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ON HUBTIC AND RESTRAINED HUBTIC OF A GRAPH

S. I. KHALAF[1], V. MATHAD[1], §

Abstract. In this article, the hubtic number of the join and corona of two connected graphs is computed. The restrained hubtic number $\xi_r(G)$ of a graph G is the maximum number such that we can partition $V(G)$ into pairwise disjoint restrained hub sets. We compute the restrained hubtic number of some standard graphs. Some bounds for $\xi_r(G)$ are obtained.

Keywords: Hub number, Hubtic number, Restrained hubtic number.

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1. Introduction

Suppose $G = (V, E)$ is a finite and an undirected graph without loops and multiple edges. We call $|V| = p$ the order of G and $|E| = q$ the size of G, where G is a (p, q) graph. In the graph G, $\delta(G)$ denotes the minimum degree among the vertices of G [4]. For definition and notation on graph theory not given here the reader is requested to see [4].

A hub set H of G is a set of vertices in G such that any two vertices $u, v \in V(G) \setminus H$ are connected by a path with all internal vertices from H. The smallest order of a hub set, is the hub number of G, which is denoted by $h(G)$. A connected hub set in G is a vertex hub set H_c such that the subgraph of G induced by H_c (denoted $G[H_c]$) is connected, the connected hub number of G, denoted $h_c(G)$, is the minimum size of a connected hub set in G [13]. For a graph G, a restrained hub set $H_r \subseteq V(G)$ of G, satisfies that for any two vertices $u, v \in V(G) \setminus H_r$, there is a path connected them with all internal vertices from H_r as well as another path with all internal vertices in $V(G) \setminus H_r$, the smallest order of H_r in G is called a restrained hub number of G and denoted by $h_r(G)$ [8]. In 2018 Shadi Khalaf, Veena Mathad and Sultan Mahde introduced a new parameter related to the hub number, which is the hubtic number $\xi(G)$ of a graph G and they defined it as, the maximum order of partition of the vertex set $V(G)$ into hub sets [6]. We mention some studies on the theory of hub [7, 9, 10, 11, 12].

[1] Department of Studies in Mathematics, University of Mysore, Manasagangotri, Mysuru, India.
e-mail: shadikhalaf1989@hotmail.com; om; ORCID: https://orcid.org/0000-0001-9077-5034.
e-mail: veena_mathad@rediffmail.com; om; ORCID: https://orcid.org/0000-0002-6621-9596.

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For graphs G and F having V_1, V_2 as disjoint vertex sets respectively, and E_1, E_2 their edge sets, respectively. Their join is denoted by $G + F$ and composed of $G \cup F$ and all edges joining V_1 with V_2 [4]. And their corona $G \circ F$, is the resultant graph of taking one script of G with order p and p scripts of F, and joining the j^{th} vertex of G to each vertex in the j^{th} script of F. The script of F whose vertices are attached to the vertex v, denote by F^v [3].

The dominating set D of a graph G is a set of vertices of G, satisfies the property for every vertex $v \notin D$, v must be adjacent to some vertex in D, the smallest order of a dominating set in G is called the domination number $\gamma(G)$ of G [5]. The maximum number of the pairwise disjoint dominating sets of G, which can be established by partition of $V(G)$ is called domatic number of G, and denoted by $d(G)$ [1].

In the last fewest years the hub theory started to get a high a attention of researchers. Moreover, a various applications of this theory started to appear. For instance, in computer network (by using the standard representation of vertices and edges), and a city's network of streets or architectures in a huge industrial area, and represent the architectures by a vertices in the corresponding graph and there will be an edge between two architectures if it is an easy transit from one to the other. If we need to fined a minimum number of servers or stations in these different types of networks, the minimum hub set of the corresponding graph will solve the problem.

In this article, motivated by this, along with high attention from researchers of the concept of domatic number, which is equivalent to the concept of hubtic number, we try to develop the hubtic number by computing the hubtic number of join and corona of two connected graphs. We introduced the restrained hubtic number of graphs. In the follow some results we used in the proof of our results.

Theorem 1.1. [13] *For a graph G, $h(G) = h_c(G) = 1$ if and only if G of the following structure:*

(1) *G is not complete graph.*
(2) *$V(G) = A \cup B \cup \{x\}$, such that $\{x\}$ is a hub set.*
(3) *x is adjacent to each vertex in A and not adjacent to any in B.*
(4) *For $v \in A$ and $u \in B$, u is adjacent to v.*
(5) *$G[B]$ is complete graph.*

Observation 1.1. [6] *For any complete graph K_p, $\xi(K_p) = p$.*

Theorem 1.2. [2] *For any connected graphs G and F such that $|V(G)| \geq 2$, $h(G \circ F) = |V(G)|$.*

Theorem 1.3. [8] *For a cycle C_p, $p \geq 3$. $h_r(C_p) = h_{rc}(C_p) = p - 3$.*

Proposition 1.1. [6] *Let G be a graph. Then, $\xi(G) \leq \delta(G) + 2$.*

Proposition 1.2. [8] *For any graph G, if v is an end vertex and H_r is a restrained hub set, then either $v \in H_r$ or $|H_r| = p - 2$.*

2. Hubtic number of join and corona of graphs

In this section, we study the hubtic number of join and corona of two connected graphs. For a vertex $v \in V(G)$, v is a hub vertex of G if $\{v\}$ is a hub set of a graph G, i.e. a vertex same to x in the Theorem 1.1.

Lemma 2.1. *Let v be a hub vertex of a graph G. Then $\xi(G) = \xi(G - v) + 1$.*

Proof. It is obvious that the hubtic partition of $G-v$ along with $\{v\}$ forms a hubtic partition of G. Therefore $\xi(G)\geq \xi(G-v)+1$. Now, suppose $\{H_1,H_2,...,H_s\}$ is a hubtic partition of G, such that $s=\xi(G)$. Without loss of generality, suppose $v\in H_1$. So $\{(H_1\setminus\{v\}),H_2,H_3,...,H_s\}$ is a hubtic partition of $G-v$. Hence $\xi(G-v)\geq s-1$. So $\xi(G)=s\leq \xi(G-v)+1$. This yields the desired conclusion. □

Theorem 2.1. *For any two connected graphs G and F with orders p,m, respectively,*

$$\xi(G\circ F)=\begin{cases} 1+m, & \text{if } p=1 \text{ and } F\cong K_m\ ;\\ 1+\xi(F), & \text{if } p=1 \text{ and } F\neq K_m;\\ 2, & \text{if } G\cong K_p \text{ and } p\geq 2.\end{cases}$$

Proof. **Case 1:** Suppose $p=1$, and F is a complete graph. Then clearly $G\circ F$ is a complete graph. Hence by Observation 1.1, the result is follow.
Case 2: If $p=1$ and F is not complete graph, let $V(G)=\{v\}$. Then by the definition of corona of two graphs, the vertex v is a hub vertex of a graph $G\circ F$, so by Lemma 2.1, $\xi(G\circ F)=\xi((G\circ F)-v)+1=\xi(F)+1$.
Case 3: Suppose G is a complete graph and $p\geq 2$. Let $H=V(G)$, then by Theorem 1.2, H is a minimum hub set of $(G\circ F)$. Let $S=V(G\circ F)\setminus V(G)$, since G is complete graph, then S is a hub set of $G\circ F$. Now assume that $Q\subset S$ is a hub set of $G\circ F$, then there is a vertex $u\in V(F^w)$ for some $w\in V(G)$ and a vertex $v\in V(G)$, such that $u,v\notin Q$ and they are not adjacent, it is obvious that w is in any path between them, but $w\notin Q$ which is a contradiction, so Q is not a hub set of $G\circ F$. Hence S is a minimal hub set of $G\circ F$. Therefore there is only two disjoint hub sets of $G\circ F$. □

Theorem 2.2. *For any two connected graphs $G_1=(p_1,q_1)$ and G_2 such that $G_1\neq K_p$ and $p_1\geq 2$, $\xi(G_1\circ G_2)=1$.*

Proof. Suppose that $G_1\neq K_p$. Consider $S=V(G_1)$, then by Theorem 1.2, S is a minimum hub set of $G_1\circ G_2$. Now, let's consider S_1 be also a hub set of $G_1\circ G_2$ such that $S_1\cap S=\phi$, the proof of the previous theorem implies that $S_1=V(G_1\circ G_2)\setminus V(G_1)$.

Since G_1 is not complete and $p_1\geq 2$, we can find at least two nonadjacent vertices in G_1 say x and y, by the definition of corona of two graphs, any path between x and y not includes any vertex from $V(G_2)$, then every path between x and y must includes vertices from S only, which is a contradiction. Hence $\xi(G_1\circ G_2)=1$. □

Theorem 2.3. *Let G and F be two connected graphs. Then*

$$\xi(G+F)=\xi(G)+\xi(F).$$

Proof. It suffices to show that any hub set of G or F is a hub set of $G+F$. Let H be a hub set of G and let $v,w\in V(G\circ F)\setminus H$. If $v,w\in V(G)$, then since H is a hub set of G there is an H-path between v and w in $G+F$, if $v\in V(G)$ and $w\in V(F)$, clearly by definition of join of two graphs, v is adjacent to w. It remains to consider the case that $v,w\in V(F)$, let $u\in H$, note that $u\in V(G)$, so u is adjacent to v and w, this implies that there is an H-path between v and w in $G+F$. Therefore, H is a hub set of $G+F$.

Finally, consider H' is a hub set of F, by using a similar argument as in the previous case we conclude that H' is a hub set of $G+F$. This establishes the proof. □

3. Restrained hubtic number

In this section we will try to develop the concept of hubtic number in graphs. In particular, we will introduce the concept of restrained hubtic number of a graph and we compute its value of some standard graphs.
We define the restrained hubtic number of a graph G as follows:

Definition 3.1. *The maximum number of the pairwise disjoint restrained hub sets of* G*, which can be established by partition of* $V(G)$*, is called restrained hubtic number of* G *and denoted by* $\xi_r(G)$*.*

Since for any complete graph K_p we have $h_r(K_p) = 0$, then the following observation is obvious.

Observation 3.1. *For any complete graph* K_p*,* $\xi_r(K_p) = p$*.*

We next determine the restrained hubtic number of a cycle graph C_p.

Proposition 3.1. *For the cycle* C_p*,* $p \geq 3$*,*

$$\xi_r(C_p) = \begin{cases} 3, & \text{if } p = 3 \text{ ;} \\ 4, & \text{if } p = 4 \text{ ;} \\ 2, & \text{if } p = 5, 6 \text{ ;} \\ 1, & \text{if } p \geq 7. \end{cases}$$

Proof. Since $C_3 \cong K_3$, then by the previous observation $\xi_r(C_3) = 3$. By Theorem 1.3, we have $h_r(C_4) = 1, h_r(C_5) = 2, h_r(C_6) = 3$ and by the definition of cycle the result for C_4, C_5, C_6 is follows. Let $p \geq 7$, then $h_r(C_p) > \frac{p}{2}$ and by the definition of restrained hubtic number we obtain the assertion. □

In the following proposition we determine the restrained hubtic number of wheel, star, double star and complete bipartite graphs. The proofs of these results are simple and are omitted.

Proposition 3.2. (1) *For the wheel graph* $W_{1,p-1}$*,* $p \geq 4$*,*

$$\xi_r(W_{1,p-1}) = \begin{cases} 4, & \text{if } p = 4 \text{ ;} \\ 5, & \text{if } p = 5 \text{ ;} \\ 3, & \text{if } p = 6, 7 \text{ ;} \\ 2, & \text{if } p \geq 8. \end{cases}$$

(2) *For the star* $K_{1,p-1}$*,* $p \geq 4$*,*

$$\xi_r(K_{1,p-1}) = 1.$$

(3) *For the double star* $S_{n,m}$*,*

$$\xi_r(S_{n,m}) = 1.$$

(4) *For the complete bipartite graph* $K_{n,m}$*,* $n, m \geq 3$*,*

$$\xi_r(K_{n,m}) = min\{n, m\}.$$

Theorem 3.1. *For any graph* G*,* $1 \leq \xi_r(G) \leq p$*. If* $h_r(G) > 0$*, then* $\xi_r(G) \leq \lfloor \frac{p}{h_r(G)} \rfloor$*.*

Proof. Let $H_r = \{H_{r_1}, H_{r_2}, ..., H_{r_m}\}$, be the restrained hubtic partition of $V(G)$ and $\xi_r(G) = m$. Clearly $|H_{r_i}| \geq h_r(G)$, $i = 1, 2, ..., m$ and we get $p = \sum_{i=1}^{r} |H_{r_i}| \geq mh_r(G)$, hence $\xi_r(G) \leq \lfloor \frac{p}{h_r(G)} \rfloor$. □

Since any restrained hub set of a graph G is a hub set of G, then it is obvious that $\xi_r(G) \leq \xi(G)$. By this inequality and by Proposition 1.1, we have the following result.

Proposition 3.3. *For any graph* G*,*

$$\xi_r(G) \leq \delta(G) + 2.$$

A graph G is called restrained hubtically full if $\xi_r(G) = \delta(G)+2$. For instance, the cycle C_4 and the wheel graph $W_{1,4}$ are both restrained hubtically full graphs.

Theorem 3.2. *If a (p, q) graph G contains an end vertex, with $p \geq 3$. Then*
$$\xi_r(G) \leq 2.$$

Proof. Suppose that v is an end vertex in a graph G and H_r is a restrained hub set of G. Then we discuss the following cases.
Case 1: $v \notin H_r$. Then by Proposition 1.2, $|H_r| = p - 2$, and we have $v, w \in V(G) \setminus H_r$ where w is adjacent to v. Note that, w is in any path between v and any other vertex of G, hence $\{w\}$ is not a restrained hub set of G. On the other hand, the set $\{v, w\}$ may be a restrained hub set of G, so $\xi_r(G) \leq 2$.
Case 2: $v \in H_r$. Let H_{r_1} be a restrained hub partite set, then $H_r \cap H_{r_1} = \phi$. This, implies that $v \notin H_{r_1}$, by following the same previous procedure the result is easy to verify. □

Proposition 3.4. *Let v be a hub vertex of a graph G. If B the set of Theorem 1.1 non empty, then*
$$\xi_r(G) \geq 2.$$

Proof. Since B non empty set, then the structure in Theorem 1.1, implies that $G - v$ is connected. Hence $\{v\}$ is a restrained hub set of G. Therefore, $V(G) \setminus \{v\}$ is a restrained hub set of G. So the result follows. □

4. CONCLUSION

In this research work, the concepts of hubtic and restrained hubtic numbers of a graph are studied. Also some bounds are obtained, and a relations between these parameters and other graph parameters are established.

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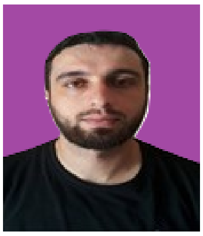

Shadi Ibrahim Khalaf was born in Palestine. He got his B. Sc. degree in mathematics in 2010, and M. Sc. degree in 2013 from An-Najah National University, Nablus, Palestine. He is right now a Ph.D. student at University of Mysore, India. He has published 5 papers in the field of graph theory.

Veena Mathad was born in India. She completed her M. Sc. (1995), and M. Phil. (1996) degrees in mathematics and was awarded her Ph.D (2005) in mathematics from Karnatak University, Dharward, India. Her research interests are graph transformations, domination in graphs, distance parameters in graphs, and stability parameters of graphs.