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GLOBAL COLOR CLASS DOMINATION PARTITION OF A GRAPH

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Abstract. Color class domination partition was suggested by E. Sampathkumar and it was studied in [1]. A proper color partition of a finite, simple graph G is called a color class domination partition (or cd-partition) if every color class is dominated by a vertex. This concept is different from dominator color partition introduced in [[2], [3]] where every vertex dominates a color class. Suppose G has no full degree vertex (that is, a vertex which is adjacent with every other vertex of the graph). Then a color class may be independent from a vertex outside the class. This leads to Global Color Class Domination Partition. A proper color partition of G is called a Global Color Class Domination Partition if every color class is dominated by a vertex and each color class is independent of a vertex outside the class. The minimum cardinality of a Global Color Class Domination Partition is called the Global Color Class Domination Partition Number of G and is denoted by $\chi_{gcd}(G)$. In this paper a study of this new parameter is initiated and its relationships with other parameters are investigated.

Keywords: Color class domination partition, Global color class domination partition, Dominator color class partition, Global color class domination number.

AMS Subject Classification: 05C69

1. Introduction

Let G be a finite, simple and undirected graph. A proper color partition of G is a partition of $V(G)$ into independent sets of G. Several types of proper color partitions have been studied earlier. One of them is dominator coloring [[2], [3]]. In this coloring, each vertex dominates a color class. The minimum cardinality of a dominator color class partition is denoted by $\chi_d(G)$. A slight variation of this coloring is called a color class domination partition. In this partition, each color class is dominated by a vertex. In graphs without any full degree vertex, Global counter part of this concept can be defined. In this paper this new concept is introduced and studied.

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2. Global color class domination partition

Definition 2.1. *Let G be a finite, simple and undirected graph. Let $\Pi = \{V_1, V_2, \ldots, V_k\}$ be a proper color partition of G. Π is called a global color class domination partition if for every color class V_i, there exists a vertex u_i which dominates V_i and there exists a vertex $w_i \notin V_i$ which is independent of V_i, $1 \le i \le k$. The minimum cardinality of a Global color class domination partition is called the Global color class domination number of G and is denoted by $\chi_{gcd}(G)$.*

If G does not have a full degree vertex, then $\Pi = \{\{v_1\}, \{v_2\}, \ldots, \{v_n\}\}$ is a global color class domination partition of G.

3. $\chi_{gcd}(G)$ for Standard Graphs

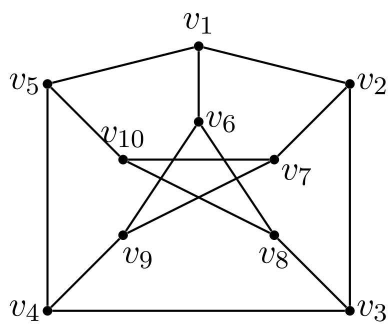
(1) $\chi_{gcd}(\overline{K_n}) = n$.
(2) $\chi_{gcd}(D_{r,s}) = 4$, $r, s \ge 1$.
(3) $\chi_{gcd}(K_{m,n}) = 4$, where $m, n \ge 2$.
(4) $\chi_{gcd}(P_n) = \begin{cases} 4 & \text{if } n = 4, 5 \\ \chi_{cd}(P_n) & \text{if } n \ge 6 \end{cases}$

$\chi_{gcd}(P_2)$ and $\chi_{gcd}(P_3)$ do not exist.

(5) $\chi_{gcd}(C_n) = \begin{cases} 4 & \text{if } n = 4 \\ 5 & \text{if } n = 5 \\ \chi_{cd}(C_n) & \text{if } n \ge 6 \end{cases}$

$\chi_{gcd}(C_3)$ does not exist.

(6) $\chi_{gcd}(P) = 5$ where P is the Petersen graph.



Petersen Graph

Here $\{\{v_1, v_3\}, \{v_2, v_4\}, \{v_5, v_6\}, \{v_7, v_8\}, \{v_9, v_{10}\}\}$ is a minimum global color class domination partition of P.

4. Main Results

Theorem 4.1. *$max\{\chi_{cd}(G), \frac{\gamma_g(G)}{2}\} \le \chi_{gcd}(G)$*

Proof. Let Π be a minimum global color class domination partition of G. Then Π is a color class domination partition of G. Therefore $\chi_{cd}(G) \le \chi_{gcd}(G)$. Let $\Pi = \{V_1, V_2, \ldots, V_k\}$ be a minimum global color partition of G. Then there exist $x_1, x_2, \ldots, x_k$ such that x_i dominates V_i, $(1 \le i \le k)$ and $y_1, y_2, \ldots, y_k$ such that y_i is independent of V_i, $(1 \le i \le k)$.

Let $S = \{x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_k\}$. Then S is a global dominating set of G. Therefore $\gamma_g(G) \leq |S| \leq 2k$. $\frac{\gamma_g(G)}{2} \leq k = \chi_{gcd}(G)$. Therefore $max\{\chi_{cd}(G), \frac{\gamma_g(G)}{2}\} \leq \chi_{gcd}(G)$. □

Remark 4.1. *Let* $G = P_6$. $\gamma_g(G) = 2$. $\chi_{gcd}(G) = \chi_{cd}(G) = \lceil\frac{n+2}{2}\rceil = 4$. *Therefore* $max\{\chi_{cd}(G), \frac{\gamma_g(G)}{2}\} = max\{\frac{2}{2}, 4\} = 4 = \chi_{gcd}(G)$.

Theorem 4.2. $\frac{n}{min(\Delta(G), n-1-\delta(G))} \leq \chi_{gcd}(G)$

Proof. Let $\Pi = \{V_1, V_2, \ldots, V_k\}$ be a minimum global color partition of G. Since each V_i is dominated by a vertex say x_i. $deg(x_i) \geq |V_i|$, $(1 \leq i \leq k)$. Therefore $|V_i| \leq \Delta(G)$, $(1 \leq i \leq k)$. That is, $\max_{1\leq i\leq k}(|V_i|) \leq \Delta(G)$. Since each V_i is independent of some y_i, $(1 \leq i \leq k)$, each V_i is dominated by y_i in $\overline{G}$, $(1 \leq i \leq k)$, therefore $|V_i| \leq deg_{\overline{G}}(y_i) \leq \Delta(\overline{G})$. $\delta(G) \leq n - \Delta(\overline{G}) - 1$. $\Delta(\overline{G}) \leq n - \delta(G) - 1$. Therefore $|V_i| \leq min\{\Delta(G), \leq n - \delta(G) - 1\}$, $(1 \leq i \leq k)$. $n = |V_1| + |V_2| + \ldots + |V_n| \leq min\{|V_1|\} + min\{|V_2|\} + \ldots + min\{|V_k|\}$. $n = k$ $min\{\Delta(G), n - \delta(G) - 1\}$. $\frac{n}{min\{\Delta(G), n-1-\delta(G)\}} \leq k = \chi_{gcd}(G)$. □

Remark 4.2. *The above bound is sharp. For: Let* $G = P_6$. $\chi_{gcd}(G) = 4$, $\Delta(G) = 2$, $\delta(G) = 1$. *Therefore* $min\{\Delta(P_6), n-1-\delta(P_6)\}$, $\frac{n}{min\Delta(P_6), n-1-\delta(P_6)} = \frac{6}{2} = 3$. $\frac{|V(P_6)|}{min\{\Delta(P_6), n-1-\delta(P_6)\}} = \chi_{gcd}(P_6)$.

Observation 4.1. *Let* $G = C_{20}$. $\chi_{gcd}(C_{20}) = \chi_{cd}(C_{20}) = \frac{20}{2} = 10$. $\chi(C_{20}) = 2$ *and* $\gamma_g(C_{20}) = 7$. *Therefore* $\chi(G) + \gamma_g(G) = 2 + 7 = 9 < \chi_{gcd}(G)$ *where* $G = C_{20}$.
Let $G = C_6$. $\chi_{gcd}(C_6) = 3$. $\chi(C_6) = 2$ *and* $\gamma_g(C_6) = 2$. *Therefore* $\chi(G) + \gamma_g(G) = 2 + 2 = 4 \geq \chi_{gcd}(G)$ *where* $G = C_6$.
Let $G = P_4$. $\chi_{gcd}(P_4) = 4$. $\chi(P_4) = 2$ *and* $\gamma_g(P_4) = 2$. *Therefore* $\chi(G) + \gamma_g(G) = 2 + 2 = 4 = \chi_{gcd}(G)$ *where* $G = P_4$. *Therefore there is no relationship between* $\chi_{gcd}(G)$ *and* $\chi(G) + \gamma_g(G)$.

Observation 4.2. *Let* G *be the disjoint union of connected graphs* $G_1, G_2, \ldots, G_k$. *Then* $\chi_{gcd}(G) = \chi_{gcd}(G_1) + \chi_{gcd}(G_2) + \ldots + \chi_{gcd}(G_k)$.

Theorem 4.3. *Let* G *have isolates. Then* $\chi_{gcd}(G) = \chi_{cd}(G)$.

Proof. Let $u_1, u_2, \ldots, u_k$ be the isolates of G. Let Π be a minimum color class domination partition of G. Since u_i, $(1 \leq i \leq k)$, are isolates, $\{u_1\}, \{u_2\}, \ldots, \{u_k\}$ all belong to Π. Therefore Π is also a global color class domination partition of G. Therefore $\chi_{gcd}(G) \leq |\Pi| = \chi_{cd}(G)$. But $\chi_{cd}(G) \leq \chi_{gcd}(G)$. Hence $\chi_{gcd}(G) = \chi_{cd}(G)$. □

Theorem 4.4. *Let* G *be a bipartite graph without isolates and the cardinalities of the bipartite sets of* G *are* ≥ 2. *Then* $\gamma(G) = \gamma_g(G) = \chi_{cd}(G) = \chi_{gcd}(G)$ *if* $N(u_i) \neq Y$ *for any* u_i *in* X *and* $N(v_i) = X$ *for some* v_i *in* Y.
If $N(u_i) = Y$ *for any* u_i *in* X *and* $N(v_i) = X$ *for some* v_i *in* Y, *then* $\gamma(G) = \gamma_g(G) = \chi_{cd}(G) = 2$ *and* $\chi_{gcd}(G) = 4$.
If $N(u_i) \neq Y$ *for any* u_i *in* X *and* $N(v_i) = X$ *for some* v_i *in* Y, *then* $\gamma(G) = \gamma_g(G) = \chi_{cd}(G) = k + 1$ *and* $\chi_{gcd}(G) = k + 2$.

Proof. Let G be a bipartite graph without isolates and let X, Y be the bipartite sets of G. Let $|X| \geq 2$, $|Y| \geq 2$. Since G is bipartite without isolates, $\overline{G} = K_r \cup K_s$. Any subset of $V(G)$ containing a vertex from X and a vertex from Y is a dominating set of $\overline{G}$. Any dominating set of G contains at least one vertex from X and at least one vertex from Y. Therefore any dominating set of G is also a dominating set of $\overline{G}$. Therefore $\gamma(G) = \gamma_g(G)$. Let $\{u_1, u_2, \ldots, u_r\}$ be a γ-set of G. Let $u_1, u_2, \ldots, u_k \in X$ and $u_{k+1}, u_{k+2}, \ldots, u_r \in Y$.

Consider $V_i = N(u_i) - \bigcup_{j=1}^{i-1} N(u_j)$. If $u_i \in X$,then $V_i \subset Y$. If $u_i \in Y$, then $V_i \subset X$. Let u_{i_1} and $u_{i_2} \in X$. Without loss of generality $i_1 < i_2$. Then $V_{i_2} \cap V_{i_1} = \phi$. If $u_{i_1} \in X$ and $u_{i_2} \in Y$, then $V_{i_2} \cap V_{i_1} = \phi$. Therefore $V_1, V_2, \ldots, V_r$ are mutually disjoint. If $u_i \in X$, $V_i \subset Y$, then V_i is independent. Therefore $\Pi = \{V_1, V_2, \ldots, V_r\}$ is a partition of G into independent sets. V_i is dominated by u_i, $(1 \leq i \leq k)$. If $N(u_i) = Y$, then $V_2, V_3, \ldots, V_k$ are empty. If $N(u_{k+1}) = X$, then $V_{k+2}, V_{k+3}, \ldots, V_r$ are empty. Therefore $\{u_1, u_{k+1}\}$ is a minimum dominating as well as global dominating set of G, that is, $\gamma(G) = \gamma_g(G) = 2$. Let $\Pi = \{V_1 - \{u_k\}, V_2 - \{u_r\}, \{u_k\}, \{u_r\}\}$ is a minimum global color class domination partition of G. Therefore $\chi_{gcd}(G) = 4$. $\Pi_1 = \{V_1, V_{k+1}\}$ is a minimum color class domination partition of G. Therefore $\chi_{cd}(G) = 2$. Suppose $N(u_1) \subsetneqq X$.But $N(u_{k+1}) = X$. Therefore $V_1 \subsetneqq Y$. Suppose $V_2 = N(u_2) - N(u_1) = \phi$. Then $N(u_2) \subset N(u_1)$. Therefore $D = \{u_1, u_3, \ldots, u_r\}$ is a dominating set of G. There $\gamma(G) < r$, a contradiction. Therefore $V_2 \neq \phi$. A similar argument shows that $V_3, V_4, \ldots, V_k$ are empty. Since $V_{k+1} = X$, $V_{k+2}, \ldots, V_r = \phi$, therefore $\Pi = \{V_1, \ldots, V_k, V_{k+1} - \{u_k\}, \{u_k\}\}$ is a minimum global color class domination partition. Therefore $\chi_{cd}(G) = k + 2$. Since $V_{k+1} = X$, $D = \{u_1, u_2, \ldots, u_k, u_{k+1}\}$ is a minimum global color class domination partition. Therefore $\chi_{gcd}(G) = k + 2$. Since $V_{k+1} = X$, $D = \{u_1, u_2, \ldots, u_k, u_{k+1}\}$ is a minimum dominating set of G. $|D| = k+1 < r$. Therefore $\gamma(G) = k + 1$, $\gamma_g(G) = k + 1$, $\chi_{cd}(G) = k + 1$, $\chi_{gcd}(G) = k + 2$.
Suppose $N(u_1) \subsetneqq Y$, $N(u_{k+1}) \subsetneqq X$. Then $V_2, \ldots, V_k, V_{k+2}, \ldots, V_r$ are non-empty.$\Pi = \{V_2, \ldots, V_k, V_{k+2}, \ldots, V_r\}$ is a minimum global color class domination partition of G. It is also a minimum color class domination partition of G. Therefore $\gamma(G) = \gamma_g(G) = \chi_{cd}(G) = \chi_{gcd}(G) = r$. □

Proposition 4.1. $\chi_{gcd}(G) = 2$ *iff* $G = \overline{K_2}$.

Proof. Suppose $\chi_{gcd}(G) = 2$. Let $\Pi = \{V_1, V_2\}$ be a χ_{gcd}-partition of G. V_1 is dominated by a vertex of V_2 or V_1 is a singleton. Since there exists a vertex in V_1 which is not adjacent with any vertex of V_2, V_1 is a singleton. Similarly V_2 is a singleton. Let $V_1 = \{u\}$, $V_2 = \{v\}$. If u and v are adjacent, then $G = K_2$ and hence G has a full degree vertex, a contradiction. Therefore u and v are not adjacent. Therefore $G = \overline{K_2}$.

The converse is obvious. □

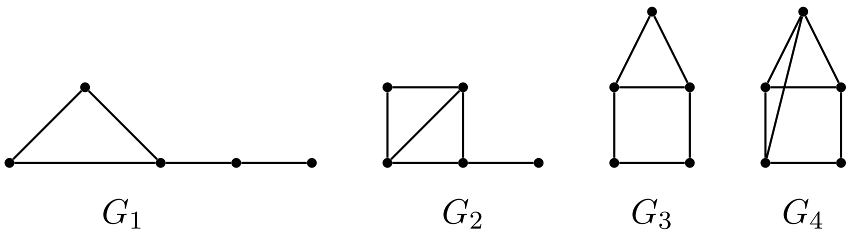
Theorem 4.5. $2 \leq \chi_{gcd}(G) \leq n$

Theorem 4.6. *Let G be disconnected. Then $\chi_{gcd}(G) = n$ iff $G = K_{r_1} \cup K_{r_2} \ldots \cup K_{r_k}$.*

Proof. Let $\chi_{gcd}(G) = n$. By hypothesis, G is disconnected. Let $G_1, G_2, \ldots, G_k$ be the components of G. Suppose G_i has two independent points u, v such that they are adjacent with a common vertex. Then $\{u, v\}$ is an element of a χ_{gcd}-partition. Therefore $\chi_{gcd}(G) \leq n$, a contradiction. Hence either G_i is complete or any two independent vertices of G_i has no common adjacent vertex. In the latter case, there exists a path of length at least three between u and v. Let $u = u_1, u_2, \ldots, u_r = v$ be a shortest path between u and v of length at least three. Then u and u_3 are independent and have a common vertex, a contradiction. Therefore G_i is complete. Therefore $G = K_{r_1} \cup K_{r_2} \ldots \cup K_{r_k}$.
The converse is obvious. □

Corollary 4.1. *If each K_{r_i} is a singleton, then $G = \overline{K_n}$.*

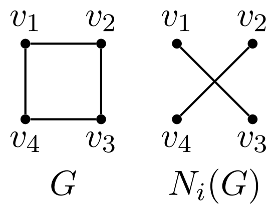
Remark 4.3. *Let G be a connected graph without full degree vertex. Suppose $|V(G)| = 3$. Then there exists no graph without full degree vertex. Let $|V(G)| = 4$. Then P_4 and C_4 are the only connected graphs without full degree vertex such that $\chi_{gcd}(G) = 4$. Let $|V(G)| = 5$. Let G_i, $1 \leq i \leq 4$ be the graphs given below:*



Then these are the four graphs without full degree vertex on five vertices such that $\chi_{gcd}(G) = 5$.

Definition 4.1. *Let* G *be a connected graph. Define* $N_i(G)$ *as follows: A vertex set of* $N_i(G)$ *is same as* $V(G)$. *Two vertices in* $N_i(G)$ *are adjacent if they are independent and they have a common adjacent vertex.*

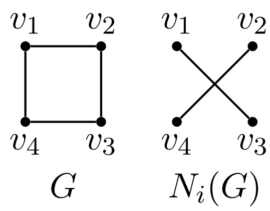
Example 4.1. *Let* $G = C_4$ *and* $N_i(G)$ *be the graphs given below:*



Theorem 4.7. *Let* G *be a connected graph without a full degree vertex. Then* $\chi_{gcd}(G) = n$ *iff for any edge* uv *in* $N_i(G)$, $\{u, v\}$ *is a maximal independent set in* G.

Proof. Suppose for any edge xy in $N_i(G)$, $\{x, y\}$ is a maximal independent set in G. Since G is connected and G has no full degree vertex, there exist two independent vertices which have a common adjacent vertex. (For : if u and v are independent and $d(u, v) = 2$, then u and v have a common vertex . Suppose $d(u, v) \geq 3$. Let $u = u_1, u_2, \ldots u_k = v$ be a shortest path between u and v. Clearly $k \geq 4$. Then u, u_3 are independent and have a common vertex u_2). Hence $N_i(G)$ has at least one edge. Let uv be an edge of $N_i(G)$. Then $\{u, v\}$ is a maximal independent set of G. Therefore there exists no vertex w in G such that w is non-adjacent with u and v. Therefore $\chi_{gcd}(G) = n$. Conversely, let G be connected without full degree vertex and $\chi_{gcd}(G) = n$. Let xy be an edge in $N_i(G)$. Then x and y have a common adjacent vertex in G. Since $\chi_{gcd}(G) = n$, x and y do not have a common non-adjacent vertex. Hence $\{x, y\}$ is a maximal independent set in G. □

Example 4.2. *Let* $G = C_4$ *and* $N_i(G)$ *be the graphs given below:*



Also $\{v_1, v_3\}$ *is a maximal independent set in* G *as well as* $\{v_2, v_4\}$. *Therefore* $\chi_{gcd}(G) = 4$.

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