TWMS J. App. Eng. Math. V.9, N.2, 2019, pp. 186-197

EXPONENTIAL DOMINATION OF TREE RELATED GRAPHS

A. AYTAÇ[1], B. ATAY ATAKUL[2], §

Abstract. The well-known concept of domination in graphs is a good tool for analyzing situations that can be modeled by networks. Although a vertex in the graph can exert influence on, or dominate, all vertices in its immediate neighbourhood, in some real world situations, this can be change. The vertex can also influence all vertices within a given distance. This situation is characterized by distance domination. The influence of the vertex in the graph doesn't extend beyond its neighbourhood and even this influence decreases with distance. Up to the present, no framework for this situation has been put forward yet. The dominating power of the vertex in the graph decreases exponentially, with distance by the factor $1/2$. Hence a vertex v can be dominated by a neighbour of v or by a number of vertices that are not too far from v. In this paper, we study the vulnerability of interconnection networks to the influence of individual vertices, using a graph-theoretic concept of exponential domination number as a measure of network robustness.

Keywords: Graph vulnerability, Network design and communication, Domination, Exponential domination number, Trees.

AMS Subject Classification: 05C40, 05C69, 68M10, 68R10

1. Introduction

Network designers attach importance of reliability and stability of a network. If the network begins losing communication links or processors, then there is a loss in its effectiveness. This event is called as vulnerability of communication networks[11, 12]. The vulnerability of communication networks measures the resistance of a network to a disruption in operation after the failure of certain processors and communication links. Network designs require greater degrees of stability and reliability or less vulnerability in communication networks.

Graph theory has become one of the most powerful mathematical tools in the analysis and study of the architecture of an interconnection network. It is well known that the underlying topology of an interconnection network is modeled by the graph. Throughout this paper, all graphs considered are simple and connected. Let $G = (V, E)$ be a simple

[1] Department of Math., Faculty of Science, Ege University, Bornova-Izmir,TURKEY.
e-mail: aysun.aytac@ege.edu.tr; ORCID: https://orcid.org/0000-0003-2086-8969.

[2] Department of Comp. and Inst. Tech. Edu., Faculty of Education, Ağrı Ibrahim Çeçen University. Ağrı,TURKEY
e-mail: batay@agri.edu.tr; ORCID: https://orcid.org/BAA26021987.

§ Manuscript received: April 19, 2017; accepted: September 18, 2018.

connected graph with a vertex set $V = V(G)$ and an edge set $E = E(G)$. For any vertex $v \in V(G)$, the *open neighbourhood* of v is $N(v) = \{u \in V(G)|uv \in E(G)\}$ and *closed neighborhood* of v is $N[v] = N(v) \cup \{v\}$. The degree of v in G denoted by $deg(v)$, is the size of its open neighborhood. The *distance* $d(u,v)$ between two vertices u and v in G is the length of a shortest path between them. The *diameter* of G, denoted by $diam(G)$ is the largest distance between two vertices in $V(G)$[13, 27].
Many graph theoretical parameters such as connectivity, toughness, integrity, binding number, domination number etc., have been used in the past to describe the stability of communication networks [2, 4, 5, 6, 7, 8, 9, 10, 24, 25, 26].
Domination in graphs is one of the concepts in graph theory which has attracted many researchers to work on it because of its many and varied applications in such fields as linear algebra and optimization, design and analysis of communication networks, and social sciences and military surveillance [23]. A set $S \subseteq V(G)$ is a dominating set if every vertex in $V(G) - S$ is adjacent to at least one vertex in S. The minimum cardinality taken over all dominating sets of G is called the domination number of G and is denoted by $\gamma(G)$.
Many variants of dominating models are available in the existing literature. The exponential domination number is one of these. It has been defined recently by Dankelmann et al. [16]. In their model, the *dominating power* of a vertex decreases exponentially, with distance by the factor $1/2$. Hence a vertex v can be dominated by a neighbour of v or by a number of vertices that are not too far from v. Such a model could be used, for example, for the analysis of dissemination of information in social networks, where the impact of the information decreases every time it is passed on.
Let $S \subseteq V(G)$. For each vertex $u \in S$ and for each $v \in V(G) - S$, we define $\overline{d}(u,v)$ to be the length of a shortest $u - v$ path in $V(G) - (S - \{u\})$ if such a path exists, and ∞ otherwise. If, for each $v \in V(G)$ we have

$$w_s(v) = \begin{cases} \sum_{u \in S} 1/2^{\overline{d}(u,v)-1}, & if\ v \notin S \\ 2, & if\ v \in S \end{cases}$$

$w_S(v) \geq 1$, then S is an exponential dominating set. The smallest cardinality of an exponential dominating set is the exponential domination number,$\gamma_e(G)$ such a set is a minimum exponential dominating set, or γ_e -set for short. If $u \in S$ and $v \in V(G) - S$ and $1/2^{\overline{d}(u,v)-1} > 1$, then we say that u exponentially dominates v. This can be thought in the following way: each vertex dominates its neighbours, $1/2$-dominates those at distance 2, and so on. Note that if S is an exponential dominating set, then every vertex of $V(G) - S$ is exponentially dominated, but the converse is not true [1, 16].
Throughout this article, the largest integer not larger than x is denoted by $\lfloor x \rfloor$ and the smallest integer not smaller than x is denoted by $\lceil x \rceil$.
The paper proceeds as follows. In Section 2, some known results are given. There are different classes of tree related graphs that have been studied for variety of purposes such as binomial tree, comet graph, complete $k - ary$ tree, E_p^t graph and regular caterpillar graph. The exponential domination number values for tree related graphs are developed in Section 3. Finally, concluding remarks of this paper are given in Section 4.

2. Basic Results

In this section, we give some known results on exponential domination number and definition of complementary prisms. We determine the exponential domination number of complementary prisms. We obtained new result.
The complement $\overline{G}$ of a simple graph G is the simple graph with vertex set $V(G)$ defined

by $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$.
Complementary prisms were first introduced by Haynes, et al. in [22]. For a graph G, its complementary prism, denoted by $G\overline{G}$, is formed from a copy of G and a copy of $\overline{G}$ by adding a perfect matching between corresponding vertices. For each $v \in V(G)$ let $\overline{v}$ denote the vertex v in the copy of $\overline{G}$. Formally $G\overline{G}$ is formed from $G \cup \overline{G}$ by adding the edge $v\overline{v}$ for every $v \in V(G)$ [21, 22].

Definition 2.1. [18] *The graph with n vertices labeled $x_1, x_2, ..., x_n$ and edges $x_1x_2, x_2x_3, ..., x_{n-1}x_n$ is called a path of length $n-1$, denoted P_n. The cycle of length n, C_n is the graph with n vertices $x_1, x_2, ..., x_n$ and the edges $x_1x_2, x_2x_3, ..., x_nx_1$.*

Theorem 2.1. [16] *For every positive integer n, $\gamma_e(P_n) = \lceil (n+1)/4 \rceil$.*

Theorem 2.2. [16] *For every positive integer $n \geq 3$,*

$$\gamma_e(C_n) = \begin{cases} 2, & if\ n = 4 \\ \lceil n/4 \rceil, & if\ n \neq 4 \end{cases}.$$

Theorem 2.3. [16] *If G is a connected graph of diameter d, then $\gamma_e(G) \geq \lceil \frac{d+2}{4} \rceil$.*

Theorem 2.4. [16] *If G is a connected graph of order n, then $\gamma_e(G) \leq \frac{2}{5}(n+2)$.*

Theorem 2.5. [16] *Let G be a connected graph of order n and T a spanning tree of G. Then $\gamma_e(G) \leq \gamma_e(T)$.*

Theorem 2.6. [16] *For every graph G, $\gamma_e(G) \leq \gamma(G)$. Also, $\gamma_e(G) = 1$ if and only if $\gamma(G) = 1$.*

Lemma 2.1. [16] *There exists a tree T of order 375 with $\gamma_e(T) = 144$.*

Lemma 2.2. *Let G be any connected graph of order n. If G has a vertex with degree $n-1$, then $\gamma_e(G) = 1$*

Proof. Let S be γ_e -set of G. If we add the vertex v with $deg(v) = n-1$ to S, then $w_S(u) = 1$ satisfies for all vertices of G. Hence, we have $\gamma_e(G) = 1$. □

Lemma 2.3. *Let G be any connected graph of order n and diameter 2. If G has not a vertex with degree $n-1$, then $\gamma_e(G) = 2$.*

Proof. Let S be γ_e -set of G. Since there is not a vertex which is adjacent to all vertices of G, $d(u, v) \leq 2$ for u in $V(G) - S$ and every v in S . Hence, the vertices of S contribute at least $1/2$ to $w_S(u)$. Thus, at least two vertices must be in S to satisfy $w_S(u) \geq 1$. Then, we have $\gamma_e(G) = 2$. □

Definition 2.2. [19] *The corona $(G_1 \circ G_2)$ of two graphs G_1 and G_2 is defined as the graph G obtained by taking one copy of G_1 (which has p_1 points) and p_1 copies of G_2, and then joining the ith point of G_1 to every point in the ith copy of G_2.*

Definition 2.3. [20] *Let G_1 and G_2 be two graphs with vertex sets are $V(G_1)$ and $V(G_2)$; edge sets are $E(G_1)$ and $E(G_2)$ respectively. Then, the join of $G_1 + G_1$ of two graphs G_1 and G_2 is the graph with vertex set $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2)\}$.*

Theorem 2.7. [3] *Let G_1 and G_2 be any two graphs. Let $(G_1 \circ G_2)$ and $(G_1 + G_2)$ be corona and join operations of G_1 and G_2, respectively.*
a) For any two graphs G_1 and G_2, $\gamma_e(G_1 \circ G_2) \geq \lceil \frac{diam(G_1 \circ G_2)}{2} \rceil$.
b) Let G_1 and G_2 be any two graphs. If $diam(G_1) < diam(G_2)$, then $\gamma_e(G_1 + G_2) = \gamma_e(G_1)$.

Theorem 2.8. [13, 27] *If G is a simple graph and $diam(G) \geq 3$, then $diam(\overline{G}) \leq 3$.*

Corollary 2.1. [13, 27] *If the diameter of G is at least 3, then $\gamma(\overline{G}) \leq 2$.*

Theorem 2.9. *Let G be a connected graph with n vertices. The exponential domination number for $G\overline{G}$ with $2n$ vertices is 2.*

Proof. The graph $G\overline{G}$ contains two subgraphs G and $\overline{G}$. Although the graph G is connected, the complement graph $\overline{G}$ can be disconnected. There are two important cases for creating $\gamma_e - set$ of $G\overline{G}$. Let S be a $\gamma_e - set$ of $G\overline{G}$. (1) The vertices of S are selected from the vertices of connected subgraphs G or $\overline{G}$, that is $S \subseteq V(G)$ or $S \subseteq V(\overline{G})$. (2) If both of subgraphs G and $\overline{G}$ are connected, then diameters of these subgraphs are compared. If $diam(G) < diam(\overline{G})$, then $S \subseteq V(G)$, otherwise $S \subseteq V(\overline{G})$. Hence, we have two cases depending on $diam(G)$.

Case 1: If $diam(G) \geq 3$, then we get $diam(\overline{G}) \leq 3$ by Theorem 2.8. Thus, vertices of S must be selected from $\overline{G}$ due to the above mentioned cases for creating S. It is clear that $\gamma(\overline{G}) \leq 2$ by Corollary 2.1. Hence, $|S| \leq 2$ by Theorem 2.6. But, S can not contain only one vertex. Because, the shortest path between a vertex u in $V(G)$ and a vertex $\overline{v}$ in $V(\overline{G}) - \{\overline{u}\}$ is 2. Thus, it is clear that $|S| = 2$. Hence, we get $\gamma_e(G\overline{G}) = 2$.

Case 2: If $diam(G) \leq 2$, then we have the following subcases.

Subcase 2.1. If there is a vertex that is adjacent to all vertices in the graph G, then $\overline{G}$ is disconnected. Hence, S is obtained from $V(G)$. The distance between two vertices in $V(G)$ and the distance between the vertices of $V(G)$ and $V(\overline{G})$ is at most two. That is, $d(u, v) = 2$ for $u \in V(G)$ and $v \in \{V(G\overline{G}) - N(\overline{u})\}$, where $\overline{u} \in V(\overline{G})$. Hence, if $v \notin S$, then S contributes $1/2$ to $w_S(v)$ and the condition $w_S(v) \geq 1$ does not satisfy. So, we must add one more vertex to S. Hence, we have $\gamma_e(G\overline{G}) = 2$.

Subcase 2.2. If there is not any vertex that is adjacent to all vertices in the graph G, then $diam(G\overline{G})$=2 or $diam(G\overline{G})$=3. If $diam(G\overline{G})$=2, then we have $\gamma_e(G\overline{G}) = 2$ by Lemma 2.3. If $diam(G\overline{G})$=3, then $\overline{G}$ is disconnected. Hence, the vertices of S are selected from G. In this case, $diam(G)$=2. Therefore, all the vertices of $G\overline{G}$ are exponentially dominated by two vertices of G and we get $\gamma_e(G\overline{G}) = 2$.

By case 1 and case 2, the exponential domination number of $G\overline{G}$ is $\gamma_e(G\overline{G}) = 2$.
This completes the proof. □

3. Exponential Domination Number in Trees

In this section, we calculate exponential domination number of trees and some related networks namely binomial tree, comet graph, complete $k - ary$ tree, and E_p^t graph.

Definition 3.1. [13] *The binomial tree B_n is an ordered tree defined recursively. The binomial tree B_0 consists of a single vertex. The binomial tree B_n consists of two binomial trees B_{n-1} that are linked together: the root of one is the leftmost child of the root of the other. The tree B_0 as a single vertex, and then the rooted tree B_{n+1} is obtained by taking one copy of each of B_0 through B_n , adding a root, and making the old roots the children of the new root.*

Theorem 3.1. *Let B_n be a binomial tree with $n \geq 3$. Then, $\gamma_e(B_n) = 2^{n-2} + 1$.*

Proof. The binomial tree B_n has 2^n vertices and B_n contains previous two subgraphs B_{n-1}. Also, B_{n-1} contains previous two subgraphs B_{n-2}. Finally, if we continue with this manner, we see that the graph B_n contains all previous binomial trees except B_0 and B_1.

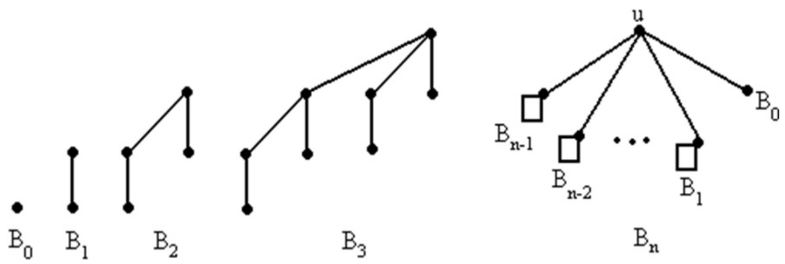

FIGURE 1. Binomial tree B_n

Hence, the resursive formula for B_n is

$$\begin{aligned} B_n &= 2(B_{n-1}) \\ &= 2(2(B_{n-2})) = 2^2(B_{n-2}) \\ &= 2^2(2(B_{n-3})) = 2^3(B_{n-3}) \\ &\dots \\ &= 2^{n-3}(2(B_{n-(n-2)})) \\ &= 2^{n-2}(B_{n-(n-2)}). \end{aligned}$$

Hence, we obtain $B_n = 2^i(B_{n-i})$ for $1 \le i \le n-2$. The proof of exponential domination number of B_n is examined depending on the value n. For $n < 3$, $\gamma_e(B_0) = \gamma_e(B_1) = 1$ and since $B_2 \cong P_4$ by Theorem 2.1 we have $\gamma_e(B_2) = \gamma_e(P_4) = 2$.

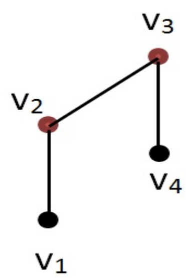

FIGURE 2. Graph B_2

From Figure 2, we can easily see that $\gamma_e - set\ S$ of B_n is $\{v_2, v_3\}$. Actually, these vertices are root vertices of B_2.

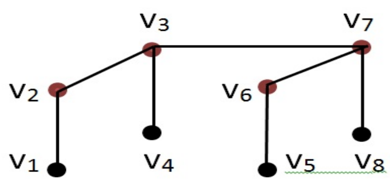

FIGURE 3. Graph B_3

From Figure 3, we can easily see that B_3 includes two subgraphs B_2. We assume that $\gamma_e - set$ S' of B_3 is formed by the minimum exponential dominating set of B_2. That is, the root vertices of each subgraph B_2 are in S. Hence, $S' = \{v_2, v_3, v_6, v_7\}$ and $w_{S'}(v) \geq 1$ for all vertices of B_3. But, in this case S' is not minimum set. Since the vertices $v_3, v_7 \in S$ are adjacent, we must remove any one from the S. Hence, we have $\gamma_e(B_3) = 2(\gamma_e(B_2) - 1) + 1 = 3$.
For $n > 3$, in a similar manner, the minimum exponential domination number of B_n is

$$\gamma_e(B_n) = 2(\gamma_e(B_{n-1}) - 1) + 1. \tag{1}$$

When we use this result in recursive formula which we have obtained for B_n, we have $\gamma_e(B_n) = 2^i(\gamma_e(B_{n-i}) - 1) + 1$ for $1 \leq i \leq n-2$. We must prove this formula by induction on i. Let $\forall n \in Z^+$ and $n \geq 3$.
When $i = 1$, we have $\gamma_e(B_n) = 2(\gamma_e(B_{n-1}) - 1) + 1$ and it is true by (1). We assume that the result is true for $i = k$ and prove it for $i = k+1$. By induction hypothesis and (1), we get

$$\begin{aligned}
\gamma_e(B_n) &= 2^k(\gamma_e(B_{n-k}) - 1) + 1 \\
&= 2^k(2(\gamma_e(B_{n-k-1}) - 1) + 1 - 1) + 1 \\
&= 2^k(2(\gamma_e(B_{n-k-1}) - 1)) + 1 \\
&= 2^{k+1}(\gamma_e(B_{n-k-1}) - 1) + 1.
\end{aligned}$$

That is, the formula is true for $i = k+1$. Hence, we have $\gamma_e(B_n) = 2^i(\gamma_e(B_{n-i}) - 1) + 1$ for $1 \leq i \leq n-2$. Initial condition $n = 2$ is achieved for $i = n-2$. We obtain the following formula.

$$\begin{aligned}
\gamma_e(B_n) &= 2^i(\gamma_e(B_{n-i}) - 1) + 1 \\
&= 2^{n-2}(\gamma_e(B_{n-(n-2)}) - 1) + 1 \\
&= 2^{n-2}(\gamma_e(B_2) - 1) + 1 \\
&= 2^{n-2}(2 - 1) + 1 \\
&= 2^{n-2} + 1.
\end{aligned}$$

The proof is completed. □

Definition 3.2. [15] *For integer $t \geq 2$ and $r \geq 1$, the comet graph $C_{t,r}$ is defined to be the graph of order $t+r$ obtained from disjoint union of a star $K_{1,t-1}$ and a path P_r with r vertices by adding an edge joining the central vertex of the star with an end-vertex of the path.*

Theorem 3.2. *The exponential domination number of a comet graph is given by $\gamma_e(C_{t,r}) = \gamma_e(P_{r+2}) = \lceil (r+3)/4 \rceil$.*

Proof. Comet graph $C_{t,r}$ includes the subgraphs P_{r+2} and $K_{1,t-1}$. One of P_{r+2} is center vertex $c \in V(K_{1,t-1})$ and the other one is pendant vertex $v \in V(K_{1,t-1})$. We know that, there is only center vertex c in $\gamma_e - set$ of $K_{1,t-1}$ and $\gamma_e(K_{1,t-1}) = 1$. Also, we know that by Theorem 2.1, the exponential domination number of P_{r+2} is $\lceil (r+3)/4 \rceil$. If one of the vertices of $\gamma_e - set$ of $C_{t,r}$ is $c \in V(K_{1,t-1})$, then all vertices of $K_{1,t-1}$ and P_{r+2} are exponentially dominated. Hence, we have

$$\gamma_e(C_{t,r}) = \lceil (r+3)/4 \rceil.$$

The proof is completed. □

Definition 3.3. [16] *The complete $k-ary$ tree H_n^k of depth n is the rooted tree in which all vertices at level $n-1$ or less have exactly k children, and all vertices at level n are leaves.*

Theorem 3.3. *The exponential domination number of a complete $k-ary$ tree H_n^k is given by $\gamma_e(H_n^k) = k^{n-1}$.*

Proof. The graph H_n^k has k children for every vertices except leaves. For example, the graph H_2^k is obtained by combining k graphs H_1^k with a root vertex. Similarly, the graph H_3^k is obtained by combining k graphs H_2^k with a root vertex. If we continue in a similar manner, we can easily see that the graph H_n^k is obtained by combining k graphs H_{n-1}^k with a root vertex. Hence, the graph H_n^k contains previous k^i graphs H_{n-i}^k for $1 \leq i \leq n-1$. Then, we have the following generalized formula.

$$\begin{aligned} H_n^k &= kH_{n-1}^k \\ &= k(k(H_{n-2}^k)) \\ &= k(k(k(H_{n-3}^k))) \\ &\dots \\ &= k^i(H_{n-i}^k). \end{aligned}$$

So, we obtain $H_n^k = k^i(H_{n-i}^k)$ for $1 \leq i \leq n-1$. The proof of exponential domination number of H_n^k is examined on the value n.

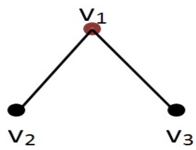

FIGURE 4. Graph H_1^2

From Figure 4, we can easily see that $\gamma_e - set$ S of H_1^2 is $\{v_1\}$. This vertex is also the root vertex.

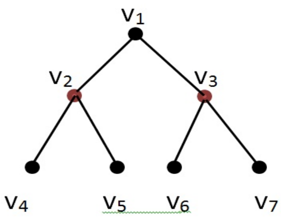

FIGURE 5. Graph H_2^2

From Figure 5, we can easily see that H_2^2 includes two subgraphs H_1^2. $\gamma_e - set$ of H_2^2 is $\{v_2, v_3\}$. The vertices of this set are the vertices of $\gamma_e - set$ of two H_1^2. Hence, for $n \geq 1$ and $1 \leq i \leq n-1$

$$\gamma_e(H_n^k) = k\gamma_e(H_{n-1}^k). \tag{2}$$

When we use this result in recursive formula which obtained for H_n^k, we have $\gamma_e(H_n^k) = k^i\gamma_e(H_{n-i}^k)$ for $n \geq 1$ and $1 \leq i \leq n-1$. We must prove this formula by induction for every value i. Let $\forall n \in Z^+$ and $n \geq 1$.

When $i = 1$, we have $\gamma_e(H_n^k) = k\ .\ \gamma_e(H_{n-1}^k)$ and it is true by (2). We assume that the result is true for $i = 5$ and prove it for $i = s+1$. By induction hypothesis and (2), we get

$$\begin{aligned}\gamma_e(H_n^k) &= k^s\gamma_e(H_{n-s}^k)\\ &= k^s(k\gamma_e(H_{n-s-1}^k))\\ &= k^{s+1}\gamma_e(H_{n-s-1}^k).\end{aligned}$$

That is, the formula is true for $i = s+1$. Hence, we have $\gamma_e(H_n^k) = k^i\gamma_e(H_{n-i}^k)$ for $n \geq 1$ and $1 \leq i \leq n-1$. Initial condition $n = 1$ is achieved for $i = n-1$. Hence, we obtain the following formula.

$$\begin{aligned}\gamma_e(H_n^k) &= k^i\gamma_e(H_{n-i}^k)\\ &= k^{n-1}\gamma_e(H_{n-(n-1)}^k)\\ &= k^{n-1}\gamma_e(H_1^k)\\ &= k^{n-1}.\end{aligned}$$

The proof is completed. □

Definition 3.4. [17]
A tree T is called a caterpillar, if removal of all its pendant vertices results in a path called the spine of T , denoted by $sp(T)$. If all vertices of $sp(T)$ have equal number of pendant vertices, then the resulting graph is called a regular caterpillar. A regular caterpillar can also be defined as the corona of two special graph types. That is, if $T_{n,m}$ is a regular caterpillar, then $T_{n,m} \cong P_n \circ mK_1$.

Theorem 3.4. *The exponential domination number of regular caterpillar $T_{n,m}$ is given by $\gamma_e(T_{n,m}) = \lceil (n+1)/2 \rceil$.*

Proof. Regular caterpillar $T_{n,m}$ has vertices of P_n (or the spine) and m the pendant vertices that attached each vertex of P_n. Let S be $\gamma_e - set$ of $T_{n,m}$. Assume that, we add all vertices of P_n to S to dominate all pendant vertices. Hence, the condition $w_S(v) \geq 1$ satisfy for all pendant vertices. But this set is not minimum exponential dominating set. Thus, S must consist of the vertices with $\overline{d}(u,v) = 2$ in $< V(T_{n,m}) - (S - \{u\}) >$, where $u \in S$, $v \in V(T_{n,m}) - S$. If $\overline{d}(u,v) = 2$, we must add a vertex w in P_n to S that this vertex is at distance 2 from u. Hence, the condition $w_S(v) = 1$ satisfy.
If n is odd, it is clear to see that the vertices which generate S are also the vertices of independent set of the spine graph P_n. Hence, for every vertex x in $V(P_n) - S$, $w_S(x) = 1$. We have $\gamma_e(T_{n,m}) = \lceil n/2 \rceil = (n+1)/2 = \lceil (n+1)/2 \rceil$.
If n is even, unlike the previous condition it is not enough to exponential dominate the last vertex v_n of spine graph P_n with $\lceil n/2 \rceil$ vertices from the graph P_n. Because, the weight of S at the vertex v_n is $1/2$. Hence, we must also add v_n to S. Thus, we have $\gamma_e(T_{n,m}) = \lceil n/2 \rceil + 1 = n/2 + 1 = n/2 + 2 = \lceil (n+1)/2 \rceil$.
Whether n is odd or even, combining two cases we have, $\gamma_e(T_{n,m}) = \lceil (n+1)/2 \rceil$.
The proof is completed. □

Definition 3.5. [14] *The graph E_n^t is a tree which has t legs and each leg has n vertices. Thus, E_n^t has $nt+2$ vertices. We have labeled the vertices of E_n^t as follows.*

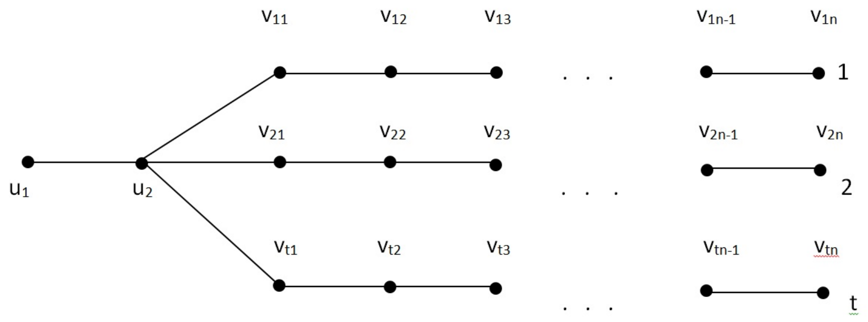

FIGURE 6. Tree E_n^t

Theorem 3.5. *The exponential domination number of tree E_n^t is given by*

$$\gamma_e(E_n^t) = \begin{cases} \begin{cases} \lceil n+1/4 \rceil t, & if\, t \leq 2^n \\ \lceil n+1/4 \rceil 2^n, & if\, t > 2^n \end{cases} & , n \equiv 2 \ (mod\ 4) \\ \begin{cases} \lfloor n+1/4 \rfloor t + 1, & if\, t < 2^{d(u_1,v)} \\ \lfloor n+1/4 \rfloor t, & if\, 2^{d(u_1,v)} \leq t < 2^{n+d(u_2,v)-1} \\ \lfloor n+1/4 \rfloor 2^{n+d(u_2,v)-1}, & if\, t \geq 2^{n+d(u_2,v)-1} \end{cases} & , n \equiv 0,1,3 \ (mod\ 4) \end{cases}$$

where,

$$d(u_1, v) = \begin{cases} 3, & if\ n \equiv 3 \ (mod\ 4) \\ 4, & if\ n \equiv 0 \ (mod\ 4) \\ 5, & if\ n \equiv 1 \ (mod\ 4) \end{cases}$$

and,
$d(u_2, v) = d(u_1, v) - 1$,
where u_1, u_2 vertices not in legs of E_n^t and v is vertex of legs.

Proof. Let S be $\gamma_e - set$ of E_n^t. For the vertex v_{in} $(1 \leq i \leq t)$ on all legs, $deg(v_{in}) = 1$. The vertex $v_{in-1}$of some legs must be added to S to exponentially dominate the vertex v_{in}. Each leg has the path graph P_n. The proof can be proved as the proof of Theorem 2.1. We start constructing S starting with the vertex v_{in-1} $(i = 1)$ on the first leg and take each vertex at distance 4 from selecting vertex on the leg. Hence, we add $\lceil (n+1)/4 \rceil$ vertices to S from the first leg in E_n^t. We use similar argument for other legs. Thus, we have four cases depending on $|V(P_n)|$ in the graph E_n^t.
For $n \equiv 0, 1, 3$ (mod 4), we get same results. Therefore we examine the proof into two cases.
Case 1: Let $n \equiv 2$ (mod 4).
If S is constructed as described above for any leg of E_n^t, then we get $S = \{v_{in-1}, ..., v_{i5}, v_{i1}\}$. Similar argument is not applied for each leg in E_n^t. Because, this approach is contradict with the definition of the minimum exponential dominating set.
It is easy to see that $u_2 \in V(E_n^t)$ is adjacent to $v_{i1} \in S$ in Figure 5. Therefore, v_{i1} dominates u_2. Since the distance between u_1 and v_{i1} is 2, we must add a vertex v_{i1} except v_{11}

to S. Therefore, the vertices of S should be taken from a certain number of legs in E_n^t. So, every vertex of $V(E_n^t) - S$ is exponentially dominated by S. This value depends on the distance between vertices of different legs.
Let v_{in} and v_{j1} be the vertices of the ith and jth legs, respectively, where $i, j \in \{1, 2, ..., t\}, i \neq j$. Note that $d(v_{j1}, v_{in}) = n - 1 + d(u_2, v_{j1}) + 1$. Since, $d(u_2, v_{j1}) = 1$, we get $d(v_{j1}, v_{in}) = n + 1$. Hence, we have two subcases depending on the number of legs in E_n^t.

Subcase 1.1. If $t < 2^{d(v_{j1}, v_{in})-1}$, that is $t < 2^n$ then the vertices of S should be taken from each leg. Hence, we have $w_s(z) \geq 1$ for every z in $V(E_n^t)$. So, we obtain $|S| = \lceil (n+1)/4 \rceil t$.

Subcase 1.2. If $t \geq 2^n$, then the vertices of S should be taken from only 2^n legs. Hence, we get $|S| = \lceil (n+1)/4 \rceil 2^n$.

Case 2: Let $n \equiv 0, 1, 3 \pmod 4$.
If S is constructed as in Case 1, then we get three different S sets depending on n mod 4. These set are
$S = \{v_{in-1}, ..., v_{i7}, v_{i3}\}$ or $S = \{v_{in-1}, ..., v_{i8}, v_{i4}\}$ or $S = \{v_{in-1}, ..., v_{i6}, v_{i2}\}$.
Hence, we must get $\lfloor (n+1)/4 \rfloor$ vertices to S from only one leg in E_n^t. The minimum distance between $u_1 \in V(E_n^t)$ and $v_{i2} \in S$ or $v_{i3} \in S$ or $v_{i4} \in S$ is $d(u_1, v_{i2}) = 3$, $d(u_1, v_{i3}) = 4$ and $d(u_1, v_{i4}) = 5$. The rest of the proof is performed as in Case1.
Note that, $d(v_{in}, v_{j2}) = n-1+d(u_2, v_{j2})+1$, $d(v_{in}, v_{j3}) = n-1+d(u_2, v_{j3})+1$, $d(v_{in}, v_{j4}) = n - 1 + d(u_2, v_{j4}) + 1$.
We denote the vertices v_{j2}, v_{j3} and v_{j4} by v to give a generalized formula. So, we have three subcases depending on the number of legs.

Subcase 2.1. If $t < 2^{d(u_1, v)}$, then the vertices of S should be taken from each leg. In this case, $w_S(u_1) \geq 1$ for $u_1, u_2 \in V(E_n^t)$ and the condition $w_S(u_2) \geq 1$ is not satisfied. Hence, we must add the vertex u_1 or u_2 to S. Therefore, we get $|S| = \lfloor (n+1)/4 \rfloor t + 1$.

Subcase 2.2. If $2^{d(u_1, v)} \leq t \leq 2^{n+d(u_2, v)-1}$ then the vertices of S should be taken from each leg. Thus, $w_S(z) \geq 1$ satisfies for every z in $V(E_n^t)$. We obtain $|S| = \lfloor (n+1)/4 \rfloor t$.

Subcase 2.3. If $t \geq 2^{n+d(u_2, v)-1}$, then the proof is similar to Subcase 1.2. Hence, we get $|S| = \lfloor (n+1)/4 \rfloor 2^{n+d(u_2, v)-1}$.
Summing Case 1 and Case 2, we obtain exponential domination number of E_n^t.
The proof is completed. □

4. Conclusion

The vulnerability of a communication can be measured by the exponential domination number of the graph describing the network. The exponential domination number has been studied as a vulnerability parameter introduced in [16]. Calculation of the exponential domination number for simple graph is important because if one can break a more complex network into smaller networks, then under some conditions the solutions for the optimization problem on the smaller networks can be combined to a solution for the optimization problem on the larger network.

References

[1] Anderson, M., Brigham, R. C., Carrington, J. R., Vitray, R. P., Yellen, J., (2009). On Exponential Domination of $C_m x C_n$, AKCE J.Graphs. Combin.,6, No. 3 341-351.
[2] Aytaç, V. and Turacı, T., (2017). Exponential Domination and Bondage Numbers in Some Graceful Cyclic Structure, Nonlinear Dynamics and Systems Theory, 17(2), 139-149.
[3] Aytac, A. and Atay, B., (2016). On Exponential Domination of Some Graphs, Nonlinear Dynamics and Systems Theory, 16(1), 12-19.

[4] Aytaç, A., Odabaş, Z.N., and Turacı, T.,(2011). The bondage number of some graphs, Comptes Rendus de Lacademie Bulgare des Sciences, vol. 64, no. 7, pp. 925930.
[5] Aytaç, A., Turacı, T. and Odabaş, Z.N.,(2013). On the bondage number of middle graphs, Mathematical Notes, vol. 93, no. 5-6, 795 801.
[6] Aytaç, A. and Odabaş, Z.N.,(2011). Residual Closeness of Wheels and Related Networks, Int. J. Found. Comput. Sci., 22, 12291240.
[7] Aytaç, A. and Berberler, Z.N.,(2017). Binding Number and Wheel Related Graphs, Int. J. Found. Comput. Sci., 28(1), 29-38.
[8] Aytaç, A. and Berberler, Z.N.,(2017). Robustness of regular caterpillars, Int. J. Found. Comput. Sci., accepted.
[9] Aytaç, A. and Odabaş Berberler, Z.N. Network robustness and residual closeness, RAIRO Operations Research, DOI: https://doi.org/10.1051/ro/2016071, appear.
[10] Aytaç, A. and Odabaş, Z.N.,(2010). Computing the rupture degree in composite graphs, Int. J.Found. Comput. Sci., 21(03), 311-319.
[11] Aytaç, A. and Odabaş, Z.N.,(2009). On Computing The Vulnerability Of Some Graphs As Average, International Journal of Pure and Applied Mathematics, 55(1), 137-146.
[12] Berberler, M.E and Berberler, Z.N.,(2017). Measuring the vulnerability in networks: a heuristic approach, ARS COMBINATORIA, accepted.
[13] Chartrand, G. and Lesniak, L., (1986). Graphs and Digraphs, Second Edition, Wadsworth. Monterey.
[14] Cormen, T., Leiserson, C.E., and Rivest, R.L., (1990). Introduction to Algorithms, The MIT Pres. (Fourth edition).
[15] Cymen, M., Pilipczuk, M. And Skrekovski, R., (2010). Relation between Randic index and average distance of trees, Institute of Mathematics, Physics and Mechanics Jadranska, 48, Slovenia, 1130p.
[16] Dankelmann, P., Day, D., Erwin, D., Mukwembi, S., Swart, H., (2009). Domination with exponential decay, Discrete Mathematics 309, 5877-5883.
[17] Gallian J.A.,(2008). A dynamic survey of graph labeling, Elect. Jour. Combin. 15, DS6.
[18] Hartsfield, N. and Ringel, Gerhard.,(1990). Pearls in Graph Theory, Academic Press, INC.
[19] Frucht, R. and Harary, F.,(1970). On the corona two graphs, Aequationes Math., vol.4, pp.322-325.
[20] Haynes, T.W., Hedetniemi, S.T., and Slater, P.j.Fundamentals of Domination in Graphs, Marcel Dekker,Inc., New York.
[21] Haynes, T.W., Henning, M. A., Van Der Merwe, L.C., (2007). The complementary product of two graphs, Bull. Instit. Combin. Appl. 51, 21-30.
[22] Haynes, T.W., Henning, M. A., Van Der Merwe, L. C.,(2009). Domination and Total Domination in Complementary Prisms, J. Comb. Optim 18, 23-37.
[23] Hedetniemi, S. T. and Laskar, R.C.,(1990).Bibliography on domination in graphs and some basic definitions of domination parameters, Discrete Mathematics, vol. 86, no. 1-3, pp. 257-277.
[24] Odabaş, Z.N. and Aytaç, A.,(2013). Residual closeness in cycles and related networks, Fundamenta Informaticae 124 (3), 297-307.
[25] Odabaş, Z.N. and Aytaç, A.,(2012). Rupture Degree and Middle Graphs, Comptes Rendus De L Academie Bulgare Des Sciences 65(3), 315-322.
[26] Turacı, T. and Aytaç, V.Residual closeness of splitting networks, Ars Combinatoria (accepted).
[27] West, D.B.,(2001). Introduction to Graph Theory (Second edition).

Aysun AYTAÇ Associate Prof., graduated from Department of Mathematics, Ege University, Izmir, Turkey in 1993. She received her MS degree and PhD in Mathematics and Computer Science from Ege University in 1996 and 2001, respectively. She is a member of Department of Mathematics, Faculty of Science, Ege University, Izmir, Turkey since 1996. Her research interests focus mainly in graph theory, especially vulnerability parameters in graphs, domination in graphs, Optimization, Combinatorics and Discrete Mathematics.

Betül ATAY ATAKUL graduated from Department of Mathematics, Ankara University, Ankara, Turkey in 2009. She received her Ms degree in Mathematics from Ankara University in 2011 and PhD degree in Mathematics and Computer Science from Ege University in 2016. She is a member of Department of Computer Education and Instructional Technology, Faculty of Education, Ağrı Ibrahim Çeçen University, Ağrı, Turkey since 2016. Her research interests focus mainly in graph theory, especially vulnerability parameters in graphs, domination in graphs, Optimization, Combinatorics and Discrete Mathematics.