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COOPERATIVE TWO-STAGE NETWORK DEA: A GOAL PROGRAMMING APPROACH

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Abstract. In this study, we present a two-stage data envelopment analysis method dealing with efficiency evaluation of decision making units with network structure. The key point of the study is that the suggested model optimizes the Black-Box and stages efficiencies simultaneously with the aim of achieving the smallest possible gap between the aspiration levels.

Keywords: Data envelopment analysis, two-stage efficiency, multi-objective model, goal programming.

AMS Subject Classification: 90BXX, 90B15, 90CXX, 90C15.

1. Introduction

The main part of the paper begins here. Data envelopment analysis (DEA), developed by Charnes et al. (1978), is a mathematical programming approach for analyzing the relative efficiency of peer decision making units (DMUs) that use multiple inputs to produce multiple outputs. Conventional DEA models treated production units only as Black Boxes with no consideration of the internal processes. Whereas, in the real world, by opening the under evaluated DMUs, they may in fact consist of internal units referred to as sub-units. Network DEA is one important branch of DEA that has been an attractive topic in literature. Most network DEA studies have focused on the two-stage structure in which two stages are connected in series. Indeed, the results obtained through two-stage DEA can be extended to more complex cases. Fre (1991) probably for the first time considered the role of sub-units as an effective factor in efficiency measures. Early studies of two-stage networks treated each stage as a DMU and calculated two stage efficiencies independently and separately; e.g. see Seiford and Zhu (1998) and Wang et al. (1997). Kao (2014) expanded DEA networks in the multi-levels model using the series system.. In all of these models, the weight for intermediate values is assumed to be the same, even it is not important to consider the intermediate values as the output of stage 1 or the input of stage 2.

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In two-stage DEA literature, the decomposition approaches, apart from the definition of the stage efficiencies, premise the definition of the overall efficiency of the DMU with a model to decompose the overall efficiency to stage efficiencies. The two basic decomposition methods are the multiplicative method of Kao and Hwang (2008) and the additive method proposed by Chen et al. (2009). However, the shortcoming of these decomposition methods is that the overall efficiency to the stage efficiencies is biased and not unique. To overcome the shortcomings of additive and the multiplicative decomposition methods, Kao et al. (2014) introduced an algorithm based on multi-objective programming. Their method includes solving several mathematical programs in order to assess the independent efficiency score of each stage. Li et al. (2012) introduced a two stage structure by assuming that the inputs to the second stage include both the outputs from the first stage and additional inputs to the second stage. Guo et al. (2016) studied additive efficiency decomposition with varying weights and the impact of weight variation on overall efficiency.

Despotis et al. (2015), based on the decomposition paradigm for the structure introduced by Li et al. (2012), presented a multi-objective programming approach, capable of dealing with general two-stage processes. Subsequently, Despotis et al. (2016) added a second phase to the last model to identify a Pareto optimal solution. It is noteworthy to say that both of these methods are nonlinear programming. Based on two last works, Despotis et al. (2016) introduced a linear programming model dealing with common weights of intermediate measures that unlike previous works do not consider external inputs in the second stage.

In this paper, a two-stage process with additional inputs to the second stage as a special kind of network (introduced by Li et al. (2012)) is considered and a new two-stage network DEA method for assessing the peer-efficiency of stage 1 and stage 2 in DEA using multi-objective programming (MOP) is proposed. To solve the suggested MOP model, a goal programming (GP) method as mentioned by Liu and Peng (2008) is used. According to the GP method, the decision maker is asked to set aspiration levels for each objective function. Following this, deviations from these aspiration levels are minimized as an optimal solution.

The remainder of this paper is structured as follows: Section 2 describes, briefly, some network DEA models. In Section 3, our proposed approach is explained. An application of electricity distribution companies of Iran and computational results are provided to illustrate our purposes in Section 4. Finally, in Section 5, concluding comments are made and future extensions are summarized.

2. Preliminaries

This section reviews two preliminary models dealing with efficiency assessments in two-stage processes.

Consider n DMUs, DMU$_j$ $(j = 1, \dots, n)$, to be evaluated. Fig 1 illustrates two specific types of two-stage network structures.

Supposes DMU$_j$ transforms m external inputs $\boldsymbol{x}_j = (x_{1j}, \dots, x_{mj})$ to s final outputs $\boldsymbol{y}_j = (y_{1j}, \dots, y_{sj})$ via D intermediate measures $\boldsymbol{z}_j = (z_{1j}, \dots, z_{Dj})$. For DMU$_j$, $\boldsymbol{x_j}$ and $\boldsymbol{z_j}$ are inputs and outputs related to the first stage, respectively. Then, $\boldsymbol{z_j}$ becomes the input vector of the second stage and $\boldsymbol{y_j}$ is the second stage's output vector. The network structure depicted in Fig 1.b, as one of the common structures in DEA network literature, allows the second stage to consume L inputs $\boldsymbol{l}_j = (l_{1j}, \dots, l_{Lj})$ from outside in addition to the intermediate measures $\boldsymbol{z}_j$. In this case, according to the basic definition of efficiency, the efficiency score of the first and the second stages of DMU$_o$, $o \in \{1, \dots, n\}$ are

calculated as their weighted outputs over their weighted inputs, as $e_o^1=\frac{\boldsymbol{\psi z_o}}{\boldsymbol{\eta x_o}}$ and $e_o^2=\frac{\boldsymbol{\omega y_o}}{\boldsymbol{\varphi z_o}+\boldsymbol{\mu l_o}}$, respectively,
where $\boldsymbol{\eta}, \boldsymbol{\psi}$, $\boldsymbol{\omega}$ and $\boldsymbol{\mu}$ are the vectors of weights associated with the measures $\boldsymbol{x}$, $\boldsymbol{z}$, $\boldsymbol{y}$ and $\boldsymbol{l}$, respectively. Then, one can evaluate the overall efficiency of the DMU$_o$ through the additive aggregation $e_o^{overall} = \frac{(e_o^1+e_o^2)}{2}$ or the multiplicative aggregation $e_o^{overall} = e_o^1.e_o^2$ of the stage efficiencies. The independent relative efficiency of stage 1 and stage 2, E_o^1

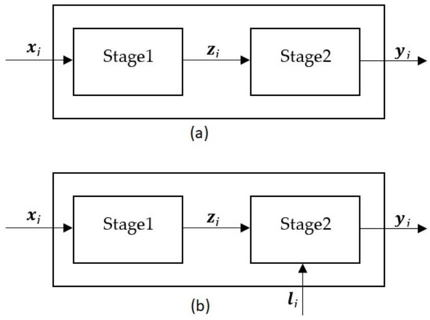

FIGURE 1. Two types of network structure, (a) Two-stage structure of DMU$_j$, (b)Two-stage structure of DMU$_j$ with additional inputs to the second stage.

and E_o^2, under constant returns to scale (CRS), can be obtained by solving the following conventional DEA models that was introduced by Charnes et al. (1978) and named the CCR model, respectively:

$$\begin{aligned} E_o^1=\max \quad & \frac{\sum_{d=1}^{D}\psi_d z_{do}}{\sum_{i=1}^{m}\eta_i x_{io}} \\ \text{s.t.} \quad & \\ & \frac{\sum_{d=1}^{D}\psi_d z_{dj}}{\sum_{i=1}^{m}\eta_i x_{ij}}\leq 1 \qquad \forall j \\ & \psi_d, \eta_i \geq 0 \qquad \forall d, i \end{aligned} \tag{1}$$

$$\begin{aligned} E_o^2= \max \quad & \frac{\sum_{r=1}^{s}\omega_r y_{ro}}{\sum_{d=1}^{D}\psi_d z_{do}+\sum_{k=1}^{L}\mu_k l_{ko}} \\ \text{s.t.} \quad & \\ & \frac{\sum_{r=1}^{s}\omega_r y_{rj}}{\sum_{d=1}^{D}\psi_d z_{dj}+\sum_{k=1}^{L}\mu_k l_{kj}}\leq 1 \qquad \forall j \\ & \psi_d, \mu_k, \omega_\mathrm{r} \geq 0 \qquad \forall d, k, r \end{aligned} \tag{2}$$

It is notable that there are two suitable points of view for calculating the weight vectors $\boldsymbol{\eta}$, $\boldsymbol{\varphi}$, $\boldsymbol{\omega}$ and $\boldsymbol{\mu}$. One is a non-cooperative approach in which overall efficiency assessments are based on a leader and follower concept, (Li et al. (2012)). In this method, one of the stages that is assumed to be more important is treated as the leader and the other as the follower. The other point of view is a cooperative approach that considers stages 1 and 2 together to reach the optimal performance of the overall DMU. For instance, in a company with a two-stage system, the marketing and production departments would cooperate with the aim of maximizing the overall profit. Liang et al. (2008) proposed a centralized approach to address the performance of two-stage network structure described in Fig 1.a. The authors suggested calculating the overall efficiency of the two-stage process as the product of the efficiencies of the two stages. In another work, Li et al. (2012)

extended the centralized method introduced by Liang et al. (2008) for the structure in Fig 1.b. They proposed a heuristic approach based on a parametric linear programing model. Despotis et al. (2015) considered a multi-objective programming framework to evaluate the cooperative efficiencies as follows:

$$\begin{array}{lll}
\max & \frac{\sum_{d=1}^{D}\psi_d z_{do}}{\sum_{i=1}^{m}\eta_i x_{io}} & \\
\max & \frac{\sum_{r=1}^{s}\omega_r y_{ro}}{\sum_{d=1}^{D}\psi_d z_{do}+\sum_{k=1}^{L}\mu_k l_{ko}} & \\
\text{s.t.} & & \\
& \frac{\sum_{d=1}^{D}\psi_d z_{dj}}{\sum_{i=1}^{m}\eta_i x_{ij}}\leq 1 & \forall j \\
& \frac{\sum_{r=1}^{s}\omega_r y_{rj}}{\sum_{d=1}^{D}\psi_d z_{dj}+\sum_{k=1}^{L}\mu_k l_{kj}}\leq 1 & \forall j \\
& \psi_d, \mu_k, \omega_r, \eta_i \geq 0 & \forall d,k,r,i
\end{array} \tag{3}$$

As inspection makes clear, the vector (E_o^1, E_o^2) is the ideal point in the objective functions space of the multi-objective model (3). The authors employed the L_∞ norm to locate the stage efficiencies as close as possible to their ideal values, by minimizing the maximum of the deviations $(E_o^1 - e_o^1)$ and $(E_o^2 - e_o^2)$ from the ideal point $\left(E_o^1, E_o^2\right)$. Accordingly, Despotis et al. (2015) proposed the following model (Despotis model hereafter):

$$\begin{array}{lll}
\min & \delta & \\
\text{s.t.} & & \\
& \sum_{d=1}^{D}\psi_d z_{do}+\delta\geq E_o^1 & \\
& \left(E_o^2-\delta\right)\left(\sum_{d=1}^{D}\psi_d z_{do}+\sum_{k=1}^{L}\mu_k l_{ko}\right)-\sum_{r=1}^{s}\omega_r y_{ro}\leq 0 & \\
& \sum_{i=1}^{m}\eta_i x_{io}=1 & \\
& \sum_{d=1}^{D}\psi_d z_{dj}-\sum_{i=1}^{m}\eta_i x_{ij}\leq 0 & \forall j \\
& \sum_{r=1}^{s}\omega_r y_{rj}-\sum_{d=1}^{D}\psi_d z_{dj}-\sum_{k=1}^{L}\mu_k l_{kj}\leq 0 & \forall j \\
& \psi_d, \mu_k, \omega_r, \eta_i\geq 0 & \forall d,k,r,i
\end{array} \tag{4}$$

where δ denotes the largest deviation. Suppose $(\boldsymbol{\psi}^*, \boldsymbol{\omega}^*, \boldsymbol{\eta}^*, \boldsymbol{\mu}^*, \boldsymbol{\delta}^*) \in \mathbb{R}^{D+s+m+L+1}$ is the optimal solution of model (4), then the efficiency scores for DMU$_o$ in the first and the second stage along with overall efficiency are calculated as follows:

$$e_{Des.,o}^1=\frac{\boldsymbol{\psi}^*\boldsymbol{z}_o}{\boldsymbol{\eta}^*\boldsymbol{x_o}}, \quad e_{Des.,o}^2=\frac{\boldsymbol{\omega}^*\boldsymbol{y}_o}{\boldsymbol{\psi}^*\boldsymbol{z}_o+\boldsymbol{\mu}^*\boldsymbol{l}_o}, \quad e_{Des.,o}^{Overall}=e_{Des.,o}^1 . e_{Des.,o}^2$$

However, model (4) is non-linear and, as it will be illustrated later in section 4, in spite of the notable idea of cooperating state of assessing efficiencies of stages in Despotis model, the results are close to the independent and there is in actuality no cooperative response.

In the next section, model (3) applying a GP approach is linearized at first and then an extension of it as a three-objective model is considered.

3. The proposed cooperative two-stage efficiencies

In this section, we formulate a new model to deal with cooperative efficiency assessments of a two-stage network with additional input to the second stage. This section is divided into two parts; first cooperation between stages to maximize both stage efficiencies is considered and the second subsection regards the cooperation between stages and the Black Box to maximize not only stages efficiencies but also overall efficiency.

3.1. **Developing the proposed model in a Two-objective programming.** Inspired by the procedure used in Liu and Peng (2008) and applying GP method, model (3) can be converted to the following model:

$$
\begin{array}{lll}
\min & \alpha^- + \alpha^+ & \\
\min & \beta^- + \beta^+ & \\
\text{s.t.} & & \\
& \frac{\sum_{d=1}^{D}\psi_d z_{do}}{\sum_{i=1}^{m}\eta_i x_{io}} + \alpha^- - \alpha^+ = A & \qquad (5a) \\
& \frac{\sum_{d=1}^{D}\psi_d z_{dj}}{\sum_{i=1}^{m}\eta_i x_{ij}} \leq 1 \qquad \forall j & \qquad (5b) \\
& \frac{\sum_{r=1}^{s}\omega_r y_{ro}}{\sum_{d=1}^{D}\psi_d z_{do} + \sum_{k=1}^{L}\mu_k l_{ko}} + \beta^- - \beta^+ = B & \qquad (5c) \\
& \frac{\sum_{r=1}^{s}\omega_r y_{rj}}{\sum_{d=1}^{D}\psi_d z_{dj} + \sum_{k=1}^{L}\mu_k l_{kj}} \leq 1 \qquad \forall j & \qquad (5d) \\
& \psi_d, \mu_k, \omega_r, \eta_i \geq \varepsilon \qquad \forall d, k, r, i & \\
& \alpha^-, \alpha^+, \beta^-, \beta^+ \geq 0 &
\end{array}
\qquad (5)
$$

where A and B are the aspiration levels and $\varepsilon > 0$ is a non-Archimedean infinitesimal value. α^-, α^+, β^- and β^+ are goal deviations. According to the definition of relative efficiency, each DMU desires to approach an unity value as the scoring efficiency (i.e. $A = B = 1$). Consequently, α^+ and β^+ as the over-achievement of goals cannot take the positive value and can be omitted. To reach the smallest gap between the aspiration levels (unity value of the efficiency score) the fractions in constraints (5a) and (5c) must be increased by increasing the numerator and/or decreasing the denominator, through adding $\overline{\alpha}^+$ and $\overline{\beta}^+$ to the numerator and taking $\overline{\alpha}^-$ and $\overline{\beta}^-$ away from denominator of these fractions, respectively. Regarding these considerations, model (5) can be written as:

$$
\begin{array}{lll}
\min & \overline{\alpha}^- + \overline{\alpha}^+ & \\
\min & \overline{\beta}^- + \overline{\beta}^+ & \\
\text{s.t.} & & \\
& \frac{\sum_{d=1}^{D}\psi_d z_{do} + \overline{\alpha}^+}{\sum_{i=1}^{m}\eta_i x_{io} - \overline{\alpha}^-} = 1 & \qquad (6a) \\
& \sum_{d=1}^{D}\psi_d z_{dj} - \sum_{i=1}^{m}\eta_i x_{ij} \leq 0 \qquad \forall j & \\
& \frac{\sum_{r=1}^{s}\omega_r y_{ro} + \overline{\beta}^+}{\sum_{d=1}^{D}\psi_d z_{do} + \sum_{k=1}^{L}\mu_k l_{ko} - \overline{\beta}^-} = 1 & \qquad (6b) \\
& \sum_{r=1}^{s}\omega_r y_{rj} - \sum_{d=1}^{D}\psi_d z_{dj} - \sum_{k=1}^{L}\mu_k l_{kj} \leq 0 \qquad \forall j & \\
& \psi_d, \mu_k, \omega_r, \eta_i \geq \varepsilon \qquad \forall d, k, r, i & \\
& \overline{\alpha}^-, \overline{\alpha}^+, \overline{\beta}^-, \overline{\beta}^+ \geq 0 &
\end{array}
\qquad (6)
$$

An easy computation shows that constraints (6a) and (6b) can be rewritten as follows, respectively:

$$
\sum_{d=1}^{D}\psi_d z_{do} - \sum_{i=1}^{m}\eta_i x_{io} + \overline{\alpha}^+ + \overline{\alpha}^- = 0
$$

$$
\sum_{r=1}^{s}\omega_r y_{ro} - \sum_{d=1}^{D}\psi_d z_{do} - \sum_{k=1}^{L}\mu_k l_{ko} + \overline{\beta}^+ + \overline{\beta}^- = 0
$$

Now, by substitution $\overline{\alpha}^{+}+\overline{\alpha}^{-}$ and $\overline{\beta}^{+}+\overline{\beta}^{-}$ with α and β, respectively, and according to the Min-Max formulation, model (6) can be simplified to the following linear model:

$$\begin{array}{lll}
\min & \delta & \\
\text{s.t.} & & \\
& \sum_{d=1}^{D}\psi_d z_{do}-\sum_{i=1}^{m}\eta_i x_{io}+\alpha=0 & \\
& \sum_{d=1}^{D}\psi_d z_{dj}-\sum_{i=1}^{m}\eta_i x_{ij}\leq 0 & \forall j \\
& \sum_{r=1}^{s}\omega_r y_{ro}-\sum_{d=1}^{D}\psi_d z_{do}-\sum_{k=1}^{L}\mu_k l_{ko}+\beta=0 & \\
& \sum_{r=1}^{s}\omega_r y_{rj}-\sum_{d=1}^{D}\psi_d z_{dj}-\sum_{k=1}^{L}\mu_k l_{kj}\leq 0 & \forall j \\
& 0\leq\alpha\leq\delta & \\
& 0\leq\beta\leq\delta & \\
& \psi_d,\mu_k,\omega_{\mathrm{r}},\eta_i\geq\varepsilon & \forall d,k,r,i
\end{array} \tag{7}$$

Theorem 3.1. *Model (7) is a feasible model.*

Proof. Let $\overline{\psi}_d=\varepsilon;\ \forall\ d\in\{1,\dots,D\},\overline{\omega}_r=\varepsilon;\ \forall\ r\in\{1,\dots,s\},\overline{\eta}_i=M;\ \forall\ i\in\{1,\dots,m\}$ and $\overline{\mu}_k=M;\ \forall\ k\in\{1,\dots,L\}$, where M is a big positive number that can be chosen by $M>\max\{\max_j\{\frac{\varepsilon\sum_{d=1}^{D}z_{dj}}{\sum_{i=1}^{m}x_{ij}}\},\ \max_j\{\frac{\varepsilon(\sum_{r=1}^{m}y_{rj}-\sum_{d=1}^{D}z_{dj})}{\sum_{k=1}^{L}l_{kj}}\}\}$. Now let $\overline{\alpha}=M\sum_{i=1}^{m}x_{io}-\varepsilon\sum_{d=1}^{D}z_{do}$, $\overline{\beta}=M\sum_{k=1}^{L}l_{ko}-\varepsilon(\sum_{r=1}^{m}y_{ro}-\sum_{d=1}^{D}z_{do})$ and $\overline{\delta}=\max\{\overline{\alpha},\overline{\beta}\}$. It can be easily seen that $(\overline{\boldsymbol{\psi}},\overline{\boldsymbol{\omega}},\overline{\boldsymbol{\eta}},\overline{\boldsymbol{\mu}},\overline{\alpha},\overline{\beta},\overline{\delta})$ is a feasible solution for model (7). □

Suppose that model (7) is solved and the optimal solution $(\boldsymbol{\psi}^*,\ \boldsymbol{\omega}^*,\ \boldsymbol{\eta}^*,\ \boldsymbol{\mu}^*,\ \alpha^*,\ \beta^*,\ \delta^*)\in\mathbb{R}^{D+s+m+L+3}$ is at hand. Now, considering the constrains of model (7) and the definitions of the efficiencies of the first and second stages as $e_o^1=\frac{\psi\ z_o}{\eta\ x_o}$ and $e_o^2=\frac{\boldsymbol{\omega}\ \boldsymbol{y}_o}{\boldsymbol{\psi}\ \boldsymbol{z}_o+\boldsymbol{\mu}\ \boldsymbol{l}_o}$, we have

$$\begin{array}{lll}
\boldsymbol{\psi}^*\boldsymbol{z}_o+\alpha^*=\boldsymbol{\eta}^*\boldsymbol{x}_o & \Rightarrow & \alpha^*=\boldsymbol{\eta}^*\boldsymbol{x}_o\left(1-e_o^1\right) \\
\boldsymbol{\omega}^*\boldsymbol{y}_o+\beta^*=\boldsymbol{\psi}^*\boldsymbol{z}_o+\boldsymbol{\mu}^*\boldsymbol{l}_o & \Rightarrow & \beta^*=(\boldsymbol{\psi}^*\boldsymbol{z}_o+\boldsymbol{\mu}^*\boldsymbol{l}_o)(1-e_o^2)
\end{array}$$

As a result, through solving model (7), the efficiency score of $\mathrm{DMU_o}$ in stage 1 and stage 2 can be calculated as follows, respectively:

$$\overline{e}_o^1=(1-\frac{\alpha^*}{\boldsymbol{\eta}^*\boldsymbol{x}_o})\ ,\ \overline{e}_o^{2^*}=(1-\frac{\beta^*}{\boldsymbol{\psi}^*\boldsymbol{z}_o+\boldsymbol{\mu}^*\boldsymbol{l}_o})$$

Consequently, the overall efficiency is defined as the product of individual efficiencies of the two stages ($\overline{e}_o^{Overall}=\overline{e}_o^1\ .\overline{e}_o^2$).

Definition 3.1. $\mathrm{DMU_o}$ *is non-dominated if and only if* $\alpha^*+\beta^*=0$ *in the optimal solution of model (7).*

3.2. **Developing the proposed model in a three-objective programming.** This section discusses the efficiency evaluation for each of the two stages, as well as the Black Box efficiency. In particular, by extending the suggested model (7), a new model with three objective functions can be proposed to optimize the efficiency of each of the two stages and the Black Box efficiency simultaneously. To do so, we consider the following basic model for DMU$_o$:

$$\begin{array}{lll}
\max & \left[\frac{\sum_{d=1}^{D}\psi_d z_{do}}{\sum_{i=1}^{m}\eta_i x_{io}},\ \frac{\sum_{r=1}^{s}\omega_r y_{ro}}{\sum_{d=1}^{D}\psi_d z_{do}+\sum_{k=1}^{L}\mu_k l_{ko}},\ \ \frac{\sum_{r=1}^{s}\omega_r y_{ro}}{\sum_{i=1}^{m}\eta_i x_{io}+\sum_{k=1}^{L}\mu_k l_{ko}}\right] & \\
\text{s.t.} & & \\
& \frac{\sum_{d=1}^{D}\psi_d z_{dj}}{\sum_{i=1}^{m}\eta_i x_{ij}}\le 1 & \forall j \\
& \frac{\sum_{r=1}^{s}\omega_r y_{rj}}{\sum_{d=1}^{D}\psi_d z_{dj}+\sum_{k=1}^{L}\mu_k l_{kj}}\le 1 & \forall j \\
& \frac{\sum_{r=1}^{s}\omega_r y_{rj}}{\sum_{i=1}^{m}\eta_i x_{ij}+\sum_{k=1}^{L}\mu_k l_{kj}}\le 1 & \forall j \\
& \psi_d,\mu_k,\omega_{\mathrm{r}},\eta_i\ge\varepsilon & \forall d,k,r,i
\end{array} \tag{8}$$

Theorem 3.2. *Model (8) is a feasible model.*

Proof. Let $\overline{\psi}_d=\varepsilon;\ \forall\ d\in\{1,\ldots,D\}$, $\overline{\omega}_r=\varepsilon;\forall\ r\in\{1,\ldots,s\}$, $\overline{\eta}_i=M;\forall\ i\in\{1,\ldots,m\}$ and $\overline{\mu}_k=M;\forall\ k\in\{1,\ldots,L\}$. Where ε and M are infinitesimal and big positive values, respectively. $(\overline{\boldsymbol{\psi}},\overline{\boldsymbol{\omega}},\overline{\boldsymbol{\eta}},\ \overline{\boldsymbol{\mu}})$ is a feasible solution for model (8). $\square$

Carrying out the same process used in the previous section and introducing the new variables α, β and ρ, model (8) can be represented as the following linear programming:

$$\begin{array}{llll}
\min & \delta & & \\
\text{s.t.} & & & \\
& \sum_{d=1}^{D}\psi_d z_{do}-\sum_{i=1}^{m}\eta_i x_{io}+\alpha=0 & & (9a) \\
& \sum_{d=1}^{D}\psi_d z_{dj}-\sum_{i=1}^{m}\eta_i x_{ij}\le 0 & \forall j & (9b) \\
& \sum_{r=1}^{s}\omega_r y_{ro}-\sum_{d=1}^{D}\psi_d z_{do}-\sum_{k=1}^{L}\mu_k l_{ko}+\beta=0 & & (9c) \\
& \sum_{r=1}^{s}\omega_r y_{rj}-\sum_{d=1}^{D}\psi_d z_{dj}-\sum_{k=1}^{L}\mu_k l_{kj}\le 0 & \forall j & (9d) \\
& \sum_{r=1}^{s}\omega_r y_{ro}-\sum_{i=1}^{m}\eta_i x_{io}-\sum_{k=1}^{L}\mu_k l_{ko}+\rho=0 & & (9e) \\
& \sum_{r=1}^{s}\omega_r y_{rj}-\sum_{i=1}^{m}\eta_i x_{ij}-\sum_{k=1}^{L}\mu_k l_{kj}\le 0 & \forall j & (9f) \\
& 0\le\alpha\le\delta & & (9g) \\
& 0\le\beta\le\delta & & (9h) \\
& 0\le\rho\le\delta & & (9i) \\
& \psi_d,\mu_k,\omega_{\mathrm{r}},\eta_i\ge\varepsilon & \forall d,k,r,i &
\end{array} \tag{9}$$

Note that the set of constraint (9f) is redundant due to constraints (9b) and (9d). Also, constraints (9a) and (9c) result $\rho=\alpha+\ \beta$ that by which it can be concluded that constraints (9g) and (9h) are redundant. Then model (9) is summarized as the following model:

$$\begin{array}{lll}
\min & \delta & \\
\text{s.t.} & & \\
& \sum_{d=1}^{D}\psi_d z_{do}-\sum_{i=1}^{m}\eta_i x_{io}+\alpha=0 & \\
& \sum_{d=1}^{D}\psi_d z_{dj}-\sum_{i=1}^{m}\eta_i x_{ij}\le 0 & \forall j \\
& \sum_{r=1}^{s}\omega_r y_{ro}-\sum_{d=1}^{D}\psi_d z_{do}-\sum_{k=1}^{L}\mu_k l_{ko}+\beta=0 & \\
& \sum_{r=1}^{s}\omega_r y_{rj}-\sum_{d=1}^{D}\psi_d z_{dj}-\sum_{k=1}^{L}\mu_k l_{kj}\le 0 & \forall j \\
& \alpha+\ \beta\le\delta & \\
& \psi_d,\mu_k,\omega_{\mathrm{r}},\eta_i\ge\varepsilon & \forall d,k,r,i \\
& \alpha,\ \beta\ge 0 &
\end{array} \tag{10}$$

In a glance, it may be thought that models (10) and (7) are the same. But, according to Fig 2, we can conclude that the feasible region of model (10) is a subset of the feasible region of model (7). Where, in Fig 2, the gray square is related to model (7) and the strips triangle is related to model (10). If the vector $(\boldsymbol{\psi}^*,\ \boldsymbol{\omega}^*,\ \boldsymbol{\eta}^*,\ \boldsymbol{\mu}^*,\alpha^*,\ \beta^*,\ \delta^*\ )\in\mathbb{R}^{D+s+m+L+3}$ is the

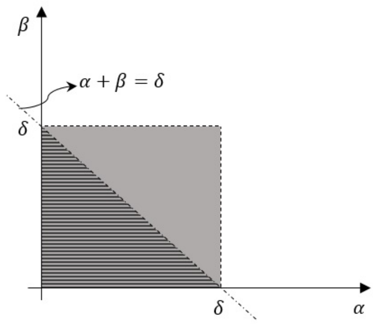

FIGURE 2. Image of feasible regions of model (7) and (10) in the space of variables α and β.

optimal solution of model (10), the efficiency score of each stage, Black Box and overall are calculated through the following equations:

$$\begin{aligned}
&\overline{\overline{e}}_o^1 = (1 - \tfrac{\alpha^*}{\boldsymbol{\eta}^* \boldsymbol{x}_o}) \\
&\overline{\overline{e}}_o^2 = (1 - \tfrac{\beta^*}{\boldsymbol{\psi}^* \boldsymbol{z}_o + \boldsymbol{\mu}^* \boldsymbol{l}_o}) \\
&\overline{\overline{e}}_o^{Black-Box} = \left(1 - \tfrac{\delta^*}{\boldsymbol{\eta}^* \boldsymbol{x}_o + \boldsymbol{\mu}^* \boldsymbol{l}_o}\right) \\
&\overline{\overline{e}}_o^{Overall} = \overline{\overline{e}}_o^1 \times \overline{\overline{e}}_o^1
\end{aligned}$$

Definition 3.2. $\mathrm{DMU_o}$ *is non-dominated in model (10), if and only if* $\delta^* = 0$ *in the optimal solution of model (10).*

In our proposed method, similar to the Despotis method, a multi-objective mathematical program is used in order to simultaneously optimize the efficiencies of sub-units (model (7)), but with the difference that the aspiration levels are considered to be 1. We also formulate another multi-objective model to simultaneously optimize the efficiencies of sub-units along with Black Box efficiency.

Remark 3.1. *The Black Box efficiency score* $\overline{\overline{e}}_j^{Black\ Box}$ *is obtained directly from the optimal solution of model (10), whereas, the overall efficiency* $\overline{\overline{e}}_j^{Overall}$ *is achieved by multiplying* $\overline{\overline{e}}_j^1$ *by* $\overline{\overline{e}}_j^2$.

4. An application in electricity industry

Tavanir[1] Company as an electricity distribution company was founded in 1992 in Iran. This Company was established to undertake the responsibility for the development of electric power generation, transmission facilities and bulk transaction of electricity with the regional electricity companies and large industries.

This section demonstrates an application of the suggested approach in the previous section in performance assessment of electricity distribution companies in Iran. The data and samples are taken from Tavanir's database for the year 2013. Sixteen electricity distribution companies were selected as DMUs in our analysis. According to the collected data, the power plant (first stage) considering the expenses and consuming available power, exports the generated electric to transmission network (second stage). Then the network,

[1]Tavanir is responsible for all activities that are associated with electric power including generation, Transmission, and distribution (www.tavanir.org.ir/en/).

using the imported energy of the system, distributes the electric energy. Table 1 presents the data set. Notice that the data were scaled.
Based upon the data set, five factors related to the performance of electricity distribution companies can be grouped into a two-stage process, as shown in Fig 3 Briefly, the following inputs, intermediate products, and final outputs are considered:
Inputs: Expenses (x_1), Power consumed in the station (x_2),
Intermediate products: Electric scale (z_1),
Additional inputs to the second stage: Imported energy (l_1),
Outputs: Electric distribution (y_1), Electric export (y_2).

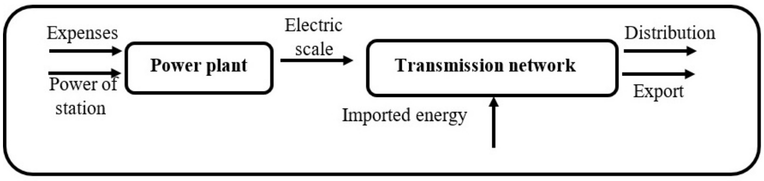

FIGURE 3. Electric Distribution Company's two-stage performance.

We now apply the mentioned assessment models in the paper to the data set and make the comparison between the proposed method (models (7) and (10)), Despotis model (4) and Independent efficiencies models (models (1) and (2)). Table 2 provides Despotis model and independent efficiencies. The efficiency score of each stage, Black Box and overall calculations via models (7) and (10) are summarized in Table 3. Note here that in Tables 2 and 3 the units are ranked based on overall efficiency.

Table 1. Electric companies data in 2013

#	DMU	Inputs			In. value*		Ex. input**	Outputs	
		x_1	x_2		z_1		l_1	y_1	y_2
1	Yazd	0.456	0.583		0.928		0.871	0.559	0.911
2	Hormozgan	0.47	0.456		0.833		0.411	0.523	0.868
3	Mazandaran	0.564	0.573		0.944		0.398	0.858	0.957
4	Gilan	0.562	0.657		0.876		0.467	1	0.865
5	Kerman	0.322	0.734		0.447		0.894	0.711	0.418
6	Fars	0.232	0.565		0.878		0.99	0.567	0.867
7	Gharb	0.354	0.653		0.833		0.57	0.296	0.878
8	Soob	0.431	0.684		0.909		0.998	0.61	0.897
9	Semnan	0.543	0.507		0.389		0.997	0.174	0.353
10	Zanjan	0.656	0.113		0.983		1	0.123	0.993
11	Khoozestan	0.34	1		1		0.61	0.185	1
12	Khorasan	1	0.748		0.834		0.066	0.558	0.809
13	Tehran	0.45	0.416		0.941		0.403	0.268	0.929
14	Bakhtar	0.343	0.426		0.404		0.566	0.652	0.348
15	Isfahan	0.546	0.815		0.962		0.559	0.396	0.98
16	Azerbaijan	0.489	0.560	0.909	0.327	0.279	0.898		

* Intermediate value

** External input

Table 2. Despotis and Independent efficiencies results.

$DMU\#$	Independent efficiencies				Despotis efficiencies			
	models (1) and (2)				model (4)			
	e_j^1	e_j^2	$e_j^{Overall}$	Rank	$e_{Des.,j}^1$	$e_{Des.,j}^2$	$e_{Des.,j}^{Overall}$	Rank
1	0.92	0.943	0.868	14	0.921	0.947	0.872	15
2	0.867	1	0.867	15	0.868	1	0.868	16
3	0.881	1	0.881	11	0.885	1.002	0.887	11
4	0.896	1	0.896	9	0.896	1.009	0.904	9
5	0.927	1	0.927	7	0.936	1.009	0.945	5
6	0.913	0.952	0.87	13	0.917	0.954	0.875	14
7	0.9	1	0.9	8	0.906	1.008	0.913	8
8	0.913	0.954	0.871	12	0.914	0.959	0.877	13
9	1	0.867	0.867	16	1.007	0.87	0.877	12
10	1	0.958	0.958	2	1.006	0.964	0.97	2
11	0.992	0.953	0.945	3	1.001	0.957	0.958	3
12	0.928	1	0.928	6	0.932	1.001	0.934	7
13	0.974	0.958	0.933	5	0.974	0.966	0.941	6
14	1	1	1	1	1	1	1	1
15	0.913	0.972	0.888	10	0.92	0.979	0.901	10
16	0.967	0.97	0.938	4	0.974	0.978	0.952	4

We also include some statistic plots to give a better view of comparisons. Fig 4(I) compares the overall efficiency obtained by independent and Despotis models. As can be seen, the results reveal the similarity of the obtained efficiencies through Despotis with the primary two-stage efficiency assessment models. In other words, Fig 4(I) indicates a small discrepancy graphically between the obtained overall independent and Despotis efficiency. In result, in spite of the notable idea of a cooperating state of assessing efficiencies of stages in Despotis method, the results are so close to the independent and there is no cooperative manner in actuality. This is due to the fact that this method tries to ignore the role of intermediate products in order to fulfill the aspirations; i.e. there is virtually no partnership among sub-units taking place.

Table 3. Resulting Efficiencies for proposed model.

$DMU\#$	model (7)				model (10)				
	$\bar{e}_j^1$	$\bar{e}_j^2$	$\bar{e}_j^{Overall}$	Rank	$\bar{\bar{e}}_j^1$	$\bar{\bar{e}}_j^2$	$\bar{\bar{e}}_j^{BlackBox}$	$\bar{\bar{e}}_j^{Overall}$	Rank
1	0.786	0.625	0.491	14	0.895	0.773	0.708	0.692	13
2	0.736	0.753	0.555	13	0.845	0.918	0.794	0.776	9
3	0.638	0.709	0.453	15	0.842	1	0.857	0.842	4
4	0.777	0.967	0.751	7	0.777	1	0.802	0.777	8
5	0.728	0.89	0.648	8	0.899	1	0.911	0.899	2
6	0.862	0.742	0.639	11	0.88	0.753	0.684	0.663	15
7	0.903	0.906	0.818	3	0.9	0.81	0.741	0.729	12
8	0.856	0.753	0.645	9	0.875	0.764	0.69	0.668	14
9	0.543	0.329	0.178	16	0.902	0.326	0.313	0.294	16
10	1	0.643	0.643	10	1	0.774	0.774	0.774	10
11	0.766	0.728	0.557	12	1	0.831	0.831	0.831	5
12	0.887	0.876	0.777	5	0.887	1	0.89	0.887	3
13	0.983	0.79	0.776	6	0.974	0.805	0.787	0.784	7
14	1	1	1	1	1	1	1	1	1
15	0.913	0.913	0.833	2	0.913	0.821	0.76	0.75	11
16	0.959	0.814	0.781	4	0.967	0.829	0.805	0.802	6

To better exhibit this, Fig 5 includes three figures, 5(I), 5(II) and 5(III), that display the efficiency distribution of stage 1, stage 2 and the overall results from independent models, model (7) and model (10), respectively. By referring to 5(I), for instance, DMU3 (Mazandaran), in spite of being efficient in stage 2, has less efficiency in stage 1 ($\sim$0.88) and the overall rank of this unit is 11. This difference in efficiency scale of stages one and two may result due to the existence of competition between sub-units or due to the lack of sufficient capacities in one of the stages. It seems that the Mazandaran company, due to of several reasons, cannot produce the desired output, but in return, due to the sufficient and available capacities in its transmission unit, the little output produced by the first stage would lead to a very high production in the second stage. Ultimately, this competitive nature of work among sub-units without resulting in improvement in the overall efficiency will result in high rank for this DMU. Returning to the results shown in table 3, there is a kind of cooperation between stages, instead of competition, with the aim of improving the overall efficiency. In other words, comparing Fig 5(I) with Figs 5(II) and 5(III), one can observe a significant increase in the overall efficiency for most DMUs. It should be noted that according to Fig 4(II), there is not much difference between peer-performance Black Box efficiency $\bar{\bar{e}}_j^{Black\ Box}$ and overall efficiency $\bar{\bar{e}}_j^{Overall}$. This shows that model (10) yields peer-performance Black Box efficiency so that we can estimate the overall efficiency without doing the additional operation. It can easily be seen that Figs 5(II) and 5(III) reveal a modification in the efficiency measure of a sub-unit in order to proportionate with the efficiency of another sub-unit by models (7) and (10) in comparison with Fig 5(I).

Consider again DMU_3. Looking at the obtained results for this unit in Tables 3, it can be seen that model (7) yields a fair efficiency score for sub-units. However, this appropriate efficiency between two stages leads to this unit having rank 15. While solving model (10) for DMU_3 leads to not only the proportionate assessment of efficiencies in stages 1 and 2, it also leads to a better rank for this unit of 4.

It is worth stressing that the obtained results after solving model (10) provide important managerial information for performance analysis and improvement. In fact, the causes of inefficiency and shortcomings of inefficient units are easy to identify when taking a managerial perspective on the distribution of efficiencies (overall and sub-unit efficiency) in Fig (7III). For instance, for the inefficient unit DMU_{10} (Zanjan) there is a significant difference between the efficiency score of stages 1 and 2. In this case, given that stage 1 is efficient, and stage 2 is inefficient, the expansion of manufacturing capacity in stage 2 could be a management strategy to improve the overall performance. This means preparing further distribution and export companies as the fundamental solution to improve the performance of the company. Furthermore, it means paving the way of distribution and export as the fundamental solution to improve the overall performance of this company. Now if there is an insignificant difference between the efficiency of the sub-units of an inefficient unit, it can be concluded that the cause of inefficiency is due to the Black Box performance of the evaluated company.

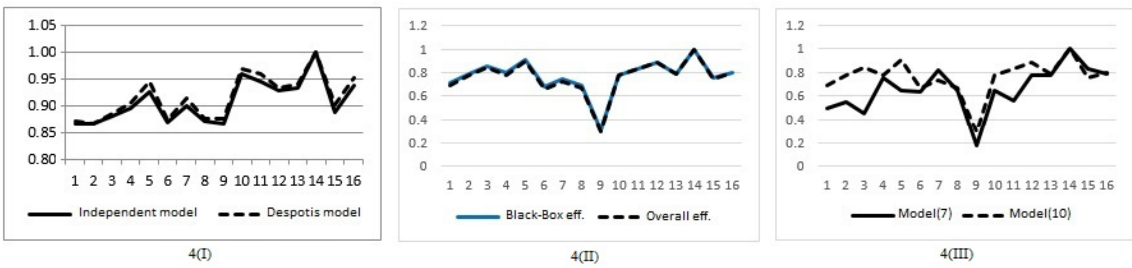

FIGURE 4. 4(I) Comparing the overall efficiency obtained by independent and Despotis models, 4(II) Comparing the results of $\bar{\bar{e}}_j^{BlackBox}$ and $\bar{\bar{e}}_j^{Overall}$ efficiencies, Comparing the results of $\bar{e}_j^{Overall}$ and $\bar{\bar{e}}_j^{Overall}$ efficiencies

5. Conclusion

Conventional DEA models treat each production unit as a Black Box with no consideration of the internal structures. Recently, network DEA models have been developed

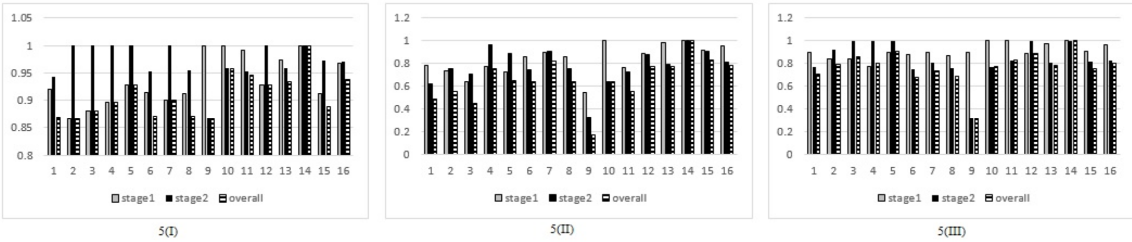

FIGURE 5. Efficiency distribution of stage one, two and overall resulted from independent models (5(I)), model (7) (5(II)), model (10) (5(III)).

to examine the efficiency of DMUs with some stages referred to as sub-units. This paper develops a method based on the MOP model for performance assessment of a specific two-stage network DEA. To this end, we first obtain a linear program by converting a MOP model with the objectives of maximizing both stage efficiencies to a linear program with respect to the GP method. We then extend the obtained model and develop a new model that optimizes the efficiency of each two stages and Black Box efficiency simultaneously. The key point here is that the suggested model optimizes the Black Box and divisional efficiencies with the aim of achieving the smallest possible gap between the aspiration levels (unity value of the efficiency score) simultaneously. In result, an important managerial point can be made that applying the suggested model causes a type of cooperation between stages, with the aim of improving the overall efficiency. To illustrate the features and the applicability of the proposed models, the performance of Iranian electricity distribution companies was studied.

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