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GEOMETRIC FUNCTION OPTIMIZATION SUBJECT TO MIXED FUZZY RELATION INEQUALITY CONSTRAINTS

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Abstract. In this paper, the mixed fuzzy relation geometric programming problem is considered. The Mixed Fuzzy Relation Inequality (MFRI) system is an importance extension of FRI. It is shown that its feasible domain is non-convex and completely determined by its maximum solution and all its minimal solutions. A combination of the components of maximum solution and one of the minimal solutions solves the optimization problem. Some simplification procedures are proposed to solve the problem. An algorithm is finally designed to solve the problem.

Keywords: Geometric programming, Mixed fuzzy relation inequality, Max-product composition, Max-hamacher composition, Non-convex optimization.

AMS Subject Classification: 90C26,90C70.

1. Introduction

One of the interesting and on-going research topics is the optimization of objective functions on the region defined by FRE or FRI system, for example, see Refs. [1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 14, 15, 16, 17, 18, 19]. Fang and Li [2] firstly studied the minimization problem of a linear objective function provided to a max-min FRE system. Fang and Li's method was improved by Wu et al. [13]. The Wu et al.'s method considers much fewer nodes with respect to Fang and Li's method. Some simplification rules were given to reduce the rate of computations to find the optimal solution [14]. A review of the done works from Fang and Li's model with different composition operators can be seen in Li and Fang [7]. Zhang et al. [18] developed the constraints of the model as FRI system. Then, some researchers improved their method to solve the problem with FRI constraints [4, 10]. However, all problems and phenomena of real-world cannot be formulated by linear objective functions. On the other hand, the development of nonlinear objective function optimization provided to FREs or FRIs is very slowly. Lu and Fang [9] firstly investigated to this topic with the max-min FRE constraints using Genetic Algorithm (GA). Then, some researchers followed their works by improving the GA [5, 6]. Recently, to overcome this topic, some researchers focused on the problem with objective functions in the different forms such as latticized linear [12, 17], linear fractional [15], quadratic

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[1], geometric [11, 16, 19]. The nonlinear optimization problem with a non-differential objective function provided to a system of Mixed FREs (MFREs) with the max-min and max-product composition operator have been considered by Li et al. [8]. They studied some properties of the problem and presented an algorithm for its resolution. Later, Feng et al. [3] investigated to the problem with the max-min and max-average composition operator in a similar method with Li et al's method. Then, an algorithm was designed to solve the problem by them. The MFRE programming with a non-differential nonlinear objective function has only been studied by Li et al. [8] and Feng et al. [3] in literature. The nonlinear programming problem with MFRI constraints have not been considered up to now. Motivated from the importance of geometric programming problem and the MFRI in theory and application, we are interested to consider an extended version of the fuzzy relation geometric programming problem. This problem is as a geometric programming problem provided to the mixed fuzzy relation inequality constraints with two operators of the max-product and max-hamacher composition. In some problems such as covering and investing problem, we need to variables which should satisfy FRI system with two different operators and FRI and FRE with an operator cannot handle such situations. We are motivated to consider the optimization problem on the defined region by the MFRI system. In this paper, we introduce the MFRI system with the max-product and max-hamacher composition operators. Then, a closed form is presented to compute its maximum solution. Moreover, its (quasi-)minimal solutions are obtained using the concept of mixed fuzzy relation inequality path. It is shown that its solution set is completely obtained by the maximum solution and the (quasi-)minimal solutions. Also, a necessary and sufficient condition is proposed for its solution existence. Then, the geometric programming problem subject to MFRI is decomposed to two sub-problems. They are solved by the maximum solution and one of the (quasi-)minimal solutions. The optimal solution of the original problem is computed based on the optimal solutions of the two sub-problems. Some simplification procedures are given to determine some components of optimal solution of the original problem. Due to NP-hardness of the problem, each simplification in this area can be very important. With regard to the above points, an algorithm is designed to solve the geometric programming problem provided to the MFRI system. The structure of this paper is as follows. The problem of geometric programming problem provided to the MFRI system is formulated in Section 2. In Section 3, the feasible solution set of its feasible domain is completely determined and some of its properties are studied. A necessary and sufficient condition is also presented for feasibility of the problem. The resolution process of the problem is expressed in Section 4. In Section 5, some simplification procedures are given to accelerate the resolution process. Then in Section 6, an algorithm is designed with regard to the mentioned points in Sections 3,4, and 5. Moreover, a numerical example is given to illustrate the algorithm. Finally, the conclusions are presented in Section 7.

2. Formulation of geometric programming problem subject to MFRI constraints

The geometric programming problem subject to the MFRI is formulated as follows:

$P:\ Min\{\ Z(x) = \beta \cdot \Pi_{j=1}^{n} x_j^{\alpha_j} | A \bullet x' \geq d^1,\ Box'' \geq d^2, C \bullet y' \leq f^1, Eoy'' \leq f^2, x \in [0,1]^n, \}$

where $\beta, \alpha_j \in R, \beta > 0, J = \{1, ...n\}, I = \{1, ..., m\}, I_1^{AB}, I_2^{AB}, K^{AB}, L^{AB}, I_1^{CE}, I_2^{CE}, K^{CE}$, and L^{CE} are index sets and $I_1^{AB} \bigcap I_2^{AB} = \emptyset, I_1^{CE} \bigcap I_2^{CE} = \emptyset, K^{AB}, L^{AB}, K^{CE}, L^{CE} \subseteq J, K^{AB} \bigcup L^{AB} = J$, and $K^{CE} \bigcup L^{CE} = J$. Moreover, assume that the following vectors and matrices are given $d^1 = [d_i^1]_{i \in I_1^{AB}}$, $d^2 = [d_i^2]_{i \in I_2^{AB}}$, $f^1 = [f_i^1]_{i \in I_1^{CE}}$, $f^2 = [f_i^2]_{i \in I_2^{CE}}$, $A = [a_{ij}]_{i \in I_1^{AB}, j \in K^{AB}}$, $B = [b_{ij}]_{i \in I_2^{AB}, j \in L^{AB}}$, $C = [c_{ij}]_{i \in I_1^{CE}, j \in K^{CE}}$, and $E = [e_{ij}]_{i \in I_2^{CE}, j \in L^{CE}}$.

The components of the vectors and matrices belong to $[0,1]$. The operators of "$\bullet$" and "o" denote the max-product and the max-hamacher product composition, respectively. We will now find the vector $x = [x_j]_{j\in J}$ such that it satisfies the constraints of problem (P) and minimizes its objective function. In the problem (P), we have $x' = [x_j]_{j\in K^{AB}}$, $x'' = [x_j]_{j\in L^{AB}}$, $y' = [x_j]_{j\in K^{CE}}$, and $y'' = [x_j]_{j\in L^{CE}}$, respectively. problem (P) can equivalently be written as follows:

$$Min\ Z(x) = \beta \cdot \Pi_{j=1}^{n} x_j^{\alpha_j}, \tag{1}$$

$$s.t.\ \max_{j\in K^{AB}}\{a_{ij}\cdot x_j\} \geq d_i^1, \forall i \in I_1^{AB}, \tag{2}$$

$$\max_{j\in L^{AB}}\{\frac{b_{ij}\cdot x_j}{b_{ij}+x_j-b_{ij}\cdot x_j}\} \geq d_i^2, \forall i \in I_2^{AB}, \tag{3}$$

$$\max_{j\in K^{CE}}\{c_{ij}\cdot x_j\} \leq f_i^1, \forall i \in I_1^{CE}, \tag{4}$$

$$\max_{j\in L^{CE}}\{\frac{e_{ij}\cdot x_j}{e_{ij}+x_j-e_{ij}\cdot x_j}\} \leq f_i^2, \forall i \in I_2^{CE}, \tag{5}$$

$$0 \leq x_j \leq 1,\ \forall j \in J. \tag{6}$$

First of all, we investigate to the structure of its feasible in the next section.

3. The structure of feasible domain of problem (1)-(6)

In this section, the structure of feasible domain of problem (1)-(6) is discussed and its solution set is completely determined. First of all, we introduce some notations as follows. Assume that a_i, b_i, c_i, and e_i are the i^th row of the matrices A, B, C, and E, respectively.

$$S(A,d^1)_i = \{x=(x',x'')\in[0,1]^{|K^{AB}|}\times[0,1]^{|L^{AB}|} | a_i \bullet x' \geq d_i^1\}, for\ each\ i \in I_1^{AB}, \tag{7}$$

$$S(B,d^2)_i = \{x=(x',x'')\in[0,1]^{|K^{AB}|}\times[0,1]^{|L^{AB}|} | b_i o x'' \geq d_i^2\}, for\ each\ i \in I_2^{AB}, \tag{8}$$

$$S(C,f^1)_i = \{x=(y',y'')\in[0,1]^{|K^{CE}|}\times[0,1]^{|L^{CE}|} | c_i \bullet y' \leq f_i^1\}, for\ each\ i \in I_1^{CE}, \tag{9}$$

$$S(E,f^2)_i = \{x=(y',y'')\in[0,1]^{|K^{CE}|}\times[0,1]^{|L^{CE}|} | e_i o y'' \leq f_i^2\}, for\ each\ i \in I_2^{CE}, \tag{10}$$

$$S(A,d^1) = \bigcap_{i\in I_1^{AB}} S(A,d^1)_i = \{x=(x',x'')\in[0,1]^{|K^{AB}|}\times[0,1]^{|L^{AB}|} | A\bullet x' \geq d^1\}, \tag{11}$$

$$S(B,d^2) = \bigcap_{i\in I_2^{AB}} S(B,d^2)_i = \{x=(x',x'')\in[0,1]^{|K^{AB}|}\times[0,1]^{|L^{AB}|} | Box'' \geq d^2\}, \tag{12}$$

$$S(C,f^1) = \bigcap_{i\in I_1^{CE}} S(C,f^1)_i = \{x=(y',y'')\in[0,1]^{|K^{CE}|}\times[0,1]^{|L^{CE}|} | C\bullet y' \leq f^1\}, \tag{13}$$

$$S(E,f^2) = \bigcap_{i\in I_2^{CE}} S(E,f^2)_i = \{x=(y',y'')\in[0,1]^{|K^{CE}|}\times[0,1]^{|L^{CE}|} | Eoy'' \leq f^2\}, \tag{14}$$

$S(A,B,d^1,d^2) = S(A,d^1)\bigcap S(B,d^2), S(C,E,f^1,f^2) = S(C,f^1)\bigcap S(E,f^2)$ and $S(A,B,C,E,d^1,d^2,f^1,f^2) = S(A,B,d^1,d^2)\bigcap S(C,E,f^1,f^2) = \{x \in [0,1]^n | A\bullet x' \geq d^1, Box'' \geq d^2, C\bullet y' \leq f^1, Eoy'' \leq f^2\}$.

With regard to relations (2)-(5), we can obtain the necessary and sufficient conditions of solution existence for set (7)-(10) in the following lemmas.

Lemma 3.1. *(1) $x \in S(A,d^1)_i$, for each $i \in I_1^{AB}$, if and only if there exists some $j_i \in K^{AB}$ such that $a_{ij_i}\cdot x_{j_i} \geq d_i^1$.*

(2) $x \in S(B, d^2)_i$, *for each* $i \in I_2^{AB}$, *if and only if there exists some* $j_i \in L^{AB}$ *such that* $(b_{ij_i} - d_i^2 + b_{ij_i} \cdot d_i^2) \cdot x_{j_i} \geq b_{ij_i} \cdot d_i^2$.

(3) $x \in S(C, f^1)_i$, *for each* $i \in I_1^{CE}$, *if and only if* $c_{ij} \cdot x_j \leq f_i^1$, $\forall j \in K^{CE}$.

(4) $x \in S(E, f^2)_i$, *for each* $i \in I_2^{CE}$, *if and only if* $(e_{ij} - f_i^2 + e_{ij} \cdot f_i^2) \cdot x_j \leq e_{ij} \cdot f_i^2$, $\forall j \in L^{CE}$.

Proof. The proofs of parts (1)-(4) can be obtained using the definitions of sets $S(A, d^1)_i$, $S(B, d^2)_i$, $S(C, f^1)_i$, and $S(E, f^2)_i$, respectively. □

It is necessary to recall the following remark.

Remark1. With regard to Lemma 3.1 (2), $x \in S(B, d^2)_i$ if and only if there exist $j \in L^{AB}$ such that $(b_{ij} - d_i^2 + b_{ij} \cdot d_i^2) \cdot x_j \geq b_{ij} \cdot d_i^2$. Since $b_{ij} \cdot d_i^2, x_j \in [0, 1]$ it is concluded that $b_{ij} - d_i^2 + b_{ij} \cdot d_i^2 \geq 0$. Therefore, the following points can be obtained:

(1) We don't consider the components b_{ij} and d_i^2 in the computations that they satisfy the relation $b_{ij} - d_i^2 + b_{ij} \cdot d_i^2 < 0$ and $b_{ij} \cdot d_i^2 > 0$. These components have no effect on finding the feasible solution set.

(2) If $b_{ij} - d_i^2 + b_{ij} \cdot d_i^2 = 0$, then $b_{ij} \cdot d_i^2 = 0$. Therefore, the relation $(b_{ij} - d_i^2 + b_{ij} \cdot d_i^2) \cdot x_j \geq b_{ij} \cdot d_i^2$ is always true, for each $x_j \in [0, 1]$. Therefore, we can remove the components b_{ij} and d_i^2 that they satisfy the relation $b_{ij} - d_i^2 + b_{ij} \cdot d_i^2 = 0$.

(3) If $d_i^2 > 0$ and $b_{ij} = 0$, then x_j can only take zero value for satisfying the relation.

Hence, we exclude the obvious cases mentioned in Remark 1 from my considerations. We are now ready to express the necessary and sufficient conditions for existence of solution set $S(A, d^1)$.

Lemma 3.2. (1) $S(A, d^1) \neq \emptyset$ *if and only if for each* $i \in I_1^{AB}$ *there exists some* $j_i \in K^{AB}$ *such that* $a_{ij_i} \geq d_i^1$.

(2) $S(B, d^2) \neq \emptyset$ *if and only if for each* $i \in I_2^{AB}$ *there exists some* $j_i \in L^{AB}$ *such that* $b_{ij_i} \geq d_i^2$.

(3) *If* $S(A, d^1) \neq \emptyset$, *then* $\bar{1} = [1, 1, ..., 1]_{1\times n}^T$ *is the greatest element in set* $S(A, d^1)$.

(4) *If* $S(B, d^2) \neq \emptyset$, *then* $\bar{1} = [1, 1, ..., 1]_{1\times n}^T$ *is the greatest element in set* $S(B, d^2)$.

Proof. The proofs of parts (1) and (3) are similar to the proofs of parts (2) and (4), respectively. We present the proof of part (2). Assume that $S(B, d^2) \neq \emptyset$ and $(x', x'') \in S(B, d^2)$. Hence, we can write $(x', x'') \in S(B, d^2)_i, \forall i \in I_2^{AB}$. Therefore, it is concluded that for each $i \in I_2^{AB}$, there exist some $j_i \in L^{AB}$ such that $(b_{ij_i} - d_i^2 + b_{ij_i} \cdot d_i^2) \cdot x_{j_i} \geq b_{ij_i} \cdot d_i^2 \geq 0$ with regard to part (2) of Lemma 3.1. Since $(x', x'') \in S(B, d^2)$, we have $x'' \in [0, 1]^{|L^{AB}|}$. Hence, we have $b_{ij_i} - d_i^2 + b_{ij_i} \cdot d_i^2 \geq b_{ij_i} \cdot d_i^2, \forall i \in I_2^{AB}$. Then it implies that there exists an $j_i \in L^{AB}$ such that $b_{ij_i} \geq d_i^2$. Conversely, assume that there exists $j_i \in L^{AB}$ such that $b_{ij_i} \geq d_i^2, \forall i \in I_2^{AB}$, or equivalently,$b_{ij_i} - d_i^2 + b_{ij_i} \cdot d_i^2 \geq b_{ij_i} \cdot d_i^2, \forall i \in I_2^{AB}$. Let $x = \bar{1} = [1, 1, ..., 1]_{1\times n}^T$. Since $x'' \in [0, 1]^{|L^{AB}|}$ and $x_{j_i} = 1 \geq \frac{b_{ij_i} \cdot d_i^2}{b_{ij_i} - d_i^2 + b_{ij_i} \cdot d_i^2}, \forall i \in I_2^{AB}$, then $x \in S(B, d^2)_i, \forall i \in I_2^{AB}$. Therefore $x \in S(B, d^2)$.

(4) It is easily proved by using part (2) and $x \in [0, 1]^n$. □

We are now ready to present the conditions of solution existence and introduce the greatest and smallest element of the sets in (21)-(24) in the following lemmas.

Lemma 3.3. (1) $S(A, B, d^1, d^2) \neq \emptyset$ *if and only if we have (a)* $\forall i \in I_1^{AB}, \exists j_i \in K^{AB}$ *s.t.* $a_{ij_i} \geq d_i^1$, *and (b)* $\forall i \in I_2^{AB}, \exists j_i \in L^{AB}$ *s.t.* $b_{ij_i} \geq d_i^2$.

(2) *If* $S(A, B, d^1, d^2) \neq \emptyset$, *then* $\bar{1} = [1, 1, ..., 1]_{1\times n}^T$ *is the greatest element in the set* $S(A, B, d^1, d^2)$.

Proof. They are direct result from Lemma 3.2. □

Lemma 3.4. *(i)*$S(C,f^1)\neq\emptyset$*, (ii)*$S(E,f^2)\neq\emptyset$*, (iii)*$S(C,E,f^1,f^2)\neq\emptyset$*, (iv)the smallest element in sets* $S(C,f^1),S(E,f^2)$*, and* $S(C,E,f^1,f^2)$ *is vector* $\overline{0}=[0,0,...,0]^T_{1\times n}$.

Proof. It is obvious that $\overline{0}$ is a feasible solution in sets $S(C,f^1),S(E,f^2)$ and $S(C,E,f^1,f^2)$. Therefore, the sets of $S(C,f^1),S(E,f^2)$ and $S(C,E,f^1,f^2)$ are not empty. On the other hand, since $x\in[0,1]^n$, it is easily seen that $\overline{0}$ is the smallest element of the sets of $S(C,f^1),S(E,f^2)$ and $S(C,E,f^1,f^2)$. □

Definition 3.1. *Let* $\overline{x}=[\overline{x_j}]_{n\times 1}$ *where*

$$\overline{x}_j=\begin{cases}\bigwedge_{i\in I_1^{CE}}\{\frac{f_i^1}{c_{ij}}|f_i^1\leq c_{ij}\},\ j\in K^{CE}\setminus L^{CE},\\ \bigwedge_{i\in I_2^{CE}}\{\frac{e_{ij}\cdot f_i^2}{e_{ij}-f_i^2+e_{ij}\cdot f_i^2}|e_{ij}\geq f_i^2\},\ j\in L^{CE}\setminus K^{CE},\\ (\bigwedge_{i\in I_1^{CE}}\{\frac{f_i^1}{c_{ij}}|f_i^1\leq c_{ij}\})\bigwedge(\bigwedge_{i\in I_2^{CE}}\{\frac{e_{ij}\cdot f_i^2}{e_{ij}-f_i^2+e_{ij}\cdot f_i^2}|e_{ij}\geq f_i^2\}),\ j\in L^{CE}\bigcap K^{CE},\end{cases}$$

where $\bigwedge\emptyset=1$.

Lemma 3.5. *(i)* $\overline{x}\in S(C,E,f^1,f^2)$*, (ii) the vector of* $\overline{x}=[\overline{x}_j]_{n\times1}$ *defined in Definition 3.1 is the maximum solution of set* $S(C,E,f^1,f^2)$.

Proof. Assume that $j\in K^{CE}\setminus L^{CE}$. If $\{\frac{f_i^1}{c_{ij}}|f_i^1\leq c_{ij}\}\neq\emptyset$, then $\overline{x}_j\leq\frac{f_i^1}{c_{ij}}$, for each $i\in I_1^{CE}$. Hence, we have $c_{ij}\cdot\overline{x}_j\leq f_i^1$, for each $i\in I_1^{CE}$. Thus, it is concluded that $\max_{j\in K^{CE}}\{c_{ij}\cdot\overline{x}_j\}\leq f_i^1$, for each $i\in I_1^{CE}$. Now, suppose that $j\in L^{CE}\setminus K^{CE}$. Then if $\{\frac{e_{ij}\cdot f_i^2}{e_{ij}-f_i^2+e_{ij}\cdot f_i^2}|e_{ij}\geq f_i^2\})\neq\emptyset$, we have $\overline{x}_j\leq\frac{e_{ij}\cdot f_i^2}{e_{ij}-f_i^2+e_{ij}\cdot f_i^2}$. Hence, $\frac{e_{ij}\cdot\overline{x}_j}{e_{ij}+\overline{x}_j-e_{ij}\cdot\overline{x}_j}\leq f_i^2,\forall i\in I_2^{CE}$. Therefore, we obtain that $\max_{j\in L^{CE}}\{\frac{e_{ij}\cdot\overline{x}_j}{e_{ij}+\overline{x}_j-e_{ij}\cdot\overline{x}_j}\}\leq f_i^2$, for each $i\in I_2^{CE}$. If $j\in L^{CE}\bigcap K^{CE}$, we can obtain a similar result with the above. Hence it is concluded that $\overline{x}\in S(C,E,f^1,f^2)$. If the sets of $\{\frac{f_i^1}{c_{ij}}|f_i^1\leq c_{ij}\}$ or $\{\frac{e_{ij}\cdot f_i^2}{e_{ij}-f_i^2+e_{ij}\cdot f_i^2}|e_{ij}\geq f_i^2\}$ be empty, the proof becomes easier and we can again obtain a similar result. Hence, $\overline{x}\in S(C,E,f^1,f^2)$.
(ii)The proof is obvious according to the definition of $\overline{x}$. □

Corollary 3.1. $S(C,E,f^1,f^2)=\{x=(y',y'')\in[0,1]^n|C\bullet y'\leq f^1,Eoy''\leq f^2\}=[\overline{0},\overline{x}]$ *where* $\overline{0}$ *and* $\overline{x}$ *are the zero vector and the maximum solution according to Definition 3.1, respectively.*

Lemma 3.6. *The maximum solution of two sets* $S(A,B,C,E,d^1,d^2,f^1,f^2)$ *and* $S(C,E,f^1,f^2)$ *is the same.*

Proof. Suppose that $\widehat{x}$ and $\overline{x}$ are the maximum solutions of sets $S(A,B,C,E,d^1,d^2,f^1,f^2)$ and $S(C,E,f^1,f^2)$, respectively. By contradiction, assume that $\widehat{x}\neq\overline{x}$. Since $S(A,B,C,E,d^1,d^2,f^1,f^2)\subseteq S(C,E,f^1,f^2)$, we have $\hat{x}\in S(C,E,f^1,f^2)$. On the other hand, $\overline{x}$ is the maximum solution of $S(C,E,f^1,f^2)$. Hence, it is concluded that $\overline{x}\geq\hat{x}$, $\hat{x}\neq\overline{x}$, and $\overline{x}\notin S(A,B,C,E,d^1,d^2,f^1,f^2)$. Since $\overline{x}\notin S(A,B,C,E,d^1,d^2,f^1,f^2)$, there exists $i\in I_1^{AB}$ or $i\in I_2^{AB}$ such that $a_i\bullet\overline{x}'<d_i^1\leq a_i\bullet\hat{x}'$ or $b_io\overline{x}''<d_i^2\leq b_io\hat{x}''$. Moreover, since $\overline{x}\geq\hat{x}$, we have $\overline{x}_j\geq\hat{x}_j$, for each $j\in K^{AB}\bigcup L^{AB}$. On the other hand, we have $0\leq a_{ij},b_{ij}\leq 1$, for each i and j. Hence, it is concluded that for each i, we have $a_i\bullet\overline{x}'\geq a_i\bullet\hat{x}'$ and $b_io\overline{x}''\geq b_io\hat{x}''$. However, these statements contradict with $a_i\bullet\overline{x}'\leq a_i\bullet\hat{x}'$ and $b_io\overline{x}''\leq b_io\hat{x}''$. Therefore $\hat{x}=\overline{x}$. □

With regard to the above lemma, we can obtain a necessary and sufficient condition for existence of solution of set $S(A,B,C,E,d^1,d^2,f^1,f^2)$ in the following lemma.

Lemma 3.7. *The set of $S(A,B,C,E,d^1,d^2,f^1,f^2)\neq\emptyset$ if and only if vector $\overline{x}$ defined by Definition 3.1 satisfies relations (2)-(6).*

Proof. if $S(A,B,C,E,d^1,d^2,f^1,f^2)$ is nonempty, then the maximum solution of $S(A,B,C,E,d^1,d^2,f^1,f^2)$ is computed by Definition 3.1 with regard to Lemma 3.6. Moreover, the vector $\overline{x}$ satisfies the relations (2)-(6) with attention to the definition of set $S(A,B,C,E,d^1,d^2,f^1,f^2)$. The proof of its converse is obvious. □

The solution set of a mixed FRI problem, i.e., $S(A,B,C,E,d^1,d^2,f^1,f^2)$, is determined by a unique maximum solution and a finite number of minimal solutions. The maximum solution is easily computed by Definition 3.1 compared with computing the minimal solutions. We know that obtaining the minimal solutions of $S(A,B,C,E,d^1,d^2,f^1,f^2)$ is more difficult. We investigate below the method for obtaining the minimal solutions of mixed FRI in details. Some concepts and theorems related to the minimal solutions are first given.

Definition 3.2. *Suppose that $\overline{x}$ is the maximum solution of set $S(C,E,f^1,f^2)$, the matrix $R^A=[r^A_{ij}]_{i\in I_1^{AB},j\in K^{AB}}$ and $R^B=[r^B_{ij}]_{i\in I_2^{AB},j\in L^{AB}}$ are called FRI characteristic matrices corresponding to the matrices of A and B, respectively, where $\forall i\in I_1^{AB}, j\in K^{AB}$, $r^A_{ij}=\begin{cases}1 & a_{ij}\cdot\overline{x}_j\geq d_i^1,\\ 0 & a_{ij}\cdot\overline{x}_j< d_i^1,\end{cases}$ and $\forall i\in I_2^{AB}, j\in L^{AB}$, $r^B_{ij}=\begin{cases}1 & \frac{b_{ij}\cdot\overline{x}_j}{b_{ij}+\overline{x}_j-b_{ij}\cdot\overline{x}_j}\geq d_i^1,\\ 0 & \frac{b_{ij}\cdot\overline{x}_j}{b_{ij}+\overline{x}_j-b_{ij}\cdot\overline{x}_j}< d_i^1.\end{cases}$*

Define a series of index sets by $J_i^A=\{j\in K^{AB}|r^A_{ij}=1\}$, for each $i\in I_1^{AB}$, and $J_i^B=\{j\in L^{AB}|r^B_{ij}=1\}$, for each $i\in I_2^{AB}$, and a series of index sets by $I_j^A=\{i\in I_1^{AB}|r^A_{ij}=1\}$, for each $j\in K^{AB}$,and $I_j^B=\{i\in I_2^{AB}|r^B_{ij}=1\}$, for $j\in L^{AB}$.

Definition 3.3. *(i)A vector $p^A=(p_i^A)_{i\in I_1^{AB}}$ is called a FRI path of inequality (2) if for any $i\in I_1^{AB}$, we have $p_i^A\in J_i^A$. Denote P^A the set of all the FRI paths of (2).*
(ii)A vector $p^B=(p_i^B)_{i\in I_2^{AB}}$ is called a FRI path of inequality (3) if for any $i\in I_2^{AB}$, we have $p_i^B\in J_i^B$. Denote P^A the set of all the FRI paths of (3).

We are now ready to present the necessary and sufficient conditions for solution existence of system (2)-(6).

Theorem 3.1. *The solution set of system (2)-(6) is not empty if and only if every row of the FRI characteristic matrices R^A and R^B has at least one non-zero component.*

Proof. (Sufficiency)suppose that every row of R^A and R^B has at least one non-zero component. For any $i\in I_1^{AB}\bigcup I_2^{AB}$, there exists $j_i\in K^{AB}\bigcup L^{AB}$ such that $r^A_{ij_i}\neq 0$ or $r^B_{ij_i}\neq 0$. Define $p=p^A\bigcup p^B$ where $p^A=(p_i^A)_{i\in I_1^{AB}}$ and $p^B=(p_i^B)_{i\in I_2^{AB}}$ and compute $x^p=(x_j^p)_{j\in K^{AB}\bigcup L^{AB}}$ as follows:

$$x_j^p=\begin{cases}\bigvee_{i\in I_1^{AB}}\{\frac{d_i^1}{a_{ij}}|p_i^A=j\}, & if\ j\in K^{AB}\setminus L^{AB},\\ \bigvee_{i\in I_2^{AB}}\{\frac{b_{ij}\cdot d_i^2}{b_{ij}-d_i^2+b_{ij}\cdot d_i^2}|p_i^B=j\}, & if\ j\in L^{AB}\setminus K^{AB},\\ (\bigvee_{i\in I_1^{AB}}\{\frac{d_i^1}{a_{ij}}|p_i^A=j\})\bigvee(\bigvee_{i\in I_2^{AB}}\{\frac{b_{ij}\cdot d_i^2}{b_{ij}-d_i^2+b_{ij}\cdot d_i^2}|p_i^B=j\}), & if\ j\in K^{AB}\bigcap L^{AB},\end{cases}$$

where $\bigvee\emptyset=0$ is defined.
For any j satisfying $x_j^p\neq 0$, we will have three cases as follows:

(1) If $j\in K^{AB}\setminus L^{AB}$, then $x_j^p=\bigvee_{i\in I_1^{AB}}\{\frac{d_i^1}{a_{ij}}|p_i^A=j\}\geq\frac{d_i^1}{a_{ij}},\forall i\in I_1^{AB}$.

(2) If $j \in L^{AB} \setminus K^{AB}$, then $x_j^p = \bigvee_{i\in I_2^{AB}}\{\frac{b_{ij}\cdot d_i^2}{b_{ij}-d_i^2+b_{ij}\cdot d_i^2}|p_i^B = j\} \geq \frac{b_{ij}\cdot d_i^2}{b_{ij}-d_i^2+b_{ij}\cdot d_i^2}, \forall i \in I_2^{AB}$.

(3) If $j \in K^{AB}\bigcap L^{AB}$, then $x_j^p(\bigvee_{i\in I_1^{AB}}\{\frac{d_i^1}{a_{ij}}|p_i^A = j\})\bigvee(\bigvee_{i\in I_2^{AB}}\{\frac{b_{ij}\cdot d_i^2}{b_{ij}-d_i^2+b_{ij}\cdot d_i^2}|p_i^B = j\}) \geq \frac{d_i^1}{a_{ij}}, \forall i \in I_1^{AB}$, or $x_j^p \geq \frac{b_{ij}\cdot d_i^2}{b_{ij}-d_i^2+b_{ij}\cdot d_i^2}, \forall i \in I_2^{AB}$.

With regard to parts (1) and (3), it is concluded that for each $j \in K^{AB}, a_{ij}\cdot x_j^{p^A} \geq d_i^1, \forall i \in I_1^{AB}$. Hence we have $\max_{j\in K^{AB}}\{a_{ij}\cdot x_j^{p^A}\} \geq d_i^1, \forall i \in I_1^{AB}$. Therefore, $A\cdot x^{p^A} \geq d$. With regard to parts (2) and (3), it is concluded that for each $j \in L^{AB}, x_j^{p^B} \geq \frac{b_{ij}\cdot d_i^2}{b_{ij}-d_i^2+b_{ij}\cdot d_i^2}, \forall i \in I_2^{AB}$, or equivalently for each $j \in L^{AB}, \frac{b_{ij}\cdot x_j^{p^B}}{b_{ij}+x_j^{p^B}-b_{ij}\cdot x_j^{p^B}} \geq d_i^2, \forall i \in I_2^{AB}$. Hence, we have $\max_{j\in L^{AB}}\{\frac{b_{ij}\cdot x_j^{p^B}}{b_{ij}+x_j^{p^B}-b_{ij}\cdot x_j^{p^B}}\} \geq d_i^2, \forall i \in I_2^{AB}$. Therefore, $Box^{p^B} \geq d^2$. If $x_j^p = \frac{d_i^1}{a_{ij}} \leq \overline{x}_j$ or $x_j^p = \frac{b_{ij}\cdot d_i^2}{b_{ij}-d_i^2+b_{ij}\cdot d_i^2} \geq \overline{x}_j, \forall j \in L^{AB}$. Thus x^P solves $C\bullet y' \leq f^1$ and $Eoy'' \leq f^2$. Consequently we conclude that x^p is a solution of (2)-(6).
(Necessity) Suppose that the solution set of (2)-(6) is not empty. If there exists a row of matrices R^A or R^B, say the i^{th} row of matrix R^A, $i \in I_1^{AB}$, whose components are all zero, then $a_{ij}\cdot\overline{x}_j < d_i^1$, for each $j \in K^{AB}$. Since an arbitrary solution x^0 of $S(C, E, f^1, f^2)$ must satisfy $x^0 \leq \overline{x}$, one has $a_{ij}\cdot\overline{x_j^0} < d_i^1$ for any $j \in K^{AB}$. Consequently, $A\bullet x^0 < d^1$. This implies that the solution set of (2)-(6) is empty. This contradicts the assumption that the solution set of (2)-(6) is not empty. We can similarly obtain the contradiction if the i^{th} row of matrix R^B, $i \in I_2^{AB}$, is empty. □

The following theorem determines the structure of feasible domain of problem (1)-(6).

Theorem 3.2. *Suppose that $S(A, B, C, E, d^1, d^2, f^1, f^2) \neq \emptyset$. Let $p \in P$ be a mixed FRI path of (1)-(6) and $\overline{x}$ be the maximum solution of (2)-(6). Define $x^p = (x_j^P)_{j\in K^{AB}\bigcup L^{AB}}$ by*

$$x_j^p = \begin{cases} \bigvee_{i\in I_1^{AB}}\{\frac{d_i^1}{a_{ij}}|p_i^A = j\}, & if\ j \in K^{AB}\setminus L^{AB},\\ \bigvee_{i\in I_2^{AB}}\{\frac{b_{ij}\cdot d_i^2}{b_{ij}-d_i^2+b_{ij}\cdot d_i^2}|p_i^B = j\}, & if\ j \in L^{AB}\setminus K^{AB},\\ (\bigvee_{i\in I_1^{AB}}\{\frac{d_i^1}{a_{ij}}|p_i^A = j\})\bigvee(\bigvee_{i\in I_2^{AB}}\{\frac{b_{ij}\cdot d_i^2}{b_{ij}-d_i^2+b_{ij}\cdot d_i^2}|p_i^B = j\}), & if\ j \in K^{AB}\bigcap L^{AB}, \end{cases} \tag{15}$$

where $\bigvee\emptyset = 0$ is defined and $j \in J$. Then the solution set of mixed FRI (2)-(6)is as follows: $S(A, B, C, E, d^1, d^2, f^1, f^2) = \bigcup_{p\in P}\{x|x^p \leq x \leq \overline{x}\}$.

Proof. Suppose that x is an arbitrary solution of system (2)-(6). Then x satisfies inequality $C\bullet y' \leq f^1$ and $Eoy'' \leq f^2$. Hence, $x \leq \overline{x}$. On the other hand, x satisfies inequality $A\bullet x' \geq d^1$ and $Box'' \geq d^2$. Therefore, we have:$\forall i \in I_1^{AB}$, $max_{j\in K^{AB}}\{a_{ij}.x_j\} \geq d_i^1$, and $\forall I_2^{AB}$, $max_{j\in L^{AB}}\{\frac{b_{ij}.x_j}{b_{ij}+x_j-b_{ij}.x_j}\} \geq d_i^2$. With regard to above statements, there exists $j_i \in K^{AB}$ such that $a_{ij_i}\cdot x_{j_i} \geq d_i^1$ and there exists $j'_t \in L^{AB}$ such that $\frac{b_{ij'_t}\cdot x_{j'_t}}{b_{ij'_t}+x_{j'_t}-b_{ij'_t}\cdot x_{j'_t}} \geq d_i^2$. Let $q_i^1 = j_i$ and $q_i^2 = j'_t$. Thus , for any $j \in K^{AB}\bigcup L^{AB}$, we have the following three cases:

(1) If $j \in K^{AB}\setminus L^{AB}$, then $x_j \geq \bigvee_{i\in I_1^{AB}}\{\frac{d_i^1}{a_{ij}}|q_i^1 = j\} \geq \bigvee_{i\in I_1^{AB}}\{\frac{d_i^1}{a_{ij}}|p_i^A = j\} = x_j^p$,

(2) If $j \in L^{AB} \backslash K^{AB}$, then $x_j \geq \bigvee_{i \in I_2^{AB}} \{\frac{b_{ij} \cdot d_i^2}{b_{ij} - d_i^2 + b_{ij} \cdot d_i^2} | q_i^2 = j\} \geq \bigvee_{i \in I_2^{AB}} \{\frac{b_{ij} \cdot d_i^2}{b_{ij} - d_i^2 + b_{ij} \cdot d_i^2} | p_i^B = j\} = x_j^p$,

(3) If $j \in K^{AB} \bigcap L^{AB}$, then $x_j \geq (\bigvee_{i \in I_1^{AB}} \{\frac{d_i^1}{a_{ij}} | q_i^1 = j\}) \bigvee (\bigvee_{i \in I_2^{AB}} \{\frac{b_{ij} \cdot d_i^2}{b_{ij} - d_i^2 + b_{ij} \cdot d_i^2} | q_i^2 = j\}) \geq (\bigvee_{i \in I_1^{AB}} \{\frac{d_i^1}{a_{ij}} | p_i^A = j\}) \bigvee (\bigvee_{i \in I_2^{AB}} \{\frac{b_{ij} \cdot d_i^2}{b_{ij} - d_i^2 + b_{ij} \cdot d_i^2} | p_i^B = j\} = x_j^p$.

Therefore, we conclude that $x \in S(A, B, C, E, d^1, d^2, f^1, f^2)$. To compute the proof, we show that for any $p \in P$, vector x^p is a solution of system (2)-(6). For any i, if $p_i = p_i^A$, then $x^p_{p_i^A} = \bigvee_{k \in I_1^{AB}} \{\frac{d_k^1}{a_{kp_k^A}} | p_k^A = p_i^A\} \geq \frac{d_i^1}{a_{ip_i^A}}$ and hence $\bigvee_{j \in K^{AB}} (a_{ij} \cdot x_j^p) \geq d_i^1$. Therefore, the inequality $A \bullet x^p \geq d^1$ holds. If $p_i = p_i^B$, then $x^p_{p_i^B} = \bigvee_{k \in I_2^{AB}} \{\frac{b_{kp_k^B} \cdot d_k^2}{b_{kp_k^B} - d_k^2 + b_{kp_k^B} \cdot d_k^2} | p_k^B = p_i^B\} \geq \frac{b_{ip_i^B} \cdot d_i^2}{b_{ip_i^B} - d_i^2 + b_{ip_i^B} \cdot d_i^2}$ and hence $\bigvee_{j \in L^{AB}} (\frac{b_{ij} \cdot x_j^p}{b_{ij} + x_j^p - b_{ij} \cdot x_j^p}) \geq d_i^2$. Therefore, the inequality $Box^p \geq d^2$. If $p_k^A \in L^{AB} \bigcap K^{AB}$, then we have $x^p_{p_k^A} = (\bigvee_{i \in I_1^{AB}} \{\frac{d_i^1}{a_{ip_i^A}} | p_i^A = p_k^A\}) \bigvee (\bigvee_{i \in I_2^{AB}} \{\frac{b_{ip_i^B} \cdot d_i^2}{b_{ip_i^B} - d_i^2 + b_{ip_i^B} \cdot d_i^2} | p_i^B = p_k^A\}) \geq \frac{d_i^1}{a_{ip_k^A}}$. We can conclude that $\bigvee_{j \in K^{AB}} (a_{ij} \cdot x_j^p) \geq d_i^1$. If $p_k^B \in L^{AB} \bigcap K^{AB}$, then we have $x^p_{p_k^B} = (\bigvee_{i \in I_1^{AB}} \{\frac{d_i^1}{a_{ip_i^A}} | p_i^A = p_k^B\}) \bigvee (\bigvee_{i \in I_2^{AB}} \{\frac{b_{ip_i^B} \cdot d_i^2}{b_{ip_i^B} - d_i^2 + b_{ip_i^B} \cdot d_i^2} | p_i^B = p_k^B\}) \geq \frac{b_{ip_i^B} \cdot d_i^2}{b_{ip_i^B} - d_i^2 + b_{ip_i^B} \cdot d_i^2}$. We can conclude that $\bigvee_{j \in L^{AB}} \{\frac{b_{ij} \cdot x_j^p}{b_{ij} + x_j^p - b_{ij} \cdot x_j^p}\} \geq d_i^2$. Therefore, we have $Box^p \geq d^2$. With regard to definition x^p, for any $j \in J$, there exist some k_j such that $x_j^p = \frac{d_{k_j}^1}{a_{k_j j}}$ and $p_{k_j} = j$ or $x_j^p = \frac{b_{k_j j} \cdot d_{k_j}^1}{b_{k_j j} - d_{k_j}^2 + b_{k_j j} \cdot d_{k_j}^2}$ and $p_{k_j} = j$. Since $a_{k_j j} \cdot \overline{x}_j \geq d_{k_j}^1$ or $\frac{b_{k_j j} \cdot \overline{x}_j}{b_{k_j j} + \overline{x}_j - b_{k_j j} \cdot \overline{x}_j} \geq d_{k_j}^2$, we have $x_j^p \leq \overline{x}_j, j \in J$, i.e., $x^p \leq \overline{x}$. This implies that x^p solves the inequality $C \bullet y' \leq f^1$ and $Eoy'' \leq f^2$. Since the solution set is not empty, vector $\overline{x}$ must be one of the solutions. Therefore, we conclude that $S(A, B, C, E, d^1, d^2, f^1, f^2)$ is the solution set of the inequalities (2)-(6). The proof is completed. □

From the above theorem, we conclude that for any $p \in P$, vector x^p is a solution of inequality (2)-(6). We call x^p a quasi-minimal solution of system (2)-(6). The above theorem also shows that $\underline{X} \subseteq \{x^p | p \in P\}$ where $\underline{X}$ denote the set of all the minimal solutions of system (2)-(6). We are now ready to present the resolution process of problem (1)-(6) in the next section.

4. The resolution process of problem (1)-(6)

In order to solve the problem (1)-(6), we first convert it into two sub-problems as follows: $SP1: \ Min\{\beta \cdot \prod_{j \in R^+} x_j^{\alpha_j} | x \in S(A, B, C, E, d^1, d^2, f^1, f^2)\}$, and $SP2: \ Min\{\beta \cdot \prod_{j \in R^-} x_j^{\alpha_j} | x \in S(C, E, f^1, f^2)\}$, where $R^+ = \{j | \alpha_j \geq 0, j \in J\}$ and $R^- = \{j | \alpha_j < 0, j \in J\}$. We now focus on the resolution of sub-problems ($SP1$) and ($SP2$), respectively, in two following lemmas.

Lemma 4.1. *The optimal solution of sub-problem ($SP2$) is $\overline{x}$, i.e., the maximum solution of $S(A, B, C, E, d^1, d^2, f^1, f^2)$ or $S(C, E, f^1, f^2)$.*

Proof. Since function $\prod_{j \in R^-} x_j^{\alpha_j}$ is a decreasing function with respect to each variable $x_j \in [0, 1]$, for $j \in R^-$, then the optimal solution of sub-problem ($SP2$) is $\overline{x}$. □

Lemma 4.2. *The optimal solution of sub-problem* $(SP1)$ *is one of the element of set* $\{x^p | p \in P\}$.

Proof. with regard to Theorem 3.1, $S(A, B, C, E, d^1, d^2, f^1, f^2) = \bigcup_{p \in P}[x^p, \overline{x}]$. On the other hand, since $\prod_{j \in R^+} x_j^{\alpha_j}$ is an increasing function with respect to each variable $x_j \in [0, 1]$, for each $j \in R^+$, then for each $x \in S(A, B, C, E, d^1, d^2, f^1, f^2)$, there exist some $p \in P$ such that $x^p \leq x$. Therefore, one of elements $x^p, p \in P$, such that $\prod_{j \in R^+}(x_j^{p^*})^{\alpha_j} = \min_{p \in P}\{\prod_{j \in R^+}(x_j^p)^{\alpha_j} | p \in P\}$ solves the sub-problem $(SP1)$. □

With regard to two Lemmas 4.1 and 4.2, we present the following theorem to find the optimal solution of the original problem (1)-(6). Let $\overline{x}^*$ and $\underline{x}^*$ be the optimal solutions of sub-problems $(SP1)$ and $(SP2)$, respectively. A new vector $x^* = (x_1^*, ..., x_n^*)^T$ is defined as:x_j^* is equal to $\underline{x}_j^*$, $if c_j \geq 0$, and $\overline{x}_j^*$, $if c_j < 0$, $\forall j \in J$. Then we have the following theorem.

Theorem 4.1. *The vector of* x^* *is an optimal solution of the problem (1)-(6).*

Proof. With regard to two Lemmas 4.1 and 4.2, we have $Z(x^*) = \beta(\prod_{j \in R^-}(x_j^*)^{\alpha_j}) \times (\prod_{j \in R^+}(x_j^*)^{\alpha_j}) = \beta(\prod_{j \in R^-}(\overline{x}_j)^{\alpha_j}) \times (\prod_{j \in R^+}(x_j^{p^*})^{\alpha_j}) \leq \beta(\prod_{j \in R^-}(x_j)^{\alpha_j}) \times (\prod_{j \in R^+}(x_j)^{\alpha_j}) = Z(x)$. For each $x \in S(A, B, C, E, d^1, d^2, f^1, f^2)$. Therefore, x^* is an optimal solution of the problem (1)-(6). □

In order to compute the optimal solution x^*, we need to find the vectors of $\overline{x}$ and x^{p^*} with regard to Theorem 4.1. The vector $\overline{x}$ is easily computed from Definition 3.1. The vector of x^{p^*} is obtained by pairwise comparison between the elements of set $\{x^p | p \in P\}$. The computation of vector x^{p^*} is usually hard and time-consuming, if the set P has many elements. Therefore some rules are proposed to simplify the problem (1)-(6). Under these rules, some of rows and columns of problem (1)-(6) are removed and its original problem is decomposed into several sub-problems with smaller dimensions. Then we can find the vector x^{p^*} in a smaller search domain and the operations accelerate the resolution process of the sub-problem $(SP1)$.

5. Some rules for reduction of problem $(SP1)$

In this section, some theorems are presented to reduced the size of problem $(SP1)$. Applying these theorems, some of the x_j^*'s of optimal of the problem $(SP1)$ can be determined immediately without solving the problem $(SP1)$. At first, it is necessary to recall the notation $A \subset B$, for the set A and B, is equivalent to $A \subseteq B$ and $A \neq B$. We assume that $S(A, B, C, E, d^1, d^2, f^1, f^2) \neq \emptyset$ in the following theorems and corollaries.

Theorem 5.1. *If for some* $i_0 \in I_1^{AB}$, *there exists* $j_0 \in K^{AB} \setminus L^{AB}$ *such that (1)* $|J_{i_0}^A| = 1$ *and* $J_{i_0}^A = \{j_{i_0}\}$, *and (2)* $\frac{d_{i_0}^1}{a_{i_0 j_0}} \geq \frac{d_i^1}{a_{i j_0}}$ *for each* $i \in I_{j_0}^A$, *then for any optimal solution* $x^* = (x_1^*, ..., x_n^*)^T$ *of problem* $(SP1)$, *we have* $x_{j_0}^* = \frac{d_{i_0}^1}{a_{i_0 j_0}}$.

Proof. Since $J_{i_0}^A = \{j_{i_0}\}$ and $\frac{d_{i_0}^1}{a_{i_0 j_0}} \geq \frac{d_i^1}{a_{i j_0}}$ for each $i \in I_{j_0}^A$, then for each FRI path $p = p^A \bigcup p^B$ where $p^A = (p_i^A)_{i \in I_1^{AB}}$, $p^B = (p_i^B)_{i \in I_2^{AB}}$, and $j_0 \in K^{AB} \setminus L^{AB}$, $x_{j_0}^p = \bigvee_{i \in I_1^{AB}}\{\frac{d_i^1}{a_{ij}} | p_i^A = j_0\} = \frac{d_{i_0}^1}{a_{i_0 j_0}}$ with regard to Theorem 3.1. Hence, for any optimal solution x^*, we have: $x_{j_0}^* = \frac{d_{i_0}^1}{a_{i_0 j_0}}$. □

Theorem 5.2. *If for some* $i_0 \in I_2^{AB}$*, there exists* $j_0 \in L^{AB} \setminus K^{AB}$ *such that (1)* $|J_{i_0}^B| = 1$ *and* $J_{i_0}^B = \{j_{i_0}\}$*, and (2)* $\frac{d_{i_0}^2 \cdot b_{i_0 j_0}}{b_{i_0 j_0} - d_{i_0}^2 + d_{i_0}^2 \cdot b_{i_0 j_0}} \geq \frac{d_i^2 \cdot b_{i j_0}}{b_{j_0} - d_i^2 + d_i^2 \cdot b_{i j_0}}$*, for each* $i \in I_{j_0}^B$*, then for any optimal solution* $x^* = (x_1^*, ..., x_n^*)^T$ *of problem (*$SP1$*), we have* $x_{j_0}^* = \frac{d_{i_0}^2 \cdot b_{i_0 j_0}}{b_{i_0 j_0} - d_{i_0}^2 + d_{i_0}^2 \cdot b_{i_0 j_0}}$.

Proof. Due to $J_{i_0}^B = \{j_{i_0}\}$ and $\frac{d_{i_0}^2 \cdot b_{i_0 j_0}}{b_{i_0 j_0} - d_{i_0}^2 + d_{i_0}^2 \cdot b_{i_0 j_0}} \geq \frac{d_i^2 \cdot b_{i j_0}}{b_{j_0} - d_i^2 + d_i^2 \cdot b_{i j_0}}$, for each $i \in I_{j_0}^B$, then for each FRI path $p = p^A \bigcup p^B$ where $p^A = (p_i^A)_{i \in I_1^{AB}}$ and $p^B = (p_i^B)_{i \in I_2^{AB}}$, and $j_0 \in K^{AB} \setminus L^{AB}$, $x_{j_0}^P = \bigvee_{i \in I_2^{AB}} \{\frac{d_i^2 \cdot b_{i j_0}}{b_{i j_0} - d_i^2 + d_i^2 \cdot b_{i j_0}} | p_i^B = j_0\} = \frac{d_{i_0}^2 \cdot b_{i_0 j_0}}{b_{i_0 j_0} - d_{i_0}^2 + d_{i_0}^2 \cdot b_{i_0 j_0}}$. □

We are now ready to design an algorithm for solving the problem (1)-(6) with regard to the mentioned points in Sections 2,3, and 4.

6. An algorithm for solving problem (1)-(6)

We present an algorithm to solve problem (1)-(6) with regard to the expressed points up to now.

Algorithm 1. Suppose that the problem (1)-(6) has been given.

Step1. Compute the maximum solution $\overline{x}$ by Definition 3.1.

Step2. Compute the characteristic matrices R^A and R^B corresponding to matrices A and B, respectively.

Step3. Check the feasibility of problem (1)-(6) with regard to Theorem 3.1. If the problem is infeasible then stop! Otherwise go to Step 4.

Step4. Create tow sub-problem ($SP1$) and ($SP2$).

Step5. The maximum solution solves the problem ($SP2$) with regard to Lemma 4.1. Then $x_j^* = \overline{x}_j$, for each $j \in J^- = \{j \in J | \alpha_j \leq 0\}$. Delete the columns corresponding to these variables from the constraints.

Step6. Check the condition of Theorem 5.1. If the conditions are satisfied, then set $x_{j_0}^* = \frac{d_{i_0}^1}{a_{i_0 j_0}}$ and delete the column corresponding to this variable from the constraints, i.e., column j_0.

Step7. Check the condition of Theorem 5.2. If the conditions are satisfied, then set $x_{j_0}^* = \frac{d_{i_0}^2 \cdot b_{i_0 j_0}}{b_{i_0 j_0} - d_{i_0}^2 + d_{i_0}^2 \cdot b_{i_0 j_0}}$ and delete the column corresponding to this variable from the constraints, i.e., column j_0.

Step8. Compute the MFRI paths of the reduced system (2)-(6) and call it as P. Find the optimal solution of reduced problem using the following relation:$\prod_{j \in R^+} (x_j^{p^*})^{\alpha_j} = \min_{p \in P} \{\prod_{j \in R^+} (x_j^p)^{\alpha_j} | p \in P\}$.

Step9. Obtain the optimal solution of problem (1)-(6)using Steps 1-8.

We now illustrate the algorithm by an example.

Example 6.1. *Consider the problem (P), where* $z = x_1^3 \times x_2^{0.2} \times x_3^{-4} \times x_4^{-1} \times x_5^{0.5} \times x_6^{-2}$,$x' = [x_1, x_4, x_5]^T$, $x'' = [x_2, x_3, x_6]^T$, $y' = [x_1, x_2, x_6]^T$, $y'' = [x_3, x_4, x_5, x_6]^T$,

$$A = \begin{pmatrix} 1 & 0.9 & 0.85 \\ 0.1 & 0.2 & 0.8 \\ 0.3 & 0.4 & 1 \end{pmatrix}, B = \begin{pmatrix} 0.3 & 0.2 & 0.5 \\ 0.6 & 0.7 & 0.4 \\ 0.8 & 0.5 & 0.1 \end{pmatrix}, d^1 = \begin{pmatrix} 0.3 \\ 0.25 \\ 0.15 \end{pmatrix}, d^2 = \begin{pmatrix} 0.18 \\ 0.3 \\ 0.4 \end{pmatrix},$$

$$C=\begin{pmatrix} 0.6 & 0.3 & 0.8 \\ 0.2 & 0.1 & 0.4 \\ 0.5 & 0.6 & 0.7 \\ 0.4 & 0.3 & 0.2 \end{pmatrix}, E=\begin{pmatrix} 0.2 & 0.4 & 0.5 & 0.3 \\ 1 & 0.6 & 0.3 & 0.5 \end{pmatrix}, f^1=\begin{pmatrix} 0.3 \\ 0.2 \\ 0.4 \\ 0.5 \end{pmatrix}, f^2=\begin{pmatrix} 0.45 \\ 0.2 \end{pmatrix}.$$

Column numbers of matrices A, B, C, and E are 1,4,5; 2,3,6; 1,2,6; and 3,4,5,6, respectively.
Step1. The maximum solution is as: $\overline{x}=[0.5, 0.6667, 0.2, 0.2308, 0.375, 0.25]^T$.
Step2. The characteristic matrices R^A *and* R^B *are as follows:*

$$R^A=\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \text{ and } R^B=\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Step3. Since each row of matrices R^A *and* R^B *has at least one non-zero component, the feasible domain of the problem is not empty.*
Step4. Sub-problem 1: $\min\{x_1^3 \times x_2^{0.2} \times x_5^{0.5} | x \in S(A,B,C,E,d^1,d^2,f^1,f^2)\}$ *and sub-problem 2:* $\min\{x_3^{-4} \times x_4^{-1} \times x_6^{-2} | x \in S(C,E,f^1,f^2)\}$.
Step5. $x_3^*=\overline{x}_3=0.2$, $x_4^*=\overline{x}_4=0.2308$, *and* $x_6^*=\overline{x}_6=0.25$. *Remove the columns 3,4, and 6 from* R^A *and* R^B.
Steps 6 and 7. The conditions of Theorem 5.2 are only hold for this problem. Then we have: $x_2^*=\max\{0.3103, 0.375, 0.4444\}=0.4444$. *Remove column 2 from updated matrix* R^B. *Hence* $R^B=\emptyset$.
Step8. $P^A=\{(1,5,1),(1,5,5),(5,5,1),(5,5,5)\}$. *The* x^p*'s corresponding to them are as:* $X^{P^A}=\{(0.5,0.3125),(0.3,0.3125),(0.3529,0.5),(0,0.3529)\}$. *Hence,* $x_1^*=0.3529$ *and* $x_5^*=0$. *Therefore, we conclude that* $x^*=[0.3529, 0.6667, 0.2, 0.2308, 0, 0.25]^T$ *with* $z^*=0$.

7. Conclusions

The geometric programming problem subject to MFRI with two operators of max-product and max-Hamacher composition is studied in this paper. The structure of feasible domain was completely determined by its unique maximal solution and minimal solutions. Then the resolution process of the problem was expressed. Moreover, some sufficient conditions were presented to simplify the problem. Finally, an algorithm was designed to solve the problem. With regard to the structure of MFRI system, the procedures are not only rules to simplify the problem. Due to NP-hardness of the problem, each simplification in this area can be very important. Obtaining other procedures in this area can be considered as a research work in future.

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