
- (ii) *a startpoint of $J$ and $F$ if $H(\{Jx\}, Fx) = 0$,*

[1] This means that $H$ can attain the value $\infty$ as it appears in the definition.

(iii) *an endpoint of $J$ and $F$ if $H(Fx, \{Jx\}) = 0$.*

The next three results are the first results of this paper. We shall not give any proof, since the proofs follow the same arguments as the proofs in [1].

**Theorem 2.2.** *(Compare*[1, Theorem 29]*) Let $(X, d)$ be a bicomplete quasi-pseudometric space. Assume $J : X \to X$ is a continuous single-valued map and let $F : X \to CB(X)$ be a set-valued map that satisfies*

$$H(Fx, Fy) \leq \psi(d(Jx, Jy)), \quad \text{for each } x, y \in X, \tag{2}$$

*where $\psi : [0, \infty) \to [0, \infty)$ is upper semicontinuous, $\psi(t) < t$ for each $t > 0$ and $\liminf_{t\to\infty}(t - \psi(t)) > 0$. Then there exists a unique $x_0 \in X$ which is both a startpoint and an endpoint of $J$ and $F$ if and only if $J$ and $F$ have the approximate mix-point property.*

**Theorem 2.3.** *(Compare*[1, Theorem 31]*) Let $(X, d)$ be a bicomplete quasi-pseudometric space. Assume $J : X \to X$ is a continuous single-valued map and let $F : X \to CB(X)$ be a set-valued map that satisfies*

$$H(Fx, Fy) \leq k(d(Jx, Jy)), \quad \text{for each } x, y \in X, \tag{3}$$

*where $k \in [0, 1)$. Then there exists a unique $x_0 \in X$ which is both a startpoint and an endpoint of $J$ and $F$ if and only if $J$ and $F$ have the approximate mix-point property.*

**Theorem 2.4.** *(Compare*[1, Corollary 30]*) Let $(X, d)$ be a bicomplete quasi-pseudometric space. Assume $J : X \to X$ is a continuous single-valued map. Let $F : X \to CB(X)$ be a set-valued map that satisfies*

$$H(Fx, Fy) \leq \psi(d(Jx, Jy)), \quad \text{for each } x, y \in X, \tag{4}$$

*where $\psi : [0, \infty) \to [0, \infty)$ is an upper semicontinuous map that satisfies $\psi(t) < t$ for each $t > 0$ and $\liminf_{t\to\infty}(t - \psi(t)) > 0$. If $J$ and $F$ have the approximate mix-point property then $F$ has a $J$-fixed point.*

**Remark 2.1.** *Observe that if we put $J = I_X$ (identity map on $X$) in Theorems 2.2, 2.3 and 2.4 respectively, we obtain* [1, Theorem 29, Theorem 31, Corollary 30] *respectively.*

## 3. More results

In [1], the proof of Theorem 2.1 basically establishes that the sets

$$C_n = \left\{ x \in X : \sup_{y \in Fx} d^s(x, y) \leq \frac{1}{n} \right\} \neq \emptyset, \qquad \text{for } n \in \mathbb{N} = \{1, 2, \cdots\},$$

form a non-increasing sequence of bounded and $\tau(d^s)$-closed sets. The conclusion follows from the Cantor intersection theorem. We shall use a similar approach in proving the next two results, with the difference that we present simpler and shorter arguments.

We now present the first non trivial generalisation of [1, Theorem 29].

**Theorem 3.1.** *Let $(X, d)$ be a bicomplete quasi-pseudometric space. Assume $J : X \to X$ is a continuous single-valued map such that $rd(x, y) \leq d(Jx, Jy)$ for some constant $r > 0$. Let $F : X \to CB(X)$ be a set-valued map that satisfies*

$$H(Fx, Fy) \leq \alpha d(Jx, Jy), \quad \text{for each } x, y \in X, \tag{5}$$

*where $\alpha \in (0, 1)$ and $r\alpha < 1$. Then there exists a unique $x_0 \in X$ which is both a startpoint and an endpoint of $J$ and $F$ if and only if $J$ and $F$ have the approximate mix-point property.*

**Proof.** It is clear that if $J$ and $F$ admit a point which is both a startpoint and an endpoint, then $J$ and $F$ have the approximate startpoint property and the approximate endpoint property, i.e the approximate mix-point property. Conversely, suppose $J$ and $F$ have the approximate mix-point property. Then

$$C_n = \left\{ x \in X : \sup_{y \in Fx} d^s(Jx, y) \leq \frac{1}{n} \right\} \neq \emptyset,$$

for each $n \in \mathbb{N}$. Also it is clear that for each $n \in \mathbb{N}$, $C_{n+1} \subseteq C_n$. Since the map $x \mapsto \sup_{y \in Fx} d^s(Jx, y)$ is $\tau(d^s)$-lower semicontinuous (as supremum of $\tau(d^s)$-continuous mappings), the $C_n$ is $\tau(d^s)$-closed.

Next we prove that for each $n \in \mathbb{N}$, $C_n$ is bounded. Indeed, for any $x, y \in C_n$,

$$\begin{aligned} d(Jx, Jy) &= H(\{Jx\}, \{Jy\}) \\ &\leq H(\{Jx\}, Fx) + H(Fx, Fy) + H(Fy, \{Jy\}) \\ &\leq \frac{2}{n} + \alpha d(Jx, Jy). \end{aligned}$$

So

$$d(Jx, Jy) \leq \frac{2}{n(1-\alpha)},$$

and since $rd(x, y) \leq d(Jx, Jy)$, we have

$$\delta(C_n) \leq \frac{2}{rn(1-\alpha)}.$$

Therefore $\lim_{n \to \infty} \delta(C_n) = 0$. It follows from the Cantor intersection theorem that $\bigcap_{n \in N} C_n = \{x_0\}$.

Thus $H(\{Jx_0\}, Fx_0) = \sup_{y \in Fx_0} d(Jx_0, y) = 0 = \sup_{y \in Fx_0} d(y, Jx_0) = H(Fx_0, \{Jx_0\})$. For uniqueness, if $z_0$ is an arbitrary startpoint and endpoint of $J$ and $F$, then $H(\{Jz_0\}, Fz_0) = 0 = H(Fz_0, \{Jz_0\})$, and so $z_0 \in \bigcap_{n \in N} C_n = \{x_0\}$.

We give the following example to illustrate our result.

**Example 3.1.**

*Indeed, consider the $T_0$-quasi-metric space $(X, d)$ where $X = \{0, 1\}$ and $d$ defined by $d(0, 1) = 0$, $d(1, 0) = 1$ and $d(x, x) = 0$ for $x = 0, 1$. Note that $(X, d)$ is bicomplete. We define on $X$ the set-valued map $F : X \to 2^X$ by $Fx = \{0\}$ and the single-valued continuous mapping $J : X \to X$ by $Jx = x^2$.*

*It is clear that for all $x, y \in X, H(Fx, Fy) = 0$.*

*For $x = 0, y = 1$, $d(x, y) = d(0, 1) = 0$, and $d(Jx, Jy) = d(0, 1) = 0$.*

*For $x = 1, y = 0$, $d(x, y) = d(1, 0) = 1$, and $d(Jx, Jy) = d(1, 0) = 1$.*

*So if we set $r = \frac{1}{2}$ and $\alpha = \frac{1}{3}$, we have that $r\alpha = \frac{1}{6} < 1$ and*

$$rd(x, y) \leq d(Jx, Jy).$$

*Moreover, the condition (5) is satisfied. Since $H(\{J0\}, F0) = 0 = H(F0, \{J0\})$, then 0 is both a startpoint and an endpoint of $J$ and $F$.*

*Observe also that:*

*-for $x = 0, Fx = \{0\}$, $\sup_{y \in Fx} d^s(Jx, y) = d^s(0, 0) = 0$,*

*-for* $x = 1, Fx = \{0\}, \sup_{y \in Fx} d^s(Jx, y) = d^s(1, 0) = 0,$ *and hence*

$$\inf_{x \in X} \sup_{y \in Fx} d^s(Jx, y) = 0.$$

*i.e.* $J$ *and* $F$ *have the approximate mix-point property.*

**Corollary 3.1.** *Let* $(X, d)$ *be a bicomplete quasi-pseudometric space. Assume* $J : X \to X$ *is a continuous single-valued map such that* $rd(x, y) \leq d(Jx, Jy)$ *for some constant* $r > 0$ *and for each* $x, y \in X$*. Let* $F : X \to CB(X)$ *be a set-valued map that satisfies*

$$H(Fx, Fy) \leq \alpha d(Jx, Jy), \quad \text{for each } x, y \in X, \tag{6}$$

*where* $\alpha \in (0, 1)$ *and* $r\alpha < 1$*. If* $J$ *and* $F$ *have the approximate mix-point property then* $F$ *has a* $J$*-fixed point.*

**Proof.**
From Theorem 3.1, we conclude that there exists $x_0 \in X$ which is both a startpoint and an endpoint for $J$ and $F$, i.e $H(\{Jx_0\}, Fx_0) = 0 = H(Fx_0, \{Jx_0\})$. The $T_0$-condition therefore guarantees the desired result.

**Theorem 3.2.** *Let* $(X, d)$ *be a bicomplete quasi-pseudometric space. Assume* $J : X \to X$ *is a continuous single-valued map such that* $rd(x, y) \leq d(Jx, Jy)$ *for some constant* $r > 0$ *and for each* $x, y \in X$*. Let* $F : X \to CB(X)$ *be a set-valued map that satisfies*

$$H(Fx, Fy) \leq \alpha[d(Jx, Fx) + d(Jy, Fy)], \quad \text{for each } x, y \in X, \tag{7}$$

*where* $\alpha \in (0, 1/2)$*. Then there exists a unique* $x_0 \in X$ *which is both a startpoint and an endpoint of* $J$ *and* $F$ *if and only if* $J$ *and* $F$ *have the approximate mix-point property.*

**Proof.** Once again, only one implication will be of interest to us, since the other one is trivial. So suppose $J$ and $F$ have the approximate mix-point property. Then we already know that the sets

$$C_n = \left\{ x \in X : \sup_{y \in Fx} d^s(Jx, y) \leq \frac{1}{n} \right\} \neq \emptyset,$$

for each $n \in \mathbb{N}$ are $\tau(d^s)$-closed and that $C_{n+1} \subseteq C_n$.

Next we prove that for each $n \in \mathbb{N}$, $C_n$ is bounded. Indeed, for any $x, y \in C_n$,

$$\begin{aligned} d(Jx, Jy) &= H(\{Jx\}, \{Jy\}) \\ &\leq H(\{Jx\}, Fx) + H(Fx, Fy) + H(Fy, \{Jy\}) \\ &\leq \frac{2}{n} + \alpha[d(Jx, Fx) + d(Jy, Fy)] \\ &\leq \frac{1}{n}(2 + 2\alpha). \end{aligned}$$

So

$$d(Jx, Jy) \leq \frac{1}{n}(2 + 2\alpha),$$

and since $rd(x, y) \leq d(Jx, Jy)$, we have

$$\delta(C_n) \leq \frac{1}{rn}(2 + 2\alpha).$$

Therefore $\lim_{n\to\infty}\delta(C_n)=0$. It follows from the Cantor intersection theorem that $\bigcap_{n\in N} C_n = \{x_0\}$. Thus $H(\{Jx_0\}, Fx_0) = \sup_{y\in Fx_0} d(Jx_0, y) = 0 = \sup_{y\in Fx_0} d(y, Jx_0) = H(Fx_0, \{Jx_0\})$. For uniqueness, if $z_0$ is an arbitrary startpoint and endpoint of $J$ and $F$, then $H(\{Jz_0\}, Fz_0) = 0 = H(Fz_0, \{Jz_0\})$, and so $z_0 \in \bigcap_{n\in N} C_n = \{x_0\}$.

**Example 3.2.** *Indeed, consider the $T_0$-quasi-metric space $(X,d)$ where $X=\{0,1\}$ and $d$ defined by $d(0,1)=0$, $d(1,0)=1$ and $d(x,x)=0$ for $x=0,1$. Note that $(X,d)$ is bicomplete. We define on $X$ the set-valued map $F : X \to 2^X$ by $Fx=\{0\}$ and the single valued continuous map $J : X \to X$ by $Jx = x^3$. It is clear that for all $x, y \in X, H(Fx, Fy) = 0$.*

*For $x=0, y=1, d(x,y)=d(0,1)=0, d(Jx,Jy)=d(0,1)=0$.*

*For $x=1, y=0, d(x,y)=d(1,0)=1, d(Jx,Jy)=d(1,0)=1$. So if we set $r=\frac{1}{2}$, we have that*

$$rd(x,y) \le d(Jx,Jy).$$

*So if we set $\alpha=\frac{1}{3}$, we have that $0<\alpha<\frac{1}{2}$. Since $H(\{J0\},F0)=0=H(F0,\{J0\})$, then $0$ is both a startpoint and an endpoint of $J$ and $F$.*

*For $x=0, y=1, \alpha[d(Jx,Fx)+d(Jy,Fy)]=\frac{1}{3}$ and*
*for $x=1, y=0, \alpha[d(Jx,Fx)+d(Jy,Fy)]=\frac{1}{3}$, so the condition (7) is satisfied.*
*Observe also that*

$$\inf_{x\in X}\sup_{y\in Fx} d^s(Jx,y)=0.$$

*i.e. $J$ and $F$ have the approximate mix-point property.*

**Corollary 3.2.** *Let $(X,d)$ be a bicomplete quasi-pseudometric space. Assume $J : X \to X$ is a continuous single-valued map such that $rd(x,y) \le d(Jx,Jy)$ for some constant $r>0$ and for each $x,y\in X$. Let $F : X \to CB(X)$ be a set-valued map that satisfies*

$$H(Fx,Fy) \le \alpha[d(Jx,Fx)+d(Jy,Fy)], \quad \text{for each } x,y\in X, \tag{8}$$

*where $\alpha \in (0,1/2)$. If $J$ and $F$ have the approximate mix-point property then $F$ has a $J$-fixed point.*

Using the same idea as in the proof of Theorem 3.1 and Theorem 3.2, one can establish the following results:

**Theorem 3.3.** *Let $(X,d)$ be a bicomplete quasi-pseudometric space. Assume $J : X \to X$ is a continuous single-valued map such that $rd(x,y) \le d(Jx,Jy)$ for some constant $r>0$ and for each $x,y\in X$. Let $F : X \to CB(X)$ be a set-valued map that satisfies*

$$H(Fx,Fy) \le \alpha[d(Jx,Fy)+d(Fx,Jy)], \quad \text{for each } x,y\in X, \tag{9}$$

*where $0<\alpha<1/2$ with $2r\alpha<1$. Then there exists a unique $x_0\in X$ which is both a startpoint and an endpoint of $J$ and $F$ if and only if $J$ and $F$ have the approximate mix-point property.*

**Theorem 3.4.** *Let $(X,d)$ be a bicomplete quasi-pseudometric space. Assume $J : X \to X$ is a continuous single-valued map such that $rd(x,y) \le d(Jx,Jy)$ for some constant $r>0$ and for each $x,y\in X$. Let $F : X \to CB(X)$ be a set-valued map that satisfies*

$$H(Fx,Fy) \le \alpha d(Jx,Jy)+Ld(Fx,Jy), \quad \text{for each } x,y\in X, \tag{10}$$

*where $\alpha > 0$ and $L \geq 0$ such that $r(\alpha + L) < 1$. Then there exists a unique $x_0 \in X$ which is both a startpoint and an endpoint of $J$ and $F$ if and only if $J$ and $F$ have the approximate mix-point property.*

## 4. Concluding remarks

All the above results remain true if instead we consider a quasi-pseudometric type space $(X, d, b)$ (see [4]). On the other side, the sets $C_n$ considered in the investigation can be made more general in the sense that we could consider sets of the form $C_\varepsilon$ where $\varepsilon > 0$. Hence we write

$$C_\varepsilon = \left\{ x \in X : \sup_{y \in Fx} d^s(Jx, y) \leq \varepsilon \right\}, \qquad \text{for any } \varepsilon > 0.$$

Therefore, the Theorem 3.1 could be reformulated as follows:

**Theorem 4.1.** *Let $(X, d, b)$ be a bicomplete quasi-pseudometric type space. Assume $J : X \to X$ is a continuous single-valued map such that $rd(x, y) \leq d(Jx, Jy)$ for some constant $r > 0$ and for each $x, y \in X$. Let $F : X \to CB(X)$ be a set-valued map that satisfies*

$$H(Fx, Fy) \leq \alpha d(Jx, Jy), \quad \textit{for each } x, y \in X, \tag{11}$$

*where $\alpha \in (0, 1)$ such that $r\alpha b^2 < 1$. Then there exists a unique $x_0 \in X$ which is both a startpoint and an endpoint of $J$ and $F$ if and only if $J$ and $F$ have the approximate mix-point property.*

In proving this theorem, the following lemma the key:

**Lemma 4.1.** *Let $(X, d, b)$ be a bicomplete quasi-pseudometric type space. Assume $J : X \to X$ is a continuous single-valued map such that $rd(x, y) \leq d(Jx, Jy)$ for some constant $r > 0$ such that $r\alpha b^2 < 1$ and for each $x, y \in X$. Let $F : X \to CB(X)$ be a set-valued map that satisfies*

$$H(Fx, Fy) \leq \alpha d(Jx, Jy), \quad \textit{for each } x, y \in X, \tag{12}$$

*where $\alpha \in (0, 1)$. Then*

$$\delta(C_\varepsilon) \leq \frac{b\varepsilon(1 + b)}{r(1 - \alpha^2 b)}, \qquad \textit{for any } \varepsilon > 0.$$

Similarly, the Theorem 3.2 could be reformulated as follows:

**Theorem 4.2.** *Let $(X, d, b)$ be a bicomplete quasi-pseudometric type space. Assume $J : X \to X$ is a continuous single-valued map such that $rd(x, y) \leq d(Jx, Jy)$ for some constant $r > 0$ and for each $x, y \in X$. Let $F : X \to CB(X)$ be a set-valued map that satisfies*

$$H(Fx, Fy) \leq \alpha[d(Jx, Fx) + d(Jy, Fy)], \quad \textit{for each } x, y \in X, \tag{13}$$

*where $\alpha \in (0, 1/2)$. Then there exists a unique $x_0 \in X$ which is both a startpoint and an endpoint of $J$ and $F$ if and only if $J$ and $F$ have the approximate mix-point property.*

The key lemma in this case is

**Lemma 4.2.** *Let $(X, d, b)$ be a bicomplete quasi-pseudometric type space. Assume $J : X \to X$ is a continuous single-valued map such that $rd(x, y) \leq d(Jx, Jy)$ for some constant $r > 0$ and for each $x, y \in X$. Let $F : X \to CB(X)$ be a set-valued map that satisfies*

$$H(Fx, Fy) \leq \alpha[d(Jx, Fx) + d(Jy, Fy)], \quad \text{for each } x, y \in X, \tag{14}$$

*where $\alpha \in (0, 1/2)$. Then*

$$\delta(C_\varepsilon) \leq \frac{b\varepsilon}{r}(1 + b + 2\alpha b), \qquad \text{for any } \varepsilon > 0.$$

In a similar manner, the Theorem 3.3 could be reformulated as follows:

**Theorem 4.3.** *Let $(X, d, b)$ be a bicomplete quasi-pseudometric type space. Assume $J : X \to X$ is a continuous single-valued map such that $rd(x, y) \leq d(Jx, Jy)$ for some constant $r > 0$ and for each $x, y \in X$. Let $F : X \to CB(X)$ be a set-valued map that satisfies*

$$H(Fx, Fy) \leq \alpha[d(Jx, Fy) + d(Fx, Jy)], \quad \text{for each } x, y \in X, \tag{15}$$

*where $0 < \alpha < 1/2$ with $2b^2 r\alpha < 1$. Then there exists a unique $x_0 \in X$ which is both a startpoint and an endpoint of $J$ and $F$ if and only if $J$ and $F$ have the approximate mix-point property.*

The key lemma is therefore:

**Lemma 4.3.** *Let $(X, d, b)$ be a bicomplete quasi-pseudometric type space. Assume $J : X \to X$ is a continuous single-valued map such that $rd(x, y) \leq d(Jx, Jy)$ for some constant $r > 0$ and for each $x, y \in X$. Let $F : X \to CB(X)$ be a set-valued map that satisfies*

$$H(Fx, Fy) \leq \alpha[d(Jx, Fy) + d(Fx, Jy)], \quad \text{for each } x, y \in X, \tag{16}$$

*where $0 < \alpha < 1/2$ with $2b^2 r\alpha < 1$. Then*

$$\delta(C_\varepsilon) \leq \frac{b\varepsilon}{r(1 - 2b^2\alpha)}(1 + b + 2\alpha b), \qquad \text{for any } \varepsilon > 0.$$

Finally the Theorem 3.4 could be reformulated as follows:

**Theorem 4.4.** *Let $(X, d, b)$ be a bicomplete quasi-pseudometric type space. Assume $J : X \to X$ is a continuous single-valued map such that $rd(x, y) \leq d(Jx, Jy)$ for some constant $r > 0$and for each $x, y \in X$. Let $F : X \to CB(X)$ be a set-valued map that satisfies*

$$H(Fx, Fy) \leq \alpha d(Jx, Jy) + Ld(Fx, Jy), \quad \text{for each } x, y \in X, \tag{17}$$

*where $\alpha > 0$ and $L \geq 0$ such that $rb^2(\alpha + bL) < 1$. Then there exists a unique $x_0 \in X$ which is both a startpoint and an endpoint of $J$ and $F$ if and only if $J$ and $F$ have the approximate mix-point property.*

The proof will be done with the use of the following lemma:

**Lemma 4.4.** *Let $(X, d, b)$ be a bicomplete quasi-pseudometric type space. Assume $J : X \to X$ is a continuous single-valued map such that $rd(x, y) \leq d(Jx, Jy)$ for some constant $r > 0$ and for each $x, y \in X$. Let $F : X \to CB(X)$ be a set-valued map that satisfies*

$$H(Fx, Fy) \leq \alpha d(Jx, Jy) + Ld(Fx, Jy), \quad \text{for each } x, y \in X, \tag{18}$$

*where $\alpha > 0$ and $L \geq 0$ such that $rb^2(\alpha + bL) < 1$. Then*

$$\delta(C_\varepsilon) \leq \frac{b\varepsilon(1+b+Lb^2)}{r(1-b^2(\alpha+bL))}, \qquad \textit{for any } \varepsilon > 0.$$

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