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ON THE ESTIMATIONS OF THE SMALL EIGENVALUES OF NON-SELF-ADJOINT STURM-LIOUVILLE OPERATORS

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Abstract. We give a new approach for the estimations of the eigenvalues of non-self-adjoint Sturm-Liouville operators with regular but not strongly regular boundary conditions. Moreover we give the error estimations.

Keywords: Eigenvalue estimations, Regular boundary conditions, Numerical methods, Sturm-Liouville Operators.

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1. Introduction

Let $T_k(q)$ and $P_k(q)$ for $k = 1, 2$ be the operators generated in $L_2[0,1]$ by the differential expression $-y'' + q(x)y$ and the following boundary conditions:

$$y_0' + \beta y_1' = 0,\ y_0 + (-1)^k y_1 = 0, \tag{1}$$

$$y_0' + (-1)^k y_1' = 0,\ y_0 + \alpha y_1 = 0 \tag{2}$$

respectively, where q is a complex-valued summable function on $[0,1]$, β and α are complex numbers satisfying $\beta \neq \pm 1$, $\alpha \neq \pm 1$ and $y_0 = y(0)$, $y_1 = y(1)$, $y_0' = y'(0)$, $y_1' = y'(1)$.

In conditions (1) and (2) if $\beta = 1$, $\beta = -1$, $\alpha = 1$ and $\alpha = -1$, then any $\lambda \in \mathbb{C}$ is an eigenvalue of infinite multiplicity of $T_1(q)$, $T_2(q)$, $P_1(q)$ and $P_2(q)$, respectively. In (1) and (2) if $\beta = -1$, $\alpha = -1$ and $k = 1$, then they are periodic boundary conditions; if $\beta = 1$, $\alpha = 1$ and $k = 2$ then they are antiperiodic boundary conditions. These boundary conditions are regular but not strongly regular. Note that the boundary conditions are strongly regular if and only if all large eigenvalues are far from each other [9]. This easify to investigate the perturbation theory and Riesz basis property. If the boundary conditions are not strongly regular then the eigenvalues are pairwise very close to each other. This situation complicates the investigation of the perturbation theory. Therefore the regular cases which are not strongly regular are still investigated. Only the special cases, the periodic and antiperiodic problems, were investigated in detail. In [8] we obtained the asymptotic formulas for the large eigenvalues by the asymptotic methods, but

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these formulas cannot be used for the small eigenvalues. In this paper we estimate the small eigenvalues of the operators T_k and P_k by the numerical methods and give the error estimations.

Let us mention the literature about the investigations of the small eigenvalues. There are a lot of papers about the estimations of the small eigenvalues of Sturm-Liouville operators with the strongly regular boundary conditions (see for example [3, 4, 10] and their references). In the numerical results about the regular but not strongly regular boundary conditions, the estimations of the small eigenvalues for the periodic and antiperiodic boundary conditions are the most widely-studied ones (see for example [5, 7] and their references). There are also many papers concerning with the estimations of the small eigenvalues for the Sturm type (separated) boundary conditions which are special cases of the strongly regular boundary conditions including the Dirichlet and Neumann boundary conditions (see for example [2, 3, 4] and their references).

We are interested in the numerical estimations of the small eigenvalues for the regular boundary conditions that are not strongly regular in cases (1) and (2). There are only two papers [1, 6] containing the estimations of the small eigenvalues for such boundary conditions. In [6], C. J. Goh, K. L. Teo and R. P. Agarwal gave some results about the estimations of the small eigenvalues in the case when the potential is real and continuous. In [1], M. H. Annaby and R. M. Asharabi, estimated the small eigenvalues for the general boundary conditions but their numerical example concerning with the operators $T_k(q)$ and $P_k(q)$ is for the potential such that the exact eigenvalues are known.

In this paper we use a method different from the methods of the papers [1] and [6], and give a new approach to get subtle estimations for the small eigenvalues when the complex-valued summable potential is in the form $q(x) = \sum_{k=1}^{\infty} q_k \cos 2\pi k x$. Note that, for this potential, it is impossible to compute the exact values of the eigenvalues.

In this paper we essentially use the following equation obtained in [8] (see (55) of [8])

$$\left[\lambda - (2\pi n)^2 - Q_n - A(\lambda)\right]\left[\lambda - (2\pi n)^2 - P_n^* - A'(\lambda)\right] - \\ - \left[P_n + B(\lambda)\right]\left[\gamma_1 n + Q_n^* + B'(\lambda)\right] = 0, \qquad (3)$$

where the terms in this equation are defined in (19), (33), (34), (36) and (38) in [8]. Nevertheless, even if we use the same equation (4) in [8] and in this paper, the methods of their investigations are absolutely different. In [8] we use the asymptotic formulas for the large eigenvalues which can not be used for the small eigenvalues. Here we use the numerical results. Moreover, this paper is the complement of [8] since they together study all eigenvalues (the large eigenvalues in [8] and the small ones in this paper).

We will focus only on the operator $T_1(q)$. The investigations of the other operators are the same. For simplicity of reading first let us give the brief scheme of the proofs of the main results. To consider the small eigenvalues, first we prove (see Theorem 2.1) that the small eigenvalues satisfy the equation (3) and using this equation we show that the eigenvalue $\lambda_{n,j}$ is the root of one of the equations:

$$\lambda = (2\pi n)^2 + \frac{1}{2}\left[\left(Q_n + P_n^* + A(\lambda) + A'(\lambda)\right) - \sqrt{\Delta(\lambda)}\right], \qquad (4)$$

$$\lambda = (2\pi n)^2 + \frac{1}{2}\left[\left(Q_n + P_n^* + A(\lambda) + A'(\lambda)\right) + \sqrt{\Delta(\lambda)}\right], \qquad (5)$$

where $\Delta(\lambda) = (Q_n - P_n^* + A(\lambda) - A'(\lambda))^2 + 4(P_n + B(\lambda))(\gamma_1 n + Q_n^* + B'(\lambda))$. To use the numerical methods, we take finite summations instead of the infinite series in the expressions (4) and (5) and show that the eigenvalues are close to the roots of the equations obtained by taking these finite summations. To find the roots of these equations, many numerical methods can be used such as the fixed point iteration and Newton method. Since it is not necessary to compute the derivatives of the functions $f_j(x)$, $j=1,2$, defined in (16), we choose the fixed point iteration method. Then using the Banach fixed point theorem, we prove that each of these equations containing the finite summations has a unique solution on the convenient set (see Theorem 2.2). Moreover we give the error estimations. To avoid eclipsing the essence by the technical details, some calculations and estimations are given in the Appendix.

For simplicity of calculations we assume that $q(x)$

$$q(x)=\sum_{k=1}^{\infty} q_k \cos 2\pi kx,\ \ \sup|q(x)|:=M<\infty,\ \ \sum_{k=1}^{\infty}|q_k|:=\frac{c}{2}<\infty,\ \ |\lambda_n(q)-\lambda_n(0)|\le M. \tag{6}$$

For $n_k\neq n$,we have

$$\left|\lambda_n-(2\pi n_k)^2\right|\ge\left|(2\pi n)^2-(2\pi n_k)^2\right|-M\ge\left|4\pi^2(n-n_k)(n+n_k)\right|-M\ge\delta(n), \tag{7}$$

where $\delta(n)=4\pi^2(2n-1)-M$.

2. Main Results

To prove one of the main results Theorem 2.1 we use the following lemmas.

Lemma 2.1. *If*

$$\delta(n)>\frac{4c}{3}, \tag{8}$$

then $R_k(\lambda_{n,j})\to 0$ *and* $R_k'(\lambda_{n,j})\to 0$ *as* $k\to\infty$ *for* $j=1,2,$ *where where*

$$R_k(\lambda_{n,j})=\sum_{n_1,\ldots,n_{k+1}}\left\{C_{k+1}(q\Psi_{n,j},\sin 2\pi n_{k+1}x)+M_{k+1}\left(q\Psi_{n,j},\varphi^*_{n_{k+1}}\right)\right\},$$

$$R_k'(\lambda_{n,j})=\sum_{n_1,\ldots,n_{k+1}}\left\{\widetilde{C}_{k+1}(q\Psi_{n,j},\sin 2\pi n_{k+1}x)+\widetilde{M}_{k+1}\left(q\Psi_{n,j},\varphi^*_{n_{k+1}}\right)\right\},$$

C_{k+1}, M_{k+1}, $\widetilde{C}_{k+1}$, $\widetilde{M}_{k+1}$ *are defined in the Appendix and* $\Psi_{n,j}$ *is the normalized eigenfunction corresponding to the eigenvalue* $\lambda_{n,j}$. *Here the summations are taken under the conditions* $n_i\neq n$ *and* $n_i=0,1,...$ *for* $i=1,2,....$

Proof. By the estimations which were done in subsection 3.1 of the Appendix, we have

$$|R_k(\lambda_{n,j})|\le\left|\sum_{n_1,\ldots,n_{k+1}}\frac{q(n,n_1)q(n_2,n_1)\ldots q(n_{k+1},n_k)\,(q\Psi_{n,j},\sin 2\pi n_{k+1}x)}{2^{k+1}\left(\lambda_{n,j}-(2\pi n_1)^2\right)\left(\lambda_{n,j}-(2\pi n_2)^2\right)\ldots\left(\lambda_{n,j}-(2\pi n_{k+1})^2\right)}\right|,$$

where $q(s,m):=(q_{s-m}-q_{s+m})$. One can easily see that there exists a nonnegative integer n_1^0 such that

$$|R_k(\lambda_{n,j})|\le\left|\left(\sum_{n_1}\frac{q(n,n_1)}{\lambda_{n,j}-(2\pi n_1)^2}\right)S(n_2,n_3,\ldots,n_{k+1})\right|$$

where

$$S(n_2,n_3,\ldots,n_{k+1})=\sum_{n_2,\ldots,n_{k+1}}\frac{2^{-(k+1)}q(n_2,n_1^0)\ldots q(n_{k+1},n_k)\,(q\Psi_{n,j},\sin 2\pi n_{k+1}x)}{\left(\lambda_{n,j}-(2\pi n_1)^2\right)\left(\lambda_{n,j}-(2\pi n_2)^2\right)\ldots\left(\lambda_{n,j}-(2\pi n_{k+1})^2\right)}.$$

It follows from (6) and (7) that

$$\left|\sum_{n_1} \frac{q(n,n_1)}{\lambda_{n,j}-(2\pi n_1)^2}\right| \leq \frac{c}{\delta\left(n\right)}.$$

Repeating this process $k+1$ times and taking into account that $\|\Psi_{n,j}\|=1$ and that

$$|(q\Psi_{n,j},\sin 2\pi n_{k+1}x)| \leq \|q\Psi_{n,j}\| \, \|\sin 2\pi n_{k+1}x\| \leq \frac{M}{\sqrt{2}},$$

we obtain

$$|R_k\left(\lambda_{n,j}\right)| \leq \frac{Mc^{k+1}}{\sqrt{2}2^{k+1}\left(\delta\left(n\right)\right)^{k+1}} = \frac{M}{\sqrt{2}2^{k+1}}\left(\frac{c}{\delta\left(n\right)}\right)^{k+1}.$$

Thus this with (8) implies $R_k\left(\lambda_{n,j}\right) \to 0$ as $k\to\infty$ for $j=1,2$. In the same way we prove the same result for $R_k'\left(\lambda_{n,j}\right)$. □

Lemma 2.2. *If (8) and the condition*

$$\delta(n) > C\left(\beta\right) M\left(\frac{1}{2}+\frac{121\left|\beta+1\right|^2}{4\pi^2\left|\beta-1\right|^2}+\left(A\left(\beta\right)\right)^2\right)^{\frac{1}{2}}, \tag{9}$$

hold, where

$$A\left(\beta\right)=\sup_{x\in[0,1]}\left|\frac{4\left(\beta+1\right)}{\beta-1}\left(x-\frac{1}{1+\beta}\right)\right|, \tag{10}$$

and $C\left(\beta\right)$ is defined in (11), then the inequality $\left|u_{n,j}\right|^2+\left|v_{n,j}\right|^2>0$ is satisfied for $j=1,2$, where $u_{n,j}$ and $v_{n,j}$ are defined by (27) in [8].

Proof. Suppose to the contrary $u_{n,j}=0$, $v_{n,j}=0$. Since the root functions of $T_1\left(0\right)$ form a Riesz basis, we have the decomposition (26) in [8] for the normalized eigenfunction $\Psi_{n,j}$ corresponding to the eigenvalue $\lambda_{n,j}$ of $T_1\left(q\right)$. To get a contradiction, it is enough to show that $\|\Psi_{n,j}\|<1$ for $j=1,2$. By the Riesz basis property, there exists a bounded and boundedly invertible operator A which takes the orthonormal basis $\{e_i : i\in\mathbb{Z}\}$ to this basis, say $Ae_{-k}=\varphi_k$ and $Ae_k=\cos 2\pi kx$. Therefore there exists $C(\beta)$ such that

$$\|A\|\leq C(\beta) \quad \& \quad A^{-1}\Psi_{n,j}\left(x\right)=\sum_{\substack{k=0\\k\neq n}}^{\infty}\left[\left(\Psi_{n,j},\sin 2\pi kx\right)e_{-k}+\left(\Psi_{n,j},\varphi_k^*\right)e_k\right]. \tag{11}$$

Therefore by (11) and Parseval's equality we have

$$\|\Psi_{n,j}\|^2 \leq \left\|AA^{-1}\Psi_{n,j}\right\|^2 \leq C^2\left\|A^{-1}\Psi_{n,j}\right\|^2 = C^2\sum_{\substack{k=0\\k\neq n}}^{\infty}\left[\left|\left(\Psi_{n,j},\sin 2\pi kx\right)\right|^2+\left|\left(\Psi_{n,j},\varphi_k^*\right)\right|^2\right]. \tag{12}$$

Now using (20) in [8], (7) and Bessel inequality we obtain that

$$\sum_{\substack{k=0\\k\neq n}}^{\infty}\left|\left(\Psi_{n,j},\sin 2\pi kx\right)\right|^2 \leq \frac{1}{\left(\delta(n)\right)^2}\sum_{\substack{k=0\\k\neq n}}^{\infty}\left|\frac{1}{\sqrt{2}}\left(q\Psi_{n,j},\sqrt{2}\sin 2\pi kx\right)\right|^2 \leq \frac{\frac{1}{2}M^2}{(\delta(n))^2}. \tag{13}$$

By (21) in [8], we have

$$\sum_{\substack{k=0\\k\neq n}}^{\infty}\left|\left(\Psi_{n,j},\varphi_k^*\right)\right|^2 \leq \sum_{\substack{k=0\\k\neq n}}^{\infty}2\left|\frac{\gamma_1 k\left(q\Psi_{n,j},\sin 2\pi kx\right)}{\left(\lambda_{n,j}-\left(2\pi k\right)^2\right)^2}\right|^2+\sum_{\substack{k=0\\k\neq n}}^{\infty}2\left|\frac{\left(q\Psi_{n,j},\varphi_k^*\right)}{\lambda_{n,j}-\left(2\pi k\right)^2}\right|^2, \tag{14}$$

where γ_1 is defined in (19) in [8]. Using (8) and taking into account that $c \geq 2M$, we obtain $\frac{12}{11}\pi^2(2n-1) > M$ and $\left|\lambda_{n,j} - (2\pi k)^2\right| > \frac{32}{11}\pi^2 k$. Therefore, using the definition of γ_1, (15) in [8] and (10) and arguing as in the proof of (13) we obtain that the first and the second term of the right side of (14) are not greater than

$$\frac{121\left|\beta+1\right|^2 M^2}{4\left|\beta-1\right|^2\pi^2(\delta(n))^2} \quad \& \quad \frac{\frac{1}{2}(A(\beta)M)^2}{(\delta(n))^2},$$

respectively.Thus using (12) we obtain

$$\left\|\Psi_{n,j}\right\| \leq \frac{1}{\delta(n)} CM\left(\frac{1}{2} + \frac{121\left|\beta+1\right|^2}{4\left|\beta-1\right|^2\pi^2} + (A(\beta))^2\right)^{\frac{1}{2}},$$

which contradicts to $\left\|\Psi_{n,j}\right\| = 1$ and completes the proof of the lemma. □

Now we are ready to prove the following theorem.

Theorem 2.1. *If (8) and (9) hold then $\lambda_{n,j}$ is an eigenvalue of T_1 if and only if it is a root of the equation (3). Moreover $\lambda \in U(n) := \left[(2\pi n)^2 - M, (2\pi n)^2 + M\right]$ is a double eigenvalue of T_1 if and only if it is a double root of (3).*

Proof. Using (8), (9) and arguing as in the proof of Theorem 1 (b) in [8], we obtain the proof. □

By Theorem 2.1, the eigenvalue $\lambda_{n,j}$ is either the root of (4) or the root of (5). To use the numerical methods, we take finite summations instead of the infinite series in the expressions (4) and (5), and get

$$\lambda = (2\pi n)^2 + \frac{1}{2}\left(Q_n + P_n^*\right) + f_j(\lambda), \tag{15}$$

for $j = 1$ and for $j = 2$, where

$$f_j(\lambda) = \frac{1}{2}\left(\widetilde{A}_{k,s}(\lambda) + A'_{k,s}(\lambda)\right) + (-1)^j\frac{1}{2}\sqrt{\Delta_{k,s}(\lambda)}, \tag{16}$$

and the functions $\widetilde{A}_{k,s}(\lambda)$, $A'_{k,s}(\lambda)$ and $\Delta_{k,s}(\lambda)$ are defined and investigated in subsection 3.1 of the Appendix (see (A.2), (A.3), and (A.4)). By (A.1) in the Appendix, (15) becomes

$$\lambda = (2\pi n)^2 + f_j(\lambda). \tag{17}$$

Now we prove that the eigenvalues of T_1 are close to the roots of (17).

Theorem 2.2. *Let (8) and (9) hold. Then for all x and y from $\left[(2\pi n)^2 - M, (2\pi n)^2 + M\right]$ the relations*

$$\left|f_j(x) - f_j(y)\right| < K_n\left|x-y\right|, \; K_n = \frac{c^2}{4(\delta(n))(\delta(n)-c)} < \frac{9}{16}, \tag{18}$$

hold for $j = 1, 2$, and for each j, (17) has a unique solution $r_{n,j}$ on $\left[(2\pi n)^2 - M, (2\pi n)^2 + M\right]$.

Moreover

$$\left|\lambda_{n,j} - r_{n,j}\right| \leq \frac{2c^{k+2}}{2^k(\delta(n))^k(2\delta(n)-c)(1-K_n)}, \; \forall j = 1, 2; \; s \geq k. \tag{19}$$

Proof. First let us prove (18) by using the mean-value theorem. For this we estimate $\left|f_j'(\lambda)\right|$. By (16) we have

$$\left|f_j'(\lambda)\right| = \left|\frac{1}{2}\left(\frac{d}{d\lambda}\widetilde{A}_{k,s}(\lambda) + \frac{d}{d\lambda}A_{k,s}'(\lambda)\right) + (-1)^j \frac{1}{4}\frac{\frac{d}{d\lambda}\Delta_{k,s}(\lambda)}{\sqrt{\Delta_{k,s}(\lambda)}}\right|$$
$$\le \frac{1}{2}\left(\left|\frac{d}{d\lambda}\widetilde{A}_{k,s}(\lambda)\right| + \left|\frac{d}{d\lambda}A_{k,s}'(\lambda)\right| + \frac{1}{2}\left|\frac{\frac{d}{d\lambda}\Delta_{k,s}(\lambda)}{\sqrt{\Delta_{k,s}(\lambda)}}\right|\right). \tag{20}$$

By the estimations (A.10), (A.11) and (A.12) in subsection 3.3 of the Appendix we prove that

$$\left|\frac{d}{d\lambda}\widetilde{A}_{k,s}(\lambda)\right| \le \frac{K_n}{2}, \quad \left|\frac{d}{d\lambda}A_{k,s}'(\lambda)\right| \le \frac{K_n}{2}, \quad \frac{\left|\frac{d}{d\lambda}\Delta_{k,s}(\lambda)\right|}{\left|\sqrt{\Delta_{k,s}(\lambda)}\right|} \le 2K_n, \tag{21}$$

respectively. Hence by (20) and (21) we obtain $\left|f_j'(\lambda)\right| \le K_n$ and since K_n can be written as

$$K_n = \frac{c^2}{4(\delta(n))^2}\sum_{j=0}^{\infty}\left(\frac{c}{\delta(n)}\right)^j,$$

we get by (8) and the geometric series formula that $K_n < \dfrac{9}{16}$.

Since the inequality

$$\left|f_j'(\lambda)\right| \le K_n < 1 \tag{22}$$

holds for all x and y from $\left[(2\pi n)^2 - M, (2\pi n)^2 + M\right]$, by the mean value theorem (18) holds, and the equation (17) has a unique solution $r_{n,j}$ on $\left[(2\pi n)^2 - M, (2\pi n)^2 + M\right]$ for each j $(j = 1, 2)$, by the contraction mapping theorem.

Now let us prove (19). Let

$$H_j(x) = x - (2\pi n)^2 - f_j(x). \tag{23}$$

Using the definition of $\{r_{n,j}\}$, we obtain $H_j(r_{n,j}) = 0$, for $j = 1, 2$. Therefore by (4) and (5) we have for $\lambda = \lambda_{n,j}$

$$\left|H_j(\lambda) - H_j(r_{n,j})\right| = \left|H_j(\lambda)\right|$$
$$= \left|\frac{1}{2}\left(A(\lambda) + A'(\lambda)\right) + (-1)^j\frac{1}{2}\sqrt{\Delta(\lambda)} - \frac{1}{2}\left(\widetilde{A}_{k,s}(\lambda) + A_{k,s}'(\lambda)\right) + (-1)^j\frac{1}{2}\sqrt{\Delta_{k,s}(\lambda)}\right|$$
$$\le \frac{1}{2}\left(\left|A'(\lambda) - A_{k,s}'(\lambda)\right| + \left|A(\lambda) - \widetilde{A}_{k,s}(\lambda)\right| + \left|\sqrt{\Delta(\lambda)} - \sqrt{\Delta_{k,s}(\lambda)}\right|\right). \tag{24}$$

First let us estimate the first term of the right-hand side of (24).From the formula (A.3) in subsection 3.1 of the Appendix and by arguing as in the proof of Lemma 2.1 and using the geometric series formula we get

$$\left|\left(A'(\lambda_{n,j}) - A_{k,s}'(\lambda_{n,j})\right)\right| \le \frac{c^{k+2}}{2^k(\delta(n))^k(2\delta(n) - c)}, \tag{25}$$

for $s \ge k$ (see (A.9) in subsection 3.2 of the Appendix).

Similarly, from the formula (A.2) in subsection 3.1 of the Appendix, for the second term of the right-hand side of (24), we get

$$\left|A(\lambda_{n,j}) - \widetilde{A}_{k,s}(\lambda_{n,j})\right| \le \frac{c^{k+2}}{2^k(\delta(n))^k(2\delta(n) - c)} \tag{26}$$

for $s \geq k$ (see (A.8) in subsection 3.2 of the Appendix). Using (A.4) and (A.5) in subsection 3.1 of the Appendix, for the third term of the right-hand side of (24) we get

$$\left|\sqrt{\Delta(\lambda_{n,j})}-\sqrt{\Delta_{k,s}(\lambda_{n,j})}\right| = \left|\left(A(\lambda_{n,j})-A'(\lambda_{n,j})-q_{2n}\right)-\left(\widetilde{A}_{k,s}(\lambda_{n,j})-A'_{k,s}(\lambda_{n,j})-q_{2n}\right)\right|$$

$$\leq \left|A(\lambda_{n,j})-\widetilde{A}_{k,s}(\lambda_{n,j})\right|+\left|A'(\lambda_{n,j})-A'_{k,s}(\lambda_{n,j})\right| \leq \frac{2c^{k+2}}{2^k(\delta(n))^k(2\delta(n)-c)} \tag{27}$$

by (25) and (26). Hence by (24)-(27) we obtain

$$|H_j(\lambda_{n,j})-H_j(r_{n,j})| \leq \frac{2c^{k+2}}{2^k(\delta(n))^k(2\delta(n)-c)}, \ \forall j=1,2. \tag{28}$$

To apply the mean value theorem we estimate $\left|H'_j(\lambda)\right|$:

$$\left|H'_j(\lambda)\right| = \left|1-f'_j(\lambda)\right| \geq \left|1-\left|f'_j(\lambda)\right|\right| \geq 1-K_n. \tag{29}$$

By the mean value formula, (28) and (29) we get

$$|H_j(\lambda_{n,j})-H_j(r_{n,j})| = \left|H'_j(\xi)\right||\lambda_{n,j}-r_{n,j}|, \quad \xi\in\left[(2\pi n)^2-M,(2\pi n)^2+M\right],$$

$$|\lambda_{n,j}-r_{n,j}| = \frac{|H_j(\lambda_{n,j})-H_j(r_{n,j})|}{\left|H'_j(\xi)\right|} \leq \frac{2c^{k+2}}{2^k(\delta(n))^k(2\delta(n)-c)(1-K_n)}, \ \forall j=1,2.$$

□

Now let us approximate $r_{n,j}$ by the fixed point iterations:

$$x_{n,i+1}=(2\pi n)^2+f_1(x_{n,i}) \quad \& \quad y_{n,i+1}=(2\pi n)^2+f_2(y_{n,i}), \tag{30}$$

where $f_j(x)$ $(j=1,2)$ is defined in (16).

First, using (A.4), (A.6) and (A.7) in the Appendix, we get

$$|f_j(\lambda_{n,j})| \leq \frac{1}{2}\left|\widetilde{A}_{k,s}(\lambda_{n,j})+A'_{k,s}(\lambda_{n,j})\right|+\left|(-1)^j\frac{1}{2}\sqrt{\Delta_{k,s}(\lambda_{n,j})}\right|$$

$$=\frac{1}{2}\left(\left|\widetilde{A}_{k,s}(\lambda_{n,j})\right|+\left|A'_{k,s}(\lambda_{n,j})\right|+\sqrt{\Delta_{k,s}(\lambda_{n,j})}\right) \leq \frac{|q_{2n}|}{2}+\frac{c^2}{2\delta(n)-c}.$$

Similarly,

$$\left|f_j\left((2\pi n)^2\right)\right| \leq \frac{1}{2}\left|\left(\widetilde{A}_{k,s}\left((2\pi n)^2\right)+A'_{k,s}\left((2\pi n)^2\right)\right)\right|+\frac{1}{2}\sqrt{\Delta_{k,s}\left((2\pi n)^2\right)} \leq$$

$$\frac{1}{2}\left(\frac{c^2}{8\pi^2(2n-1)}\sum_{j=0}^{k}\left(\frac{c}{8\pi^2(2n-1)}\right)^j+|q_{2n}|+\frac{c^2}{8\pi^2(2n-1)-c}\right) \leq \frac{|q_{2n}|}{2}+\frac{c^2}{8\pi^2(2n-1)-c}. \tag{31}$$

Theorem 2.3. *If (8) and (9) hold then for the sequence $\{x_{n,i}\}$ and $\{y_{n,i}\}$ defined by (30), the following estimations hold:*

$$|x_{n,i}-r_{n,1}| \leq K_n^i\left(\frac{|q_{2n}|}{2(1-K_n)}+\frac{c^2}{(1-K_n)(8\pi^2(2n-1)-c)}\right), \tag{32}$$

$$|y_{n,i}-r_{n,2}| \leq K_n^i\left(\frac{|q_{2n}|}{2(1-K_n)}+\frac{c^2}{(1-K_n)(8\pi^2(2n-1)-c)}\right), \tag{33}$$

for $i=1,2,3,\ldots$, where K_n is defined in (18).

Proof. Without loss of generality we can take $x_{n,0} = (2\pi n)^2$. By (23) and (30) we have

$$|x_{n,i} - r_{n,1}| = \left|(2\pi n)^2 + f_1(x_{n,i-1}) - \left((2\pi n)^2 + f_1(r_{n,1})\right)\right|$$
$$= |f_1(x_{n,i-1}) - f_1(r_{n,1})| < K_n |x_{n,i-1} - r_{n,1}| < K_n^i |x_{n,0} - r_{n,1}|.$$

Therefore it is enough to estimate $|x_{n,0} - r_{n,1}|$. By definitions of $r_{n,j}$ and $x_{n,0}$ we obtain

$$r_{n,1} - x_{n,0} = f_1(r_{n,1}) + (2\pi n)^2 - x_{n,0} = f_1(r_{n,1}) - f_1(x_{n,0}) + f_1\left((2\pi n)^2\right)$$

and by the mean value theorem there exists $x \in \left[(2\pi n)^2 - M, (2\pi n)^2 + M\right]$ such that $f_1(r_{n,1}) - f_1(x_{n,0}) = f_1'(x)(r_{n,1} - x_{n,0})$. These two equalities imply that $(r_{n,j} - x_{n,0})(1 - f_1'(x)) = f_1\left((2\pi n)^2\right)$. Hence by (22) and (31) we get

$$(r_{n,1} - x_{n,0}) \le \frac{f_1\left((2\pi n)^2\right)}{1 - K_n} \le \left(\frac{|q_{2n}|}{2(1 - K_n)} + \frac{c^2}{(1 - K_n)(8\pi^2(2n-1) - c)}\right)$$

and

$$|x_{n,i} - r_{n,1}| \le K_n^i \left(\frac{|q_{2n}|}{2(1 - K_n)} + \frac{c^2}{(1 - K_n)(8\pi^2(2n-1) - c)}\right).$$

One can easily show in a similar way to (32) that for the iteration (30),

$$|y_{n,i} - r_{n,2}| \le K_n^i \left(\frac{|q_{2n}|}{2(1 - K_n)} + \frac{c^2}{(1 - K_n)(8\pi^2(2n-1) - c)}\right).$$

□

Thus by (19), (32) and (33) we have the approximations $x_{n,i}$ and $y_{n,i}$ for $\lambda_{n,1}$ and $\lambda_{n,2}$, respectively, with the errors

$$|\lambda_{n,1} - x_{n,i}| < \frac{c^{k+2}}{2^k (\delta(n))^k (2\delta(n) - c)(1 - K_n)} + K_n^i \left(\frac{|q_{2n}|}{2(1 - K_n)} + \frac{c^2}{(1 - K_n)(8\pi^2(2n-1) - c)}\right),$$

$$|\lambda_{n,2} - y_{n,i}| < \frac{c^{k+2}}{2^k (\delta(n))^k (2\delta(n) - c)(1 - K_n)} + K_n^i \left(\frac{|q_{2n}|}{2(1 - K_n)} + \frac{c^2}{(1 - K_n)(8\pi^2(2n-1) - c)}\right).$$

Remark 2.1. *If $q(x) = \sum_{k=1}^{p} q_k \cos(2k\pi x)$, where p is a finite positive integer, then it follows from the formulas (30) that for $n \ge s + p + 1$*

$$x_{n,i} = y_{n,i} = (2\pi n)^2,$$

since the multiplicands $(q_{n-n_1} - q_{n+n_1})$ and $q(n, n_1)$ in $f_1(x_{n,i})$ and $f_2(y_{n,i})$ are zero.

3. Appendix: Calculations and Estimations

3.1. **Calculations of the terms of (3) for the case (6).**

When $q(x)$ has the form (6), one can easily verify that

$$Q_n - P_n^* = -2c_{2n} = -q_{2n}, \quad Q_n + P_n^* = 0. \tag{A.1}$$

Therefore the functions used in (3) have the form:

$$a_1(\lambda) = \frac{q(n, n_1)}{2\left(\lambda - (2\pi n_1)^2\right)}, \; b_1(\lambda) = 0, \; a_{k+1}(\lambda) = \frac{q(n_{k+1}, n_k)}{2\left(\lambda - (2\pi n_{k+1})^2\right)}, \quad b_{k+1}(\lambda) = 0,$$

$$A_{k+1}(\lambda)=\frac{\gamma_1 n_{k+1} q\left(n_{k+1},n_k\right)}{2\left(\lambda-\left(2\pi n_{k+1}\right)^2\right)^2},\quad B_{k+1}(\lambda)=\frac{q\left(n_{k+1},n_k\right)}{2\left(\lambda-\left(2\pi n_{k+1}\right)^2\right)};\ k=1,2,\ldots,$$

$$C_1(\lambda)=\frac{q_{n-n_1}-q_{n+n_1}}{2\left(\lambda-\left(2\pi n_1\right)^2\right)},\quad M_1(\lambda)=0,\ C_{k+1}(\lambda)=a_1a_2\ldots a_k a_{k+1},\quad M_{k+1}(\lambda)=0$$

$$\widetilde{C}_1(\lambda)=\frac{\gamma_1 n_1 q\left(n,n_1\right)}{2\left(\lambda-\left(2\pi n_1\right)^2\right)^2},\quad \widetilde{M}_1(\lambda)=\frac{q\left(n,n_1\right)}{2\left(\lambda-\left(2\pi n_1\right)^2\right)},$$

for $k=1,2,\ldots,$ and

$$\widetilde{M}_{k+1}(\lambda)=\widetilde{M}_k B_{k+1}=B_1B_2\ldots B_{k+1}$$
$$=\frac{q\left(n,n_1\right)q\left(n_2,n_1\right)\ldots q\left(n_{k+1},n_k\right)}{2^{k+1}\left(\lambda-\left(2\pi n_1\right)^2\right)\left(\lambda-\left(2\pi n_2\right)^2\right)\ldots\left(\lambda-\left(2\pi n_{k+1}\right)^2\right)};\ k=1,2,\ldots.$$

Using these functions, we obtain the followings:

$$\sum_{n_1,\ldots,n_m=1}^{s}\left[C_m\left(q\varphi_n,\sin 2\pi n_m x\right)+M_m\left(q\varphi_n,\varphi_{n_m}^*\right)\right]=\sum_{n_1,\ldots,n_m=1}^{s}a_1a_2\ldots a_m\left(q\varphi_n,\sin 2\pi n_m x\right)$$
$$=\sum_{n_1,\ldots,n_m=1}^{s}\left\{\frac{q\left(n,n_1\right)q\left(n_2,n_1\right)\ldots q\left(n_m,n_{m-1}\right)q\left(n,n_m\right)}{2^{m+1}\left(\lambda-\left(2\pi n_1\right)^2\right)\left(\lambda-\left(2\pi n_2\right)^2\right)\ldots\left(\lambda-\left(2\pi n_m\right)^2\right)}\right\}:=\alpha_{m,s}(\lambda),$$

$$\beta_{m,s}(\lambda):=\sum_{n_1,\ldots,n_m=1}^{s}\left[C_m\left(q\cos 2\pi n x,\sin 2\pi n_m x\right)+M_m\left(q\cos 2\pi n x,\varphi_{n_m}^*\right)\right]=0,$$

$$\widetilde{A}_{k,s}(\lambda):=\sum_{m=1}^{k}\alpha_{m,s}(\lambda)$$
$$=\sum_{n_1=1}^{s}\frac{\left(q\left(n,n_1\right)\right)^2}{2^2\left(\lambda-\left(2\pi n_1\right)^2\right)}+\sum_{n_1,n_2=1}^{s}\frac{q\left(n,n_1\right)q\left(n_2,n_1\right)q\left(n,n_2\right)}{2^3\left(\lambda-\left(2\pi n_1\right)^2\right)\left(\lambda-\left(2\pi n_2\right)^2\right)}+ \tag{A.2}$$
$$+\ldots+\sum_{n_1,\ldots,n_k=1}^{s}\frac{q\left(n,n_1\right)q\left(n_2,n_1\right)\ldots q\left(n_k,n_{k-1}\right)q\left(n,n_k\right)}{2^{k+1}\left(\lambda-\left(2\pi n_1\right)^2\right)\left(\lambda-\left(2\pi n_2\right)^2\right)\ldots\left(\lambda-\left(2\pi n_k\right)^2\right)},$$

$$\widetilde{B}_{k,s}(\lambda):=\sum_{m=1}^{k}\beta_{m,s}(\lambda)=0,$$

$$\alpha'_{m,s}(\lambda):=\sum_{n_1,\ldots,n_m=1}^{s}\left[\widetilde{C}_m\left(q\cos 2\pi n x,\sin 2\pi n_m x\right)+\widetilde{M}_m\left(q\cos 2\pi n x,\varphi_{n_m}^*\right)\right]$$
$$=\sum_{n_1,\ldots,n_m=1}^{s}\frac{q\left(n,n_1\right)q\left(n_2,n_1\right)\ldots q\left(n_m,n_{m-1}\right)}{2^{m+1}\left(\lambda-\left(2\pi n_1\right)^2\right)\left(\lambda-\left(2\pi n_2\right)^2\right)\ldots\left(\lambda-\left(2\pi n_m\right)^2\right)}q\left(n,n_m\right),$$

$$A'_{k,s}(\lambda) := \sum_{m=1}^{k} \alpha'_{m,s}(\lambda)$$

$$= \sum_{n_1=1}^{s} \frac{q(n,n_1)^2}{2^2\left(\lambda-(2\pi n_1)^2\right)} + \sum_{n_1,n_2=1}^{s} \frac{q(n,n_1)\,q(n_2,n_1)\,q(n,n_2)}{2^3\left(\lambda-(2\pi n_1)^2\right)\left(\lambda-(2\pi n_2)^2\right)} + \tag{A.3}$$

$$+\ldots+\sum_{n_1,\ldots,n_k=1}^{s} \frac{q(n,n_1)\,q(n_2,n_1)\ldots q(n_k,n_{k-1})\,q(n,n_k)}{2^{k+1}\left(\lambda-(2\pi n_1)^2\right)\left(\lambda-(2\pi n_2)^2\right)\ldots\left(\lambda-(2\pi n_k)^2\right)},$$

and using also (A.1), we have

$$\Delta_{k,s}(\lambda) := \left(Q_n - P_n^* + \widetilde{A}_{k,s}(\lambda) - A'_{k,s}(\lambda)\right)^2 + 4\left(P_n + \widetilde{B}_{k,s}(\lambda)\right)\left(\gamma_1 n + Q_n^* + B'_{k,s}(\lambda)\right) \tag{A.4}$$

$$= \left(Q_n - P_n^* + \widetilde{A}_{k,s}(\lambda) - A'_{k,s}(\lambda)\right)^2 = \left(\widetilde{A}_{k,s}(\lambda) - A'_{k,s}(\lambda) - q_{2n}\right)^2,$$

and

$$\Delta(\lambda) = \left(A(\lambda) - A'(\lambda) - q_{2n}\right)^2. \tag{A.5}$$

3.2. **Estimations of $\widetilde{A}_{k,s}$, $A'_{k,s}$, $A-\widetilde{A}_{k,s}$ and $A'-A'_{k,s}$.** Using (6) and (7), in (A.2) and (A.3), we obtain

$$\left|\widetilde{A}_{k,s}(\lambda)\right| \le \frac{c^2}{2^2\delta(n)} + \frac{c^3}{2^3(\delta(n))^2} + \ldots + \frac{c^{k+1}}{2^{k+1}(\delta(n))^k} = \frac{c^2}{2^2\delta(n)} \sum_{j=0}^{k} \left(\frac{c}{2\delta(n)}\right)^j, \tag{A.6}$$

$$\left|A'_{k,s}(\lambda)\right| \le \frac{c^2}{2^2\delta(n)} + \frac{c^3}{2^3(\delta(n))^2} + \ldots + \frac{c^{k+1}}{2^{k+1}(\delta(n))^k} \le \frac{c^2}{2(2\delta(n)-c)}, \tag{A.7}$$

by the geometric series formula.

By (A.2) and (A.3), and using the definitions of $A(\lambda_{n,j})$, $\widetilde{A}_k(\lambda_{n,j})$, $A'(\lambda_{n,j})$ and $A'_k(\lambda_{n,j})$, for $s \ge k$ we obtain

$$\left|A(\lambda_{n,j}) - \widetilde{A}_{k,s}(\lambda_{n,j})\right| \le \left|A(\lambda_{n,j}) - \widetilde{A}_k(\lambda_{n,j})\right| +$$

$$+\left|\sum_{m=1}^{k} \sum_{n_1,\ldots,n_m=s+1}^{\infty} \frac{q(n,n_1)\,q(n_2,n_1)\ldots q(n_m,n_{m-1})\,q(n,n_m)}{2^{m+1}\left(\lambda-(2\pi n_1)^2\right)\left(\lambda-(2\pi n_2)^2\right)\ldots\left(\lambda-(2\pi n_m)^2\right)}\right|$$

$$\le \frac{c^{k+2}}{2^{k+1}(\delta(n))^{k+1}} + \frac{c^{k+3}}{2^{k+2}(\delta(n))^{k+2}} + \cdots = \frac{c^{k+2}}{2^k(\delta(n))^k(2\delta(n)-c)}, \tag{A.8}$$

and

$$\left|\left(A'(\lambda_{n,j}) - A'_{k,s}(\lambda_{n,j})\right)\right| \le \frac{c^{k+2}}{2^k(\delta(n))^k(2\delta(n)-c)}. \tag{A.9}$$

3.3. **Estimations of the derivatives.** We use the following estimations for the proof of Theorem 2.2:

$$\left|\frac{d}{d\lambda}C_1(\lambda)\right| = \left|\frac{d}{d\lambda}a_1(\lambda)\right| = \left|-\frac{q(n,n_1)}{2\left(\lambda-(2\pi n_1)^2\right)^2}\right|, \quad \left|\frac{d}{d\lambda}a_{k+1}(\lambda)\right| = \left|-\frac{q(n_{k+1},n_k)}{2\left(\lambda-(2\pi n_{k+1})^2\right)^2}\right|,$$

$$\left|\frac{d}{d\lambda}C_{k+1}\left(\lambda\right)\right| = \left|\left(\frac{d}{d\lambda}C_k\right)a_{k+1} + C_k\left(\frac{d}{d\lambda}a_{k+1}\right)\right|$$
$$\leq \frac{(k+1)\,q\left(n,n_1\right)q\left(n_2,n_1\right)\dots q\left(n_{k+1},n_k\right)}{2^{k+1}\left(\delta\left(n\right)\right)^{k+2}};\ k=1,2,\dots,$$

$$\left|\frac{d}{d\lambda}\alpha_{k,s}\left(\lambda\right)\right| = \left|\sum_{n_1,\dots,n_k=1}^{s}\left(\frac{d}{d\lambda}C_k\right)\left(q\varphi_n,\sin 2\pi n_k x\right)\right|$$
$$\leq \sum_{n_1,\dots,n_k=1}^{s}\frac{k\left|q\left(n,n_1\right)\right|\left|q\left(n_2,n_1\right)\right|\dots\left|q\left(n_k,n_{k-1}\right)\right|}{2^k\left(\delta\left(n\right)\right)^{k+1}}\frac{\left|q\left(n,n_k\right)\right|}{2},$$

$$\left|\frac{d}{d\lambda}\widetilde{A}_{k,s}\left(\lambda\right)\right| = \left|\sum_{m=1}^{k}\frac{d}{d\lambda}\alpha_{m,s}\left(\lambda\right)\right| \leq \sum_{m=1}^{k}\frac{mc^{m+1}}{2^{m+1}\left(\delta\left(n\right)\right)^{m+1}} \leq \frac{c^2}{2^3\delta\left(n\right)\left(\delta\left(n\right)-c\right)}, \quad \text{(A.10)}$$

$$\left|\frac{d}{d\lambda}\widetilde{M}_1\left(\lambda\right)\right| = \left|\frac{d}{d\lambda}B_1\left(\lambda\right)\right| = \left|-\frac{q\left(n,n_1\right)}{2\left(\lambda-\left(2\pi n_1\right)^2\right)^2}\right|,$$

$$\left|\frac{d}{d\lambda}B_{k+1}\left(\lambda\right)\right| = \left|-\frac{\left(q\cos 2\pi n_{k+1}x,\varphi^*_{n_k}\right)}{\left(\lambda-\left(2\pi n_{k+1}\right)^2\right)^2}\right| = \frac{q\left(n_{k+1},n_k\right)}{2\left(\lambda-\left(2\pi n_{k+1}\right)^2\right)^2};\ k=1,2,\dots,$$

$$\left|\frac{d}{d\lambda}\widetilde{M}_{k+1}\left(\lambda\right)\right| = \left|\frac{d}{d\lambda}\left(\widetilde{M}_k B_{k+1}\right)\right| = \left|\left(\frac{d}{d\lambda}\widetilde{M}_k\right)B_{k+1} + \widetilde{M}_k\left(\frac{d}{d\lambda}B_{k+1}\right)\right|$$
$$\leq \frac{(k+1)\,q\left(n,n_1\right)q\left(n_2,n_1\right)\dots q\left(n_{k+1},n_k\right)}{2^{k+1}\left(\delta\left(n\right)\right)^{k+2}};\ k=1,2,\dots,$$

$$\left|\frac{d}{d\lambda}\alpha'_{k,s}\left(\lambda\right)\right| = \left|\sum_{n_1,\dots,n_k=1}^{s}\left(\frac{d}{d\lambda}\widetilde{M}_k\right)\left(q\cos 2\pi n x,\varphi^*_{n_k}\right)\right|$$
$$\leq \sum_{n_1,\dots,n_k=1}^{s}\frac{kq\left(n,n_1\right)q\left(n_2,n_1\right)\dots q\left(n_k,n_{k-1}\right)}{2^k\left(\delta\left(n\right)\right)^{k+1}}\left(q\cos 2\pi n x,\varphi^*_{n_k}\right);\ k=1,2,\dots,$$

$$\left|\frac{d}{d\lambda}A'_{k,s}\left(\lambda\right)\right| = \left|\sum_{m=1}^{k}\frac{d}{d\lambda}\alpha'_{m,s}\left(\lambda\right)\right| \leq \sum_{m=1}^{k}\frac{mc^{k+1}}{2^{m+1}\left(\delta\left(n\right)\right)^{m+1}} \leq \frac{c^2}{2^3\delta\left(n\right)\left(\delta\left(n\right)-c\right)}, \quad \text{(A.11)}$$

and

$$\frac{\left|\frac{d}{d\lambda}\Delta_{k,s}\left(\lambda\right)\right|}{\left|\sqrt{\Delta_{k,s}\left(\lambda\right)}\right|} = \frac{2\left|Q_n-P^*_n+\widetilde{A}_{k,s}\left(\lambda_n\right)-A'_{k,s}\left(\lambda_n\right)\right|\left|\frac{d}{d\lambda}\widetilde{A}_{k,s}\left(\lambda\right)-\frac{d}{d\lambda}A'_{k,s}\left(\lambda\right)\right|}{\left|Q_n-P^*_n+\widetilde{A}_{k,s}\left(\lambda_n\right)-A'_{k,s}\left(\lambda_n\right)\right|}$$
$$\leq 2\left(\left|\frac{d}{d\lambda}\widetilde{A}_{k,s}\left(\lambda\right)\right| + \left|\frac{d}{d\lambda}A'_{k,s}\left(\lambda\right)\right|\right) \leq \frac{c^2}{2\delta\left(n\right)\left(\delta\left(n\right)-c\right)}. \quad \text{(A.12)}$$

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