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NUMERICAL SOLUTION OF TIME-FRACTIONAL ORDER FOKKER-PLANCK EQUATION

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Abstract. In this article, new iterative method (NIM) is employed to find the numerical solution of linear and nonlinear time-fractional order Fokker-Planck equation (FPE), which is applied in many fields of engineering and applied science. The introduced technique renders an analytical solution in the form of a convergent series with easily computable components without using any restrictive assumptions. Three numerical examples are tested using this method. Plotted graph illustrate the efficiency and accuracy of the proposed method.

Keywords: Fokker-Planck equation; New iterative method (NIM); Analytic solution, Caputo fractional derivative.

AMS Subject Classification: 74S30.

1. Introduction

In day-today life fractional calculus has become an integral part due to its application in practical life. There are near about 300 years when many mathematicians and people from other fields started working for the development of this branch due to of its applications in many field of applied sciences and engineering. Fractional differential equations are the generalized form of classical differential equations of integer order through the applications of fractional calculus. The non-local property of fractional differential equations is the main advantage in forming numerous mathematical models as the subsequent phase of the system depends not only upon its current state but also upon all of its proceeding states. It is very difficult to find the analytic solution of a fractional differential equations. So many techniques like variational iteration method [1-7], differential transform method [8], homotopy peturbation transform method [9], homotopy analysis transform method [10-12], q-homotopy analysis transform method (q-HATM) [13-15] and local fractional homotopy perturbation Sumudu transform algorithm (LFHPSTA)[16] are developed by many peoples to find numerical solution of fractional differential equations due of their importance in many field like anomalous diffusion, viscoelastic materials, polymers, control

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theory, signal theory, viscoelastic materials, finance and many other fields of science and engineering.

In this paper, New iterative method (NIM) is applied to find the numerical solution of time- fractional Fokker-Planck equations. Recently, Daftardar-Gejji, Jafari and others [17-22] proposed a new iterative method. NIM is applied to find the numerical solution of ordinary, partial differential equations and fractional order differential equations. Bhalekar and Daftardar-Gejji applied new iterative method to find numerical solution of fractional evolution equations [23].

An important role of Fokker Plank equation is to represent the Brownian motion of particles. FPE characterize the variety of probability of a random function of time and space variable. So we can say that FPE is used to characterize the solute transport. Generalized form of the FPE to describe the motion of a concentration field g(x,t) of time variable t and space variable x is as [24,25]

$$\frac{\partial g}{\partial t} = [-\frac{\partial}{\partial x} L(x,t,g) + \frac{\partial^2}{\partial x^2} K(x,t,g)]g(x,t), \tag{1}$$

with given condition $g(x,0) = h(x),\ x \in R$, where $L > 0$ is the coefficient of drift and $K > 0$ is the coefficient of diffusion which may depend on time. The more general form of FPE is its nonlinear form which play an important role in many nonlinear phenomena like neuroscience, population dynamics, plasma physics, engineering, biophysics and surface physics.

In this article our aim is to use new iterative method to solve time fractional FPE

$$D_t^\alpha g = [-D_x L + D_x^2 K]g,\ 0 < \alpha \leq 1, \tag{2}$$

where α is the order of time fractional derivative, $g(x,t)$ vanishes for negative value of x and t when $\alpha = 1$, it becomes classical nonlinear FPE. Due to broad applications of the fractional Fokker-Planck equations, it has been studied by several researchers in [26-40]. Analytic solutions of fractional order FPE has been find out by different techniques such as iterative Laplace transform method [26], Homotopy perturbation method (HPM) [39], homotopy perturbation transform method (HPTM) [40], finite element method [41] and residual power series method(RPSM) [42].

2. Basic definition of fractional calculus

Definition 2.1. A real function $f(t),\ t > 0$ is said to be in the space $C_\alpha,\ \alpha \in R$ if there exists a real number $p(> \alpha)$, such that $f(t) = t^p f_1(t)$, where $f_1 \in C[0, \infty]$ clearly $C_\alpha \subset C_\beta$ if $\beta \leq \alpha[37-39]$ and is said to be in the space $C_\alpha^m,\ m \in N \cup \{0\}$, if $f^{(m)} \in C_\alpha[43-45]$.

Definition 2.2. The left sided Riemann-Liouville fractional integral [43-45] of order $\alpha > 0$ of a function $f \in C_\mu$, $\mu \geq -1$ is defined as:

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}}\, d\tau \text{ where } I^0 f(t) = f(t).$$

Definition 2.3. The left sided Caputo fractional derivative of $f, f \in C_{-1}^m, m \in \mathrm{IN} \cup \{0\}$ [43-45],

$$D_t^\mu f(t) = \begin{cases} I^{m-\mu} f^{(m)}(t)\ , & m-1 < \mu < m,\ m \in N \\ \frac{d^m(f(t))}{dt^m}\ , & \mu = m. \end{cases}$$

Definition 2.4. The Mittag-Leffler function $E_\alpha(z)$ with $\alpha > 0$ is defined by the following series representation, valid in the whole complex plane [43-45]:

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(1+\alpha n)}, \alpha > 0, z \in C.$$

3. Description of New iterative method

Daftardar-Gejji and Jafari [17-21] have considered the following functional equation $g(t) = h + M(g)$, where M is nonlinear function defined from Banach space $A \to A$, h is a known function . Our goal is to solve $g(t)$ in the form of a series as $g(t) = \sum_{k=0}^{\infty} g_k(t)$. So nonlinear operator can be written as

$$M(\sum_{k=0}^{\infty} g_k(t)) = M(g_0) + \sum_{k=1}^{\infty}\{M(\sum_{i=0}^{k} g_i) - M(\sum_{i=0}^{k-1} g_i)\}, so$$

$$\sum_{k=0}^{\infty} g_k = h + M(g_0) + \sum_{k=1}^{\infty}\{M(\sum_{i=0}^{k} g_i) - M(\sum_{i=0}^{k-1} g_i)\}.$$

We can define iterations as

$$\begin{cases} g_0 = h, \\ g_1 = M(g_0), \\ g_{k+1} = M(g_0 + ... + g_k) - M(g_0 + ... + g_{k-1}). \end{cases} \tag{3}$$

Then

$(g_1 + ... + g_{k+1}) = M(g_1 + ...g_k)$, if M is contraction mapping,
i.e. $||M(x) - M(y)|| \leq p||x - y||, 0 < p < 1$, then $||g_{k+1}|| \leq p^n||g_0||$ and then $\sum_{k=0}^{\infty} g_k(t)$ is absolutely and uniformly convergent series.
Now our aim is to solve time fractional FPE

$$D_t^\alpha g = [-D_x L + D_x^2 K]g, 0 < \alpha \leq 1, g(x,0) = h(x), so$$

$$g_0 = h(x), g_1 = M(g_0) = I_t^\alpha([-D_x L + D_x^2 M]g_0),$$

$$g_{k+1} = I_t^\alpha([-D_x L + D_x^2 M](g_0 + ... + g_k)) - I_t^\alpha([-D_x L + D_x^2 M](g_0 + ... + g_{k-1})),$$

$$|g_k| \leq |h(x)|\,|t^{k\alpha}/\Gamma(1+k\alpha)|.$$

Thus

$$\sum_{k=0}^{\infty} g_k \leq \sum_{k=0}^{\infty} |g_k| \leq \sum_{k=0}^{\infty} \left|\frac{t^{k\alpha}}{\Gamma(1+k\alpha)}\right| \leq |h(x)| \sum_{n=0}^{\infty} \left|\frac{t^{k\alpha}}{\Gamma(1+k\alpha)}\right|.$$

Now RHS is absolutely convergent series and then by comparison test LHS is also convergent. Hence we can conclude that solution of the FPE $g(x,t) = h(x)E_\alpha(t^\alpha)$, where $E_\alpha(t^\alpha)$ is Mittag- Leffler function, is convergent.

4. Numerical Examples

In this section, new iterative method is illustrated by some examples.
Example 4.1. We consider the following linear time- fractional Fokker-Planck equation

$$\frac{\partial^\alpha g}{\partial t^\alpha} = -\frac{\partial(xg)}{\partial x} + \frac{\partial^2(\frac{x^2 g}{2})}{\partial x^2},\ x,\ t > 0,\ 0 < \alpha \leq 1,\ g(x,0) = 0. \tag{4}$$

The exact solution of the problem for $\alpha = 1$ is given as $g(x,t) = x\,e^t$.
With the help of given condition, we assume $g_0(x,t) = x$.

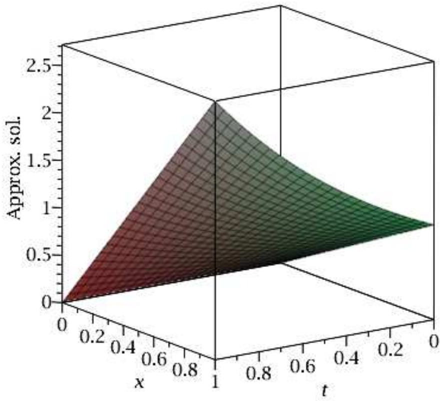

FIGURE 1. Approx. sol. for Ex. 4.1.

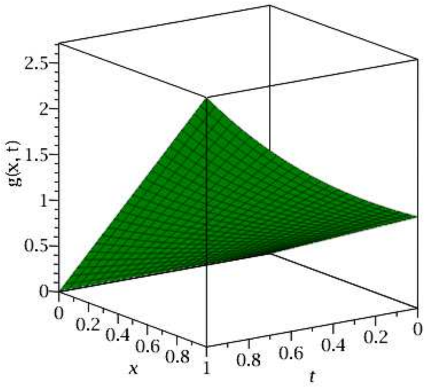

FIGURE 2. Exact sol. for Ex. 4.1.

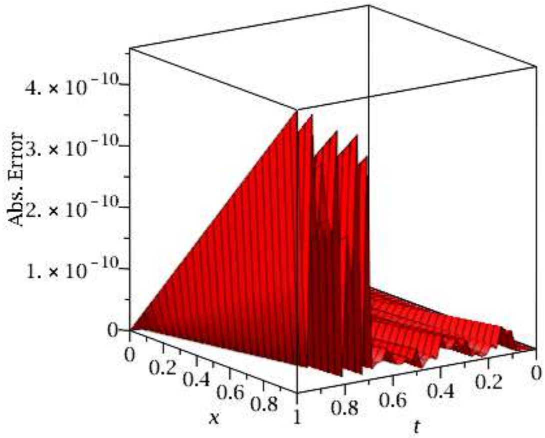

FIGURE 3. Abs. error = $|g_{exa.} - g_{app.}|_{\alpha=1}$ for Ex. 4.1..

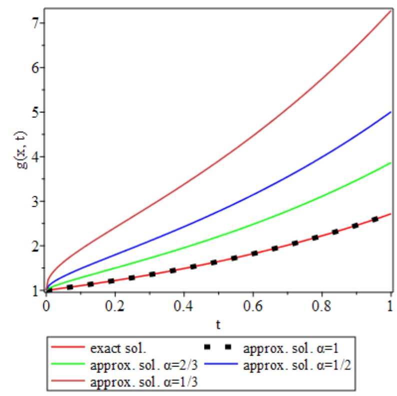

FIGURE 4. Comparison between exact and approx. sol. for Ex. 4.1.

Then applying new iterative method, we get

$g_1(x,t) = M(g_0(x,t)) = I_t^{\alpha}[-\frac{\partial(x\, g_0(x,t))}{\partial x} + \frac{\partial^2(\frac{x^2}{2}\ g_0(x,t))}{\partial x^2}] = x\frac{t^{\alpha}}{\Gamma(1+\alpha)}$,

$g_2(x,t) = M\{g_0(x,t) + g_1(x,t)\} - M\{g_0(x,t)\} = x\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}$,

$g_n(x,t) = M\{g_0(x,t) + ... + g_{n-1}(x,t)\} - M\{g_0(x,t) + ... + g_{n-2}(x,t)\} = x\frac{t^{n\alpha}}{\Gamma(1+n\alpha)}$.

Thus $g(x,t) = \sum_{n=0}^{\infty} g_n(x,t) = x\, E_{\alpha}(t^{\alpha})$, thus when $\alpha = 1$, it becomes $g(x,t) = x\, e^t$, which is the exact solution of classical Fokker Planck equation (4).

Example 4.2. Consider the linear time-fractional Fokker -Planck equation as follows:

$$\frac{\partial^{\alpha} g}{\partial t^{\alpha}} = -\frac{\partial(\frac{x\, g}{6})}{\partial x} + \frac{\partial^2(\frac{x^2\, g}{12})}{\partial x^2},\ x,\ t > 0,\ 0 < \alpha \leq 1,\ g(x,0) = x^2. \qquad (5)$$

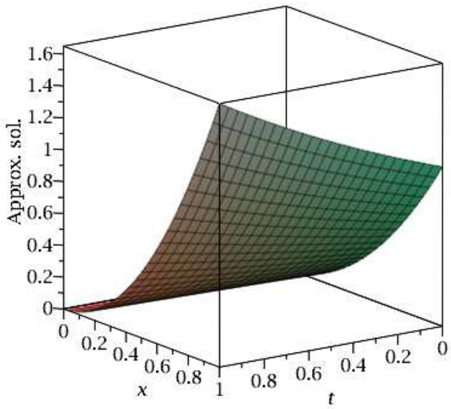

FIGURE 5. Approx. solution. for Ex. 4.2.

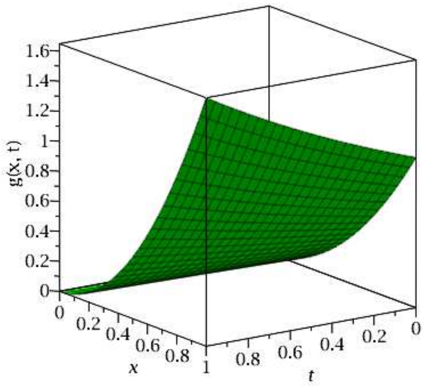

FIGURE 6. Exact solution. for Ex. 4.2.

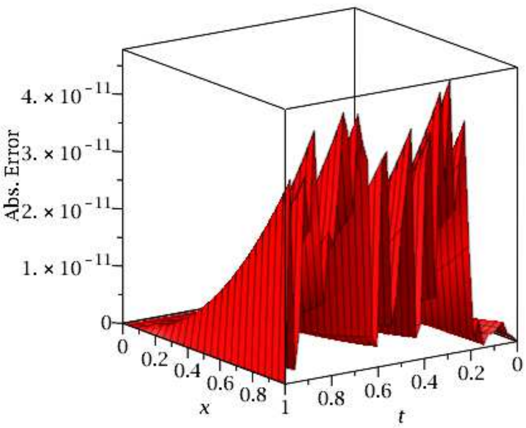

FIGURE 7. Abs. error $= |g_{exa.} - g_{app.}|_{\alpha=1}$. for Ex. 4.2.

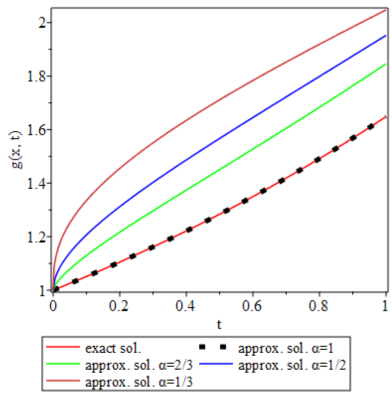

FIGURE 8. Comp. between exact and apprx. sol. for Ex. 4.2.

The exact solution of the problem for $\alpha = 1$ is given as $g(x,t) = x^2 e^{t/2}$.
With the help of given condition, we assume $g_0(x,t) = x^2$.
Then applying new iterative method, we get

$g_1(x,t) = M(g_0(x,t)) = I_t^{\alpha}[-\frac{\partial(\frac{x\, g_0(x,t)}{6})}{\partial x} + \frac{\partial^2(\frac{x^2 g_0(x,t)}{12})}{\partial x^2}] = \frac{1}{2}x^2\frac{t^{\alpha}}{\Gamma(1+\alpha)}$,

$g_2(x,t) = M\{g_0(x,t) + g_1(x,t)\} - M\{g_0(x,t)\} = (\frac{1}{2})^2 x^2 \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}$,

$g_n(x,t) = M\{g_0(x,t) + ... + g_{n-1}(x,t)\} - M\{g_0(x,t) + ... + g_{n-2}(x,t)\} = (\frac{1}{2})^n x^2 \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}$.

Thus $g(x,t) = \sum_{n=0}^{\infty} g_n(x,t) = x^2 E_{\alpha}(\frac{t^{\alpha}}{2})$, thus when $\alpha = 1$, it becomes $g(x,t) = x^2 e^{t/2}$, which is the exact solution of classical Fokker Planck equation (5).

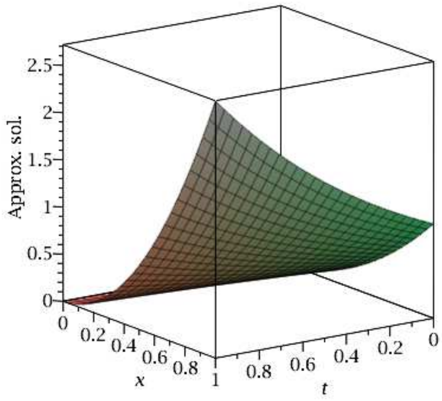

FIGURE 9. Approx. sol. for Ex. 4.3.

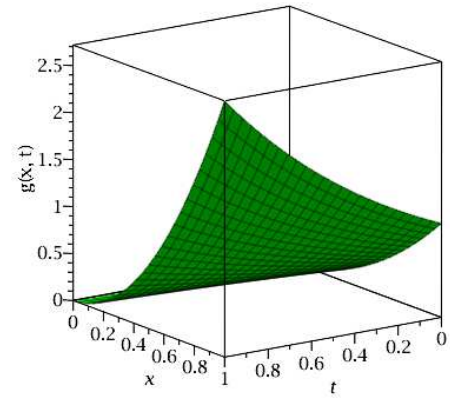

FIGURE 10. Exact sol. for Ex. 4.3.

Example 4.3. Finally, consider the nonlinear time-fractional Fokker-Planck equation

$$\frac{\partial^\alpha g}{\partial t^\alpha} = -\frac{\partial(\frac{4g^2}{x} - \frac{gx}{3})}{\partial x} + \frac{\partial^2 g^2}{\partial x^2},\ x,\ t > 0,\ 0 < \alpha \leq 1,\ g(x,0) = x^2. \quad (6)$$

The exact solution of the problem for $\alpha = 1$ is given as $g(x,t) = x^2 e^t$.
With the help of given conditions, we assume $g_0(x,t) = x^2$.
Then applying new iterative method, we get

$g_1(x,t) = M(g_0(x,t)) = I_t^\alpha[-\frac{\partial(4\frac{g_0^2(x,t)}{x} - x\frac{g_0(x,t)}{3})}{\partial x} + \frac{\partial^2(g_0^2(x,t))}{\partial x^2}] = x^2 \frac{t^\alpha}{\Gamma(1+\alpha)}$,
$g_2(x,t) = M\{g_0(x,t) + g_1(x,t)\} - M\{g_0(x,t)\} = x^2 \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}$,
$g_n(x,t) = M\{g_0(x,t) + ... + g_{n-1}(x,t)\} - M\{g_0(x,t) + ... + g_{n-2}(x,t)\} = x^2 \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}$,
Thus $g(x,t) = \sum_{n=0}^{\infty} g_n(x,t) = x^2 E_\alpha(t^\alpha)$, thus when $\alpha = 1$, it becomes $g(x,t) = x^2 e^t$, which is the exact solution of classical Fokker Planck equation (6).

5. Numerical results and discussion

Figures $1-2$, $5-6$ and $9-10$ shows the plot of approximate solution and exact solution. From these we observe that exact solution and approximate solution obtained by new iterative method (NIM) is same. Figures $3, 7$ and 11 denotes the absolute error between exact solution and approximate solution for $\alpha = 1$ which is negligible. Figures $4, 8$ and 12 shows the comparison between exact and approximate solution for $\alpha = 1$ at $x = 1$ and plot of approximate solution for different values of $\alpha = 2/3, 1/2$ and $1/3$. We conclude that g is strictly increasing for different values of t between 0 to 1 also g is strictly increasing as α is decreasing from 1 to 0.

6. Conclusion

In the present work, a new iterative method (NIM) is applied to obtain the approximate analytic solution of linear and nonlinear time fractional Fokker-Planck equation. We obtain high accuracy solution without using any restrictive assumptions like adomain polynomial, perturbed parameter, He's polynomial, Lagrange's multiplier etc. It is very

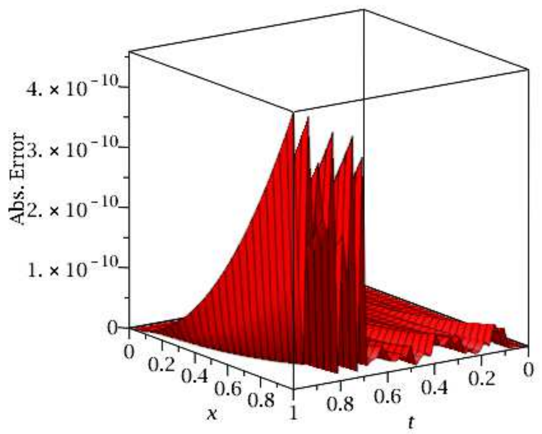

FIGURE 11. Abs. error = $|g_{exa.} - g_{app.}|_{\alpha=1}$ for Ex. 4.3.

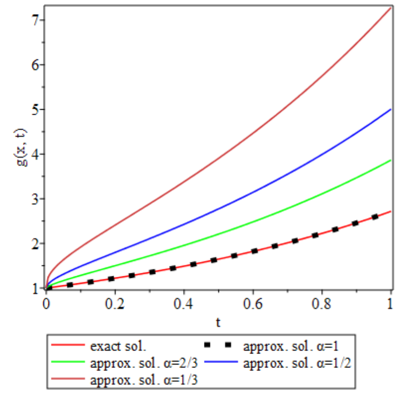

FIGURE 12. Comp. between exact and apprx. sol. for Ex. 4.3.

simple in use but of high accuracy method for the numerical solution of linear and non-linear fractional partial differential equation.

7. Acnowledgement

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