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GENERALIZATION OF SOME INEQUALITIES FOR THE POLAR DERIVATIVE OF POLYNOMIALS WITH RESTRICTED ZEROS

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Abstract. If $p(z)$ is a polynomial of degree n, then Govil [N. K. Govil, *Some inequalities for derivative of polynomials*, J. Approx. Theory, **66** (1991) 29-35.] proved that if $p(z)$ has all its zeros in $|z| \leq k,\ (k \geq 1)$, then

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{1+k^n} \left\{ \max_{|z|=1} |p(z)| + \min_{|z|=k} |p(z)| \right\}.$$

In this article, we obtain a generalization of above inequality for the polar derivative of a polynomial. Also we extend some inequalities for a polynomial of the form $p(z) = z^s \left(a_0 + \sum_{\nu=t}^{n-s} a_\nu z^\nu \right),\ t \geq 1,\ 0 \leq s \leq n-1$, which having no zeros in $|z| < k,\ k \geq 1$ except s-fold zeros at the origin.

Keywords: Polynomial, Inequality, Maximum modulus, Polar Derivative, Restricted Zeros.

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1. Introduction

Let $p(z)$ be a polynomial of degree n, then according to Bernstein's inequality on the derivative of a polynomial, we have

$$\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)|, \tag{1}$$

equality holds in (1) if $p(z)$ has all its zeros at the origin.
The inequality (1) can be sharpened, if we restrict ourselves to the class of polynomials having no zeros in $|z| < 1$, in fact, P. Erdös conjectured and later Lax [10] proved that if $p(z) \neq 0$ in $|z| < 1$, then (1) can be replaced by

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{2}$$

The result is best possible and equality holds in (2) for a polynomial which has all its zeros on $|z| = 1$.

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If the polynomial $p(z)$ has all its zeros in $|z| \leq 1$, then it was proved by Turán [13] that

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{2} \max_{|z|=1} |p(z)|, \tag{3}$$

with equality for those polynomials, which have all their zeros on $|z| = 1$.
For a polynomial $p(z)$ of degree n which having no zeros in $|z| < k,\ k \geq 1$, inequality (2) was generalized by Malik [11] who proved that

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{1+k} \max_{|z|=1} |p(z)|. \tag{4}$$

The inequality (4) is sharp and equality holds for $p(z) = (z+k)^n$.
As a generalization of (4), Aziz and Shah [2] proved that if $p(z)$ has no zero in $|z| < k,\ k \geq 1$, except s-fold zeros at the origin, then

$$\max_{|z|=1} |p'(z)| \leq \frac{n+ks}{1+k} \max_{|z|=1} |p(z)|\,. \tag{5}$$

If the polynomial $p(z)$ has all its zeros in $|z| \leq k, k \geq 1$, then it was proved by Govil [9] that

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{1+k^n} \left\{ \max_{|z|=1} |p(z)| + \min_{|z|=k} |p(z)| \right\}. \tag{6}$$

The result is best possible and equality holds in (6) for $p(z) = z^n + k^n$.
Gardner et al. [8] proved that if the polynomial $p(z) = a_0 + \sum_{\nu=t}^{n} a_\nu z^\nu$, having no zeros in $|z| < k,\ k \geq 1$, then

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{1+s_0} \left\{ \max_{|z|=1} |p(z)| - \min_{|z|=k} |p(z)| \right\}, \tag{7}$$

where $s_0 = k^{t+1} \left\{ \frac{(\frac{t}{n})\frac{|a_t|}{|a_0|-m}k^{t-1}+1}{(\frac{t}{n})\frac{|a_t|}{|a_0|-m}k^{t+1}+1} \right\}$.

Let α be a complex number. For a polynomial $p(z)$ of degree n, $D_\alpha p(z)$, the polar derivative of $p(z)$ is defined [12] as

$$D_\alpha p(z) = np(z) + (\alpha - z)p'(z).$$

It is easy to see that

$$\lim_{|\alpha|\to\infty} [\frac{D_\alpha p(z)}{\alpha}] = p'(z).$$

In order to extend inequality (6) for the polar derivative, Aziz and Rather[1] proved that if $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k$ where $k \geq 1$, then for every real or complex number α with $|\alpha| \geq k$,

$$\max_{|z|=1} |D_\alpha p(z)| \geq \frac{n(|\alpha|-k)}{1+k^n} \max_{|z|=1} |p(z)|. \tag{8}$$

As a refinement and generalization of inequality (8), Dewan et al. [7], proved that if $p(z)$ is a polynomial of degree n, which has all its zeros in $|z| \leq k$, where $k \geq 1$, with s-fold zeros at the origin, where $0 \leq s \leq n$, then for every real or complex number α with $|\alpha| \geq k$

$$\max_{|z|=1} |D_\alpha p(z)| \geq \frac{n}{1+k^{n-s}} \left\{ (|\alpha|-k) \max_{|z|=1} |p(z)| + (\frac{|\alpha|}{k^s} + \frac{1}{k^{n-1}}) \min_{|z|=k} |p(z)| \right\}. \tag{9}$$

Also inequality (7) extended by Dewan et al. [6] for the polar derivative of a polynomial. They proved that if $p(z) = a_0 + \sum_{\nu=t}^{n} a_\nu z^\nu$, is a polynomial of degree n which does not vanish in $|z| < k,\ k \geq 1$, then for $|\alpha| \geq 1$

$$\max_{|z|=1} |D_\alpha p(z)| \leq \frac{n}{1+s_0} \left\{ (|\alpha|+s_0) \max_{|z|=1} |p(z)| - (|\alpha|-1) \min_{|z|=k} |p(z)| \right\}, \tag{10}$$

where s_0 is as defined in (7).

The following result, propose a refinement to inequality (9). In a precise set up, we have

Theorem 1. Let $p(z)=\sum_{\nu=0}^{n} a_\nu z^\nu$, be a polynomial of degree $n\geq 3$, which has all its zeros in $|z|\leq k,\ k\geq 1$ with s-fold zeros at the origin $(0\leq s\leq n-3)$, then for every real or complex number α with $|\alpha|\geq k$

$$\begin{aligned}\max_{|z|=1}|D_\alpha p(z)| &\geq \frac{n}{1+k^{n-s}}\left\{(|\alpha|-k)\max_{|z|=1}|p(z)|+\left(\frac{|\alpha|}{k^s}+\frac{1}{k^{n-1}}\right)\min_{|z|=k}|p(z)|\right\}+\\ &+\frac{1}{k^{n-1}}\left\{\frac{2(k^{n-1}-1)}{(n+1)}|na_0+\alpha a_1|+\left(\frac{k^{n-1}-1}{n-1}-\frac{k^{n-3}-1}{n-3}\right)|(n-1)a_1+2\alpha a_2|\right\}\\ &+\frac{2n(|\alpha|-k)}{k(1+k^{n-s})}\left\{\frac{|a_{n-1}|}{(n-s+1)}\left[\frac{k^{n-s}-1}{n-s}-(k-1)\right]+\frac{|a_{n-2}|}{k}\times\right.\\ &\left.\left[\frac{(k^{n-s}-1)-(n-s)(k-1)}{(n-s)(n-s-1)}-\frac{(k^{n-s-2}-1)-(n-s-2)(k-1)}{(n-s-2)(n-s-3)}\right]\right\}\end{aligned}\tag{11}$$

for $n>3$
and

$$\begin{aligned}\max_{|z|=1}|D_\alpha p(z)| &\geq \frac{n}{1+k^{n-s}}\left\{(|\alpha|-k)\max_{|z|=1}|p(z)|+\left(\frac{|\alpha|}{k^s}+\frac{1}{k^{n-1}}\right)\min_{|z|=k}|p(z)|\right\}+\\ &+\frac{k-1}{2k^{n-1}}\left\{(k+1)|na_0+\alpha a_1|+(k-1)|(n-1)a_1+2\alpha a_2|\right\}\\ &+\frac{2n(|\alpha|-k)}{k(1+k^{n-s})}\left\{\frac{|a_{n-1}|}{(n-s+1)}\left[\frac{k^{n-s}-1}{n-s}-(k-1)\right]+\frac{|a_{n-2}|(k-1)^{n-s}}{k(n-s)(n-s-1)}\right\}.\end{aligned}\tag{12}$$

for $n=3$.

Dividing both sides of the inequalities (11) and (12) by $|\alpha|$ and letting $|\alpha|\to\infty$, we have the following refinement of the inequality (6).

Corollary 1.1. *If $p(z)=\sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n, having all its zeros in $|z|\leq k,\ k\geq 1$ with s-fold zeros at the origin, then*

$$\begin{aligned}\max_{|z|=1}|p'(z)| &\geq \frac{n}{1+k^{n-s}}\left\{\max_{|z|=1}|p(z)|+\frac{1}{k^s}\min_{|z|=k}|p(z)|\right\}+\\ &+\frac{2}{k^{n-1}}\left\{\frac{(k^{n-1}-1)|a_1|}{n+1}+\left(\frac{k^{n-1}-1}{n-1}-\frac{k^{n-3}-1}{n-3}\right)|a_2|\right\}\\ &+\frac{n}{1+k^{n-s}}\left\{\frac{4k^{n-1}|a_{n-1}|}{(n-s+1)}\left[\frac{k^{n-s}-1}{n-s}-(k-1)\right]+\frac{4k^{n-2}|a_{n-2}|}{2k^n}\times\right.\\ &\left.\left[\frac{(k^{n-s}-1)-(n-s)(k-1)}{(n-s)(n-s-1)}-\frac{(k^{n-s-2}-1)-(n-s-2)(k-1)}{(n-s-2)(n-s-3)}\right]\right\}\end{aligned}\tag{13}$$

for $n > 3$
and

$$\begin{aligned}\max_{|z|=1}|p'(z)| \geq & \frac{n}{1+k^{n-s}}\left\{\max_{|z|=1}|p(z)| + \frac{1}{k^s}\min_{|z|=k}|p(z)|\right\} + \\ & + \frac{k-1}{2k^{n-1}}\{(k+1)|a_1| + 2(k-1)|a_2|\} \\ & + \frac{2n}{k(1+k^{n-s})}\left\{\frac{|a_{n-1}|}{(n-s+1)}\left[\frac{k^{n-s}-1}{n-s} - (k-1)\right] + \frac{|a_{n-2}|(k-1)^{n-s}}{k(n-s)(n-s-1)}\right\}.\end{aligned} \tag{14}$$

for $n = 3$.

Next we consider a polynomial of degree n, having no zeros in $|z| < k,\ k \geq 1$, except s-fold zeros at the origin and prove the following generalization of inequality (10) .

Theorem 2. If $p(z) = z^s(a_0 + \sum_{\nu=t}^{n-s} a_\nu z^\nu),\ t \geq 1,\ 0 \leq s \leq n-1$ is a polynomial of degree n having s-fold zeros at the origin and the remaining $(n-s)$ zeros in $|z| \geq k,\ k \geq 1$, then for every α with $|\alpha| \geq 1$, we have

$$\begin{aligned}\max_{|z|=1}|D_\alpha p(z)| \leq & \frac{n(|\alpha| + \Lambda_t) + s\Lambda_t(|\alpha|-1)}{1+\Lambda_t}\max_{|z|=1}|p(z)| \\ & - \frac{(n-s)(|\alpha|-1)}{k^s(1+\Lambda_t)}\min_{|z|=k}|p(z)|,\end{aligned} \tag{15}$$

where $\Lambda_t = k^{t+1}\left\{\frac{(\frac{t}{n-s})\frac{k^s|a_t|}{k^s|a_0|-m}k^{t-1}+1}{(\frac{t}{n-s})\frac{k^s|a_t|}{k^s|a_0|-m}k^{t+1}+1}\right\}$, and $m = \min_{|z|=k}|p(z)|$.

Remark 1.1. *Clearly for $s = 0$, inequality (15) reduce to inequality (10).*

Dividing both side of (15) by $|\alpha|$ and let $|\alpha| \to \infty$, then we have

Corollary 1.2. *If $p(z)$ is a polynomial of degree n having no zero in $|z| < k,\ k \geq 1$ except s-fold zeros at the origin, then*

$$\max_{|z|=1}|p'(z)| \leq \frac{n + s\Lambda_t}{1+\Lambda_t}\max_{|z|=1}|p(z)| - \frac{n-s}{k^s(1+\Lambda_t)}\min_{|z|=k}|p(z)|, \tag{16}$$

where Λ_t is as defined in Theorem 2.

Remark 1.2. *Inequality (16) is a refinement of inequality (5), since by applying Lemma 2.4 for the polynomial $p(z)/z^s$ of degree $(n-s)$, we can conclude that $k \leq \Lambda_t$ which is equivalent to $\frac{n+s\Lambda_t}{1+\Lambda_t} \leq \frac{n+ks}{1+k}$. For $s = 0$ the inequality (16) reduce to inequality (7).*

2. Lemmas

For the proofs of these theorems, we need the following lemmas. The first lemma is due to Aziz and Rather [1].

Lemma 2.1. *If $p(z)$ is a polynomial of degree n, has all its zeros in $|z| \leq 1$, then for every $|\alpha| \geq 1$,*

$$\max_{|z|=1}|D_\alpha p(z)| \geq \frac{n}{2}\{(|\alpha|-1)\max_{|z|=1}|p(z)| + (|\alpha|+1)\min_{|z|=1}|p(z)|\} \tag{17}$$

The following lemma is due to Dewan, Kaur and Mir [4].

Lemma 2.2. *If $p(z)$ is a polynomial of degree n, then for $R \geq 1$,*

$$\max_{|z|=R} |p(z)| \leq R^n \max_{|z|=1} |p(z)| - \frac{2(R^n-1)}{n+2}|p(0)| - \left[\frac{R^n-1}{n} - \frac{R^{n-2}-1}{n-2}\right]|p'(0)| \tag{18}$$

if $n > 2$, and

$$\max_{|z|=R} |p(z)| \leq R^n \max_{|z|=1} |p(z)| - \frac{R-1}{2}[(R+1)|p(0)| + (R-1)|p'(0)|] \tag{19}$$

if $n = 2$.

The following lemma is due to Dewan, Singh and Mir [5].

Lemma 2.3. *If $p(z)$ is a polynomial of degree n, having no zeros in $|z| < 1$, then for $R \geq 1$,*

$$\max_{|z|=R} |p(z)| \leq \frac{R^n+1}{2} \max_{|z|=1} |p(z)| - \frac{R^n-1}{2} \min_{|z|=1} |p(z)| - \frac{2}{n+1}\left[\frac{(R^n-1)}{n} - (R-1)\right]|p'(0)| - \left[\frac{(R^n-1)-n(R-1)}{n(n-1)} - \frac{(R^{n-2}-1)-(n-2)(R-1)}{(n-2)(n-3)}\right]|p''(0)| \tag{20}$$

if $n > 3$, and

$$\max_{|z|=R} |p(z)| \leq \frac{R^n+1}{2} \max_{|z|=1} |p(z)| - \frac{R^n-1}{2} \min_{|z|=1} |p(z)| - \frac{2}{n+1}\left[\frac{R^n-1}{n} - (R-1)\right]|p'(0)| - \frac{(R-1)^n}{n(n-1)}|p''(0)|] \tag{21}$$

if $n = 3$.

The following lemma is due to Gardner, Govil and Weems [8].

Lemma 2.4. *If $p(z) = a_0 + \sum_{\nu=t}^{n} a_\nu z^\nu$, is a polynomial of degree n having no zeros in $|z| < k$, $k \geq 1$, then*

$$k^t \leq s_0, \tag{22}$$

where s_0 is given in (7).

3. Proofs of the theorems

Proof of the Theorem 1. Let $G(z) = p(kz)$. Since $p(z)$ has all its zeros in $|z| \leq k$, then $G(z)$ has all its zeros in $|z| \leq 1$. Now for α, where $|\alpha| \geq k$ and using Lemma 2.1 to the polynomial $G(z)$, we have

$$\max_{|z|=1} |D_{\frac{\alpha}{k}} G(z)| \geq \frac{n}{2}\left\{\left(\frac{|\alpha|}{k} - 1\right) \max_{|z|=1} |G(z)| + \left(\frac{|\alpha|}{k} + 1\right) \min_{|z|=1} |G(z)|\right\}. \tag{23}$$

By replacing $G(z) = p(kz)$ in above inequality we get,

$$\max_{|z|=k} |D_\alpha p(z)| \geq \frac{n}{2}\left\{\left(\frac{|\alpha|-k}{k}\right) \max_{|z|=k} |p(z)| + \left(\frac{|\alpha|+k}{k}\right) \min_{|z|=k} |p(z)|\right\}. \tag{24}$$

Now by using Lemma 2.2 for the polynomial $D_\alpha p(z)$, which is of degree $n-1$, where $n-1>2$, we conclude that

$$\max_{|z|=k}|D_\alpha p(z)| \leq k^{n-1}\max_{|z|=1}|D_\alpha p(z)| - \frac{2(k^{n-1}-1)}{n+1}|na_0+\alpha a_1| - \left[\frac{k^{n-1}-1}{n-1}-\frac{k^{n-3}-1}{n-3}\right]|(n-1)a_1+2\alpha a_2|. \tag{25}$$

By using R.H.S of the inequality (25) for the inequality (24), we have

$$\max_{|z|=1}|D_\alpha p(z)| \geq \frac{n}{2}\left\{\left(\frac{|\alpha|-k}{k^n}\right)\max_{|z|=k}|p(z)| + \left(\frac{|\alpha|+k}{k^n}\right)\min_{|z|=k}|p(z)|\right\} + \frac{2(k^{n-1}-1)}{(n+1)k^{n-1}}|na_0+\alpha a_1| + \frac{1}{k^{n-1}}\left[\frac{k^{n-1}-1}{n-1}-\frac{k^{n-3}-1}{n-3}\right]|(n-1)a_1+2\alpha a_2|. \tag{26}$$

Now by the hypothesis the polynomial $p(z)$ having all zeros in $|z|\leq k,\ k\geq 1$ with s-fold zeros at the origin, therefore $q(z)=z^n p(1/z)$ is a polynomial of degree $(n-s)$ which does not vanish in $|z|<1/k$, where $1/k\leq 1$. By using the variable z/k instead of z, we conclude that the polynomial $q(z/k)$ is a polynomial of degree $(n-s)$, having no zeros in $|z|<1$. Now we can use Lemma 2.3 for the polynomial $q(z/k)$, then we have

$$\max_{|z|=k}|q(z/k)| \leq \frac{k^{n-s}+1}{2}\max_{|z|=1}|q(z/k) - \frac{k^{n-s}-1}{2}\min_{|z|=1}|q(z/k)| - \frac{2|a_{n-1}|}{(n-s+1)k}\left[\frac{k^{n-s}-1}{n-s}-(k-1)\right] - \frac{2|a_{n-2}|}{k^2}\times\left\{\frac{(k^{n-s}-1)-(n-s)(k-1)}{(n-s)(n-s-1)} - \frac{(k^{n-s-2}-1)-(n-s-2)(k-1)}{(n-s-2)(n-s-3)}\right\}. \tag{27}$$

Since $q(z/k)=(z/k)^n p(k/z)$, therefore $\max_{|z|=k}|q(z/k)|=\max_{|z|=1}|p(z)|$, $\max_{|z|=1}|q(z/k)|=(1/k^n)\max_{|z|=k}|p(z)|$, and $\min_{|z|=1}|q(z/k)|=(1/k^n)\min_{|z|=k}|p(z)|$. By replacing these in (27), we have

$$\max_{|z|=1}|p(z)| \leq \frac{k^{n-s}+1}{2k^n}\max_{|z|=k}|p(z) - \frac{k^{n-s}-1}{2k^n}\min_{|z|=k}|p(z)| - \frac{2|a_{n-1}|}{(n-s+1)k}\left\{\frac{k^{n-s}-1}{n-s}-(k-1)\right\} - \frac{2|a_{n-2}|}{k^2}\times\left\{\frac{(k^{n-s}-1)-(n-s)(k-1)}{(n-s)(n-s-1)} - \frac{(k^{n-s-2}-1)-(n-s-2)(k-1)}{(n-s-2)(n-s-3)}\right\}. \tag{28}$$

or

$$\max_{|z|=k}|p(z)| \geq \frac{2k^n}{k^{n-s}+1}\max_{|z|=1}|p(z) + \frac{k^{n-s}-1}{k^{n-s}+1}\min_{|z|=k}|p(z)| + \frac{4k^{n-1}|a_{n-1}|}{(n-s+1)(k^{n-s}+1)}\left[\frac{k^{n-s}-1}{n-s}-(k-1)\right] + \frac{4k^{n-2}|a_{n-2}|}{k^{n-s}+1}\times\left\{\frac{(k^{n-s}-1)-(n-s)(k-1)}{(n-s)(n-s-1)} - \frac{(k^{n-s-2}-1)-(n-s-2)(k-1)}{(n-s-2)(n-s-3)}\right\}. \tag{29}$$

By using the inequality (29) for $\max\limits_{|z|=k}|p(z)|$ in the inequality (26), we have

$$\begin{aligned}\max_{|z|=1}|D_\alpha p(z)| \geq & \frac{n}{1+k^{n-s}}\left\{(|\alpha|-k)\max_{|z|=1}|p(z)| + \left(\frac{|\alpha|}{k^s}+\frac{1}{k^{n-1}}\right)\min_{|z|=k}|p(z)|\right\} + \\ & + \frac{1}{k^{n-1}}\left\{\frac{2(k^{n-1}-1)}{(n+1)}|na_0+\alpha a_1| + \left(\frac{k^{n-1}-1}{n-1}-\frac{k^{n-3}-1}{n-3}\right)|(n-1)a_1+2\alpha a_2|\right\}. \\ & + \frac{2n(|\alpha|-k)}{k(1+k^{n-s})}\left\{\frac{|a_{n-1}|}{(n-s+1)}\left[\frac{k^{n-s}-1}{n-s}-(k-1)\right]+\frac{|a_{n-2}|}{k}\times\right. \\ & \left.\left[\frac{(k^{n-s}-1)-(n-s)(k-1)}{(n-s)(n-s-1)}-\frac{(k^{n-s-2}-1)-(n-s-2)(k-1)}{(n-s-2)(n-s-3)}\right]\right\}.\end{aligned} \tag{30}$$

The inequality completes the proof of the Theorem 1 in the case $n > 3$. For the case $n = 3$, we have the similar proof as above, only it is enough to use inequalities (19) and (21) instead of inequalities (18) and (20), respectively. This completes the proof of Theorem 1. □

Proof of the Theorem 2. Let $p(z) = z^s h(z)$, where $h(z) = a_0 + \sum\limits_{\nu=t}^{n-s} a_\nu z^\nu$, is a polynomial of degree $(n-s)$ having no zeros in $|z| \leq k,\ k \geq 1$. Applying inequality (10) to the polynomial $h(z)$, we get

$$\max_{|z|=1}|D_\alpha h(z)| \leq \frac{n-s}{1+s_0'}\{(|\alpha|+s_0')\max_{|z|=1}|h(z)| - (|\alpha|-1)m'\}, \tag{31}$$

where $s_0' = k^{t+1}\left\{\frac{(\frac{t}{n-s})\frac{|a_t|}{|a_0|-m'}k^{t-1}+1}{(\frac{t}{n-s})\frac{|a_t|}{|a_0|-m'}k^{t+1}+1}\right\}$ and $m' = \min_{|z|=k}|h(z)|$.

On the other hand

$$\begin{aligned}&D_\alpha p(z) = np(z) + (\alpha - z)p'(z) = \\ &nz^s h(z) + (\alpha - z)(sz^{s-1}h(z) + z^s h'(z)) = z^s D_\alpha h(z) + \alpha s z^{s-1} h(z).\end{aligned}$$

Therefore

$$zD_\alpha p(z) = z^{s+1}D_\alpha h(z) + \alpha s p(z).$$

Hence for $|z| = 1$, we have

$$|D_\alpha p(z)| \leq |D_\alpha h(z)| + s|\alpha||p(z)|.$$

which implies

$$\max_{|z|=1}|D_\alpha p(z)| \leq \max_{|z|=1}|D_\alpha h(z)| + s|\alpha|\max_{|z|=1}|p(z)|.$$

Since $\max_{|z|=1}|p(z)| = \max_{|z|=1}|h(z)|$ and $\min_{|z|=k}|h(z)| = \frac{1}{k^s}\min_{|z|=k}|p(z)|$, which on using in (31) gives

$$\begin{aligned}\max_{|z|=1}|D_\alpha p(z)| \leq & \frac{n(|\alpha|+\Lambda_t)+s\Lambda_t(|\alpha|-1)}{1+\Lambda_t}\max_{|z|=1}|p(z)| \\ & - \frac{(n-s)(|\alpha|-1)}{k^s(1+\Lambda_t)}\min_{|z|=k}|p(z)|\},\end{aligned} \tag{32}$$

Hence the proof of Theorem 2 is complete. □

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