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EXISTENCE OF NONOSCILLATORY SOLUTIONS OF SECOND-ORDER NEUTRAL DIFFERENTIAL EQUATIONS

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ABSTRACT. We obtain some sufficient conditions for the existence of nonoscillatory solutions of nonlinear second order neutral differential equation with forcing term. Our results improve and extend some existing results. Examples are also included to illustrate our results.

Keywords: Fixed point, Second-order, Nonoscillatory solution.

AMS Subject Classification: 34K11, 34K40.

1. Introduction

In this work, we study the second-order neutral nonlinear differential equation

$$\big(r(t)\,(x(t)-p(t)x(t-\tau))'\big)' + f_1(t,x(\sigma_1(t))) - f_2(t,x(\sigma_2(t))) = g(t), \tag{1}$$

where $p, g \in C([t_0,\infty),\mathbb{R})$, $\tau > 0$, $r \in C([t_0,\infty),(0,\infty))$ and $\sigma_i \in C([t_0,\infty),\mathbb{R})$ with $\lim_{t\to\infty}\sigma_i(t) = \infty$, $i = 1,2$.

Throughout this paper, we assume that $f_i(t,x) \in C([t_0,\infty)\times\mathbb{R},\mathbb{R})$ is a nondecreasing in x for $i = 1,2$, $xf_i(t,x) > 0$ for $x \neq 0$, $i = 1,2$, and satisfies

$$|f_i(t,x) - f_i(t,y)| \le q_i(t)|x-y| \quad \text{for} \quad t \in [t_0,\infty) \quad \text{and} \quad x,y \in [a,b], \tag{2}$$

where $q_i \in C([t_0,\infty),\mathbb{R}^+)$, $i = 1,2$, and $[a,b]$ $(0 < a < b$ or $a < b < 0)$ is any closed interval. Furthermore, suppose that

$$\int_{t_0}^{\infty}\int_{t_0}^{s}\frac{q_i(u)}{r(s)}duds < \infty, \quad i = 1,2, \tag{3}$$

$$\int_{t_0}^{\infty}\int_{t_0}^{s}\frac{|f_i(u,d)|}{r(s)}duds < \infty \quad \text{for some } d \neq 0,\ i = 1,2, \tag{4}$$

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and

$$\int_{t_0}^{\infty}\int_{t_0}^{s}\frac{|g(u)|}{r(s)}duds<\infty \tag{5}$$

hold. The motivation of this paper comes from the work of Yang, Zhang and Ge in [10], where they investigated the existence of nonoscillatory solutions of the following equations

$$(x(t)-p(t)x(t-\tau))''+f_1(t,x(\sigma_1(t)))-f_2(t,x(\sigma_2(t)))=0 \tag{6}$$

and

$$(x(t)-p(t)x(t-\tau))''+f_1(t,x(\sigma_1(t)))-f_2(t,x(\sigma_2(t)))=g(t), \tag{7}$$

where $f_i(t,x)$, $i=1,2$ are nondecreasing in x. For the other works and related books concerning existence of nonoscillatory solutions of neutral differential and difference equations, we refer to [1–9] and references cited therein.

When we take $r(t)=1$ and $g(t)=0$, and $r(t)=1$ in equation (1), we obtain (6) and (7), respectively. That means the results in current paper is more general than the results in [10].

The purpose of this paper is to present some new sufficient conditions for the existence of nonoscillatory solutions of (1).

Let $T_0=\min\{t_1-\tau,\inf_{t\ge t_1}\sigma_1(t),\inf_{t\ge t_1}\sigma_2(t)\}$ for $t_1\ge t_0$. By a solution of equation (1), we mean a function $x\in C([T_0,\infty),\mathbb{R})$ in the sense that both $x(t)-p(t)x(t-\tau)$ and $r(t)(x(t)-p(t)x(t-\tau))'$ are continuously differentiable on $[t_1,\infty]$ and such that equation (1) is satisfied for $t\ge t_1$.

As is customary, a solution of (1) is said to be oscillatory if it has arbitrarily large zeros. Otherwise the solution is called nonoscillatory.

Throughout this paper, we suppose that X is the set of all continuous and bounded functions on $[t_0,\infty)$ with the sup norm.

2. Main Results

Theorem 2.1. *Assume that (3)-(5) hold and $0\le p(t)\le p<1$. Then (1) has a bounded nonoscillatory solution.*

Proof. Suppose (4) holds with $d>0$, the case $d<0$ can be treated similarly. Set

$$A=\{x\in X: N_1\le x(t)\le d,\quad t\ge t_0\},$$

where N_1 is a positive constant such that

$$N_1<(1-p)d.$$

It is clear that A is a closed, bounded and convex subset of X. In view of (3)-(5) there exists a $t_1>t_0$ sufficiently large such that $t-\tau\ge t_0$, $\sigma_1(t)\ge t_0$, $\sigma_2(t)\ge t_0$ for $t\ge t_1$ and

$$p+2\int_{t_1}^{\infty}\int_{t_1}^{s}\frac{q_i(u)}{r(s)}duds\le\theta_1<1,\quad i=1,2, \tag{8}$$

where θ_1 is a constant,

$$\int_{t_1}^{\infty}\int_{t_1}^{s}\frac{1}{r(s)}[f_1(u,d)+|g(u)|]duds\le(1-p)d-\alpha \tag{9}$$

and

$$\int_{t_1}^{\infty}\int_{t_1}^{s}\frac{1}{r(s)}[f_2(u,d)+|g(u)|]duds\le\alpha-N_1, \tag{10}$$

where $\alpha \in \Big(N_1, (1-p)d\Big)$. Define a mapping $S : A \longrightarrow X$ as follows:

$$(Sx)(t) = \begin{cases} \alpha + p(t)x(t-\tau) + \int_t^\infty \frac{1}{r(s)} \int_{t_1}^s [f_1(u, x(\sigma_1(u))) \\ -f_2(u, x(\sigma_2(u))) - g(u)]duds, \ t \geq t_1 \\ (Sx)(t_1), \ t_0 \leq t \leq t_1. \end{cases}$$

Clearly Sx is continuous. For every $x \in A$ and $t \geq t_1$, using (9), we have

$$\begin{aligned} (Sx)(t) &= \alpha + p(t)x(t-\tau) + \int_t^\infty \int_{t_1}^s \frac{1}{r(s)}[f_1(u, x(\sigma_1(u))) - f_2(u, x(\sigma_2(u))) - g(u)]duds \\ &\leq \alpha + pd + \int_{t_1}^\infty \int_{t_1}^s \frac{1}{r(s)}[f_1(u,d) + |g(u)|]duds \\ &\leq d \end{aligned}$$

and taking (10) into account, we have

$$\begin{aligned} (Sx)(t) &= \alpha + p(t)x(t-\tau) + \int_t^\infty \int_{t_1}^s \frac{1}{r(s)}[f_1(u, x(\sigma_1(u))) - f_2(u, x(\sigma_2(u))) - g(u)]duds \\ &\geq \alpha - \int_{t_1}^\infty \int_{t_1}^s \frac{1}{r(s)}[f_2(u,d) + |g(u)|]duds \\ &\geq N_1. \end{aligned}$$

Then $SA \subset A$. Now we show that S is a contraction mapping on A. In fact, for $x, y \in A$ and $t \geq t_1$, in view of (2) and (8), we have

$$\begin{aligned} |(Sx)(t) - (Sy)(t)| &\leq p|x(t-\tau) - y(t-\tau)| \\ &+ \sum_{i=1}^2 \int_t^\infty \frac{1}{r(s)} \int_{t_1}^s |f_i(u, x(\sigma_i(u))) - f_i(u, y(\sigma_i(u)))|duds \\ &\leq p|x(t-\tau) - y(t-\tau)| \\ &+ \sum_{i=1}^2 \int_{t_1}^\infty \frac{1}{r(s)} \int_{t_1}^s q_i(u)|x(\sigma_i(u)) - y(\sigma_i(u)))|duds \\ &\leq \|x-y\|\Big[p + \sum_{i=1}^2 \int_{t_1}^\infty \int_{t_1}^s \frac{q_i(u)}{r(s)}duds\Big] \\ &\leq \theta_1\|x-y\|, \end{aligned}$$

where we used sup norm. Then it follows that

$$\|Sx - Sy\| \leq \theta_1\|x-y\|.$$

Since $\theta_1 < 1$, S is a contraction mapping on A. Consequently, S has the unique fixed point $x \in A$ such that $Sx = x$, which is obviously a positive solution of (1). This completes the proof. □

Theorem 2.2. *Assume that (3)-(5) hold and* $1 < p_1 \leq p(t) \leq p_2 < \infty$. *Then (1) has a bounded nonoscillatory solution.*

Proof. Suppose (4) holds with $d > 0$, the case $d < 0$ can be treated similarly. Set

$$A = \{x \in X : N_2 \leq x(t) \leq d, \quad t \geq t_0\},$$

where N_2 is a positive constant such that

$$p_2 N_2 < (p_1 - 1)d.$$

It is obvious that A is a closed, bounded and convex subset of X. In view of (3)-(5) there exists a $t_1 > t_0$ sufficiently large such that $\sigma_1(t+\tau) \geq t_0$, $\sigma_2(t+\tau) \geq t_0$ for $t \geq t_1$ and

$$\frac{1}{p_1}\Big[1 + 2\int_{t_1}^{\infty}\int_{t_1}^{s}\frac{q_i(u)}{r(s)}duds\Big] \leq \theta_2 < 1, \quad i = 1, 2, \tag{11}$$

where θ_2 is a constant,

$$\int_{t_1}^{\infty}\int_{t_1}^{s}\frac{1}{r(s)}[f_1(u,d) + |g(u)|]duds \leq \alpha - p_2N_2 \tag{12}$$

and

$$\int_{t_1}^{\infty}\int_{t_1}^{s}\frac{1}{r(s)}[f_2(u,d) + |g(u)|]duds \leq (p_1 - 1)d - \alpha, \tag{13}$$

where $\alpha \in \Big(p_2N_2, (p_1 - 1)d\Big)$. Define a mapping $S : A \longrightarrow X$ as follows:

$$(Sx)(t) = \begin{cases} \frac{1}{p(t+\tau)}\Big[\alpha + x(t+\tau) - \int_{t+\tau}^{\infty}\frac{1}{r(s)}\int_{t_1+\tau}^{s}[f_1(u, x(\sigma_1(u))) \\ -f_2(u, x(\sigma_2(u))) - g(u)]duds\Big], \ t \geq t_1 \\ (Sx)(t_1), \ t_0 \leq t \leq t_1. \end{cases}$$

Clearly, Sx is continuous. For every $x \in A$ and $t \geq t_1$, using (13), we have

$$\begin{aligned} (Sx)(t) &= \frac{1}{p(t+\tau)}\Big[\alpha + x(t+\tau) - \int_{t+\tau}^{\infty}\int_{t_1+\tau}^{s}\frac{1}{r(s)}[f_1(u, x(\sigma_1(u))) \\ &- f_2(u, x(\sigma_2(u))) - g(u)]duds\Big] \\ &\leq \frac{1}{p_1}\Big[\alpha + d + \int_{t_1}^{\infty}\int_{t_1}^{s}\frac{1}{r(s)}[f_2(u,d) + |g(u)|]duds\Big] \\ &\leq d \end{aligned}$$

and taking (12) into account, we have

$$\begin{aligned} (Sx)(t) &= \frac{1}{p(t+\tau)}\Big[\alpha + x(t+\tau) - \int_{t+\tau}^{\infty}\int_{t_1+\tau}^{s}\frac{1}{r(s)}[f_1(u, x(\sigma_1(u))) \\ &- f_2(u, x(\sigma_2(u))) - g(u)]duds\Big] \\ &\geq \frac{1}{p(t+\tau)}\Big[\alpha - \int_{t_1+\tau}^{\infty}\int_{t_1+\tau}^{s}\frac{1}{r(s)}[f_1(u,d) + |g(u)|]duds\Big] \\ &\geq \frac{1}{p_2}\Big[\alpha - \int_{t_1}^{\infty}\int_{t_1}^{s}\frac{1}{r(s)}[f_1(u,d) + |g(u)|]duds\Big] \\ &\geq N_2. \end{aligned}$$

Then $SA \subset A$. Now we show that S is a contraction mapping on A. In fact, for $x, y \in A$ and $t \geq t_1$, in view of (2) and (11), we have

$$\begin{aligned} |(Sx)(t) - (Sy)(t)| &\leq \frac{1}{p(t+\tau)}\Big[|x(t+\tau) - y(t+\tau)| \\ &\quad + \sum_{i=1}^{2}\int_{t+\tau}^{\infty}\frac{1}{r(s)}\int_{t_1+\tau}^{s}|f_i(u, x(\sigma_i(u))) - f_i(u, y(\sigma_i(u)))|duds\Big] \\ &\leq \frac{\|x-y\|}{p_1}\Big[1 + \sum_{i=1}^{2}\int_{t_1}^{\infty}\int_{t_1}^{s}\frac{q_i(u)}{r(s)}duds\Big] \\ &\leq \theta_2\|x-y\|, \end{aligned}$$

where we used sup norm. This immediately implies that

$$\|Sx - Sy\| \leq \theta_2\|x - y\|.$$

Since $\theta_2 < 1$, S is a contraction mapping on A. Consequently, S has the unique fixed point $x \in A$ such that $Sx = x$, which is obviously a positive solution of (1). This completes the proof. □

Theorem 2.3. *Assume that (3)-(5) hold and $-1 < -p \leq p(t) \leq 0$. Then (1) has a bounded nonoscillatory solution.*

Proof. Suppose (4) holds with $d > 0$, the case $d < 0$ can be treated similarly. Set

$$A = \{x \in X : N_3 \leq x(t) \leq d, \quad t \geq t_0\},$$

where N_3 is a positive constant such that

$$N_3 + pd < d.$$

It is clear that A is a closed, bounded and convex subset of X. In view of (3)-(5) there exists a $t_1 > t_0$ sufficiently large such that $t - \tau \geq t_0$, $\sigma_1(t) \geq t_0$, $\sigma_2(t) \geq t_0$ for $t \geq t_1$ and

$$p + 2\int_{t_1}^{\infty}\int_{t_1}^{s}\frac{q_i(u)}{r(s)}duds \leq \theta_3 < 1, \quad i = 1, 2, \tag{14}$$

where θ_3 is a constant,

$$\int_{t_1}^{\infty}\int_{t_1}^{s}\frac{1}{r(s)}[f_1(u, d) + |g(u)|]duds \leq d - \alpha \tag{15}$$

and

$$\int_{t_1}^{\infty}\int_{t_1}^{s}\frac{1}{r(s)}[f_2(u, d) + |g(u)|]duds \leq \alpha - N_3 - pd, \tag{16}$$

where $\alpha \in \Big(N_3 + pd, d\Big)$. Define a mapping $S : A \longrightarrow X$ as follows :

$$(Sx)(t) = \begin{cases} \alpha + p(t)x(t-\tau) + \int_t^{\infty}\frac{1}{r(s)}\int_{t_1}^{s}[f_1(u, x(\sigma_1(u))) \\ -f_2(u, x(\sigma_2(u))) - g(u)]duds, \; t \geq t_1 \\ (Sx)(t_1), \; t_0 \leq t \leq t_1. \end{cases}$$

Clearly, Sx is continuous. For every $x \in A$ and $t \geq t_1$, using (15), we have

$$\begin{aligned}(Sx)(t) &= \alpha + p(t)x(t-\tau) + \int_t^\infty \int_{t_1}^s \frac{1}{r(s)}[f_1(u, x(\sigma_1(u))) - f_2(u, x(\sigma_2(u))) - g(u)]duds \\ &\leq \alpha + \int_{t_1}^\infty \int_{t_1}^s \frac{1}{r(s)}[f_1(u, d) + |g(u)|]duds \\ &\leq d\end{aligned}$$

and taking (16) into account, we have

$$\begin{aligned}(Sx)(t) &= \alpha + p(t)x(t-\tau) + \int_t^\infty \int_{t_1}^s \frac{1}{r(s)}[f_1(u, x(\sigma_1(u))) - f_2(u, x(\sigma_2(u))) - g(u)]duds \\ &\geq \alpha - pd - \int_{t_1}^\infty \int_{t_1}^s \frac{1}{r(s)}[f_2(u, d) + |g(u)|]duds \\ &\geq N_3.\end{aligned}$$

Then $SA \subset A$. Now we show that S is a contraction mapping on A. In fact, for $x, y \in A$ and $t \geq t_1$, in view of (2) and (14), we have

$$\begin{aligned}|(Sx)(t) - (Sy)(t)| &\leq |p(t)||x(t-\tau) - y(t-\tau)| \\ &\quad + \sum_{i=1}^2 \int_t^\infty \frac{1}{r(s)} \int_{t_1}^s |f_i(u, x(\sigma_i(u))) - f_i(u, y(\sigma_i(u)))|duds \\ &\leq p\|x-y\| + \sum_{i=1}^2 \int_{t_1}^\infty \frac{1}{r(s)} \int_{t_1}^s q_i(u)|x(\sigma_i(u)) - y(\sigma_i(u)))|duds \\ &\leq \|x-y\|\Big[p + \sum_{i=1}^2 \int_{t_1}^\infty \int_{t_1}^s \frac{q_i(u)}{r(s)}duds\Big] \\ &\leq \theta_3\|x-y\|,\end{aligned}$$

where we used sup norm. This implies that

$$\|Sx - Sy\| \leq \theta_3\|x-y\|.$$

Since $\theta_3 < 1$, S is a contraction mapping on A. Consequently, S has the unique fixed point $x \in A$ such that $Sx = x$, which is obviously a positive solution of (1). This completes the proof. □

Theorem 2.4. *Assume that (3)-(5) hold and* $-\infty < -p_1 \leq p(t) \leq -p_2 < -1$. *Then (1) has a bounded nonoscillatory solution.*

Proof. Suppose (4) holds with $d > 0$, the case $d < 0$ can be treated similarly. Set

$$A = \{x \in X : N_4 \leq x(t) \leq d, \quad t \geq t_0\},$$

where N_4 is a positive constant such that

$$p_1 N_4 + d < p_2 d.$$

It is obvious that A is a closed, bounded and convex subset of X. In view of (3)-(5) there exists a $t_1 > t_0$ sufficiently large such that $\sigma_1(t+\tau) \geq t_0$, $\sigma_2(t+\tau) \geq t_0$ for $t \geq t_1$ and

$$\frac{1}{p_2}\Big[1 + 2\int_{t_1}^\infty \int_{t_1}^s \frac{q_i(u)}{r(s)}duds\Big] \leq \theta_4 < 1, \quad i = 1, 2, \tag{17}$$

where θ_4 is a constant,

$$\int_{t_1}^{\infty}\int_{t_1}^{s}\frac{1}{r(s)}[f_1(u,d)+|g(u)|]duds \le p_2 d-\alpha \tag{18}$$

and

$$\int_{t_1}^{\infty}\int_{t_1}^{s}\frac{1}{r(s)}[f_2(u,d)+|g(u)|]duds \le \alpha-p_1N_4-d, \tag{19}$$

where $\alpha\in\Big(p_1N_4+d,p_2d\Big)$. Define a mapping $S:A\longrightarrow X$ as follows:

$$(Sx)(t)=\begin{cases} -\frac{1}{p(t+\tau)}\Big[\alpha-x(t+\tau)+\int_{t+\tau}^{\infty}\frac{1}{r(s)}\int_{t_1+\tau}^{s}[f_1(u,x(\sigma_1(u))) \\ -f_2(u,x(\sigma_2(u)))-g(u)]duds\Big],\ t\ge t_1 \\ (Sx)(t_1),\ t_0\le t\le t_1. \end{cases}$$

Clearly, Sx is continuous. For every $x\in A$ and $t\ge t_1$, using (18), we have

$$\begin{aligned}(Sx)(t) &= -\frac{1}{p(t+\tau)}\Big[\alpha-x(t+\tau)+\int_{t+\tau}^{\infty}\int_{t_1+\tau}^{s}\frac{1}{r(s)}[f_1(u,x(\sigma_1(u))) \\ &- f_2(u,x(\sigma_2(u)))-g(u)]duds\Big] \\ &\le \frac{1}{p_2}\Big[\alpha+\int_{t_1}^{\infty}\int_{t_1}^{s}\frac{1}{r(s)}[f_1(u,d)+|g(u)|]duds\Big] \\ &\le d\end{aligned}$$

and taking (19) into account, we have

$$\begin{aligned}(Sx)(t) &= -\frac{1}{p(t+\tau)}\Big[\alpha-x(t+\tau)+\int_{t+\tau}^{\infty}\int_{t_1+\tau}^{s}\frac{1}{r(s)}[f_1(u,x(\sigma_1(u))) \\ &- f_2(u,x(\sigma_2(u)))-g(u)]duds\Big] \\ &\ge -\frac{1}{p(t+\tau)}\Big[\alpha-d-\int_{t_1+\tau}^{\infty}\int_{t_1+\tau}^{s}\frac{1}{r(s)}[f_2(u,d)+|g(u)|]duds\Big] \\ &\ge \frac{1}{p_1}\Big[\alpha-d-\int_{t_1}^{\infty}\int_{t_1}^{s}\frac{1}{r(s)}[f_2(u,d)+|g(u)|]duds\Big] \\ &\ge N_4.\end{aligned}$$

Then $SA\subset A$. Now we show that S is a contraction mapping on A. In fact, for $x,y\in A$ and $t\ge t_1$, in view of (2) and (17), we have

$$\begin{aligned}|(Sx)(t)-(Sy)(t)| &\le \frac{1}{|p(t+\tau)|}\Big[|x(t+\tau)-y(t+\tau)| \\ &\quad +\sum_{i=1}^{2}\int_{t+\tau}^{\infty}\frac{1}{r(s)}\int_{t_1+\tau}^{s}|f_i(u,x(\sigma_i(u)))-f_i(u,y(\sigma_i(u)))|duds\Big] \\ &\le \frac{\|x-y\|}{p_2}\Big[1+\sum_{i=1}^{2}\int_{t_1}^{\infty}\int_{t_1}^{s}\frac{q_i(u)}{r(s)}duds\Big] \\ &\le \theta_4\|x-y\|,\end{aligned}$$

where we used sup norm. This implies that

$$\|Sx-Sy\|\le\theta_4\|x-y\|.$$

Since $\theta_4 < 1$, S is a contraction mapping on A. Consequently, S has the unique fixed point $Sx = x$, which obviously a positive solution of (1). This completes the proof. □

Example 2.1. *Consider the equation*

$$(t^2(x(t) - \frac{1}{t}x(t-1))')' + \frac{t-3}{(t-4)^2}x(t-3) - \frac{2t}{(t-2)^3}x^3(t-1) = \frac{1}{t-4}, \quad t_0 > 4. \tag{20}$$

Note that $r(t) = t^2$, $p(t) = \frac{1}{t}$, $\tau = 1$, $\sigma_1(t) = t-3$, $\sigma_2(t) = t-1$, $f_1(t,x) = \frac{t-3}{(t-4)^2}x$, $f_2(t,x) = \frac{2t}{(t-2)^3}x^3$ *and* $g(t) = \frac{1}{t-4}$. *It is easy to verify that the conditions of Theorem 2.1 are all satisfied and* $x(t) = 1 - \frac{1}{t}$ *is a nonoscillation solution of (20).*

Example 2.2. *Consider the equation*

$$(\exp(t)(x(t) - (\exp(-t)+2)x(t-1))')' + \exp(-t-2)x(t-2) - \exp(-t-3)x(t-3)$$
$$= -\exp(-t-3)(2\exp(4) - \exp(1) + 1). \tag{21}$$

Note that $r(t) = \exp(t)$, $p(t) = \exp(-t) + 2$, $\tau = 1$, $\sigma_1(t) = t-2$, $\sigma_2(t) = t-3$, $f_1(t,x) = \exp(-t-2)x$, $f_2(t,x) = \exp(-t-3)x$ *and* $g(t) = -\exp(-t-3)(2\exp(4) - \exp(1) + 1)$. *We can check that the conditions of Theorem 2.2 are all satisfied and* $x(t) = \exp(-t) + 1$ *is a nonoscillation solution of (21).*

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