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RAMSEY NUMBERS FOR CLASS OF EDGELESS, COMPLETE AND STAR GRAPHS

KALIRAJ K.[1], NARESH KUMAR H.[2], §

Abstract. For any graph class $\mathcal{G}$ of any two positive integers i and j, the Ramsey number $R_{\mathcal{G}}(i,j)$ is the smallest integer such that every graph in $\mathcal{G}$ on atleast $R_{\mathcal{G}}(i,j)$ vertices has a clique of size i or an independent set of size j. In this paper, we found the Ramsey numbers for the graph class of edgeless graphs, complete graphs, star graphs and class of all edgeless and complete graphs.

Keywords: Graph class and Ramsey numbers.

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1. Introduction

Ramsey theory is an important subfield of combinatorics that studies how a large system can be ensured that it contains some particular structure. Since the begining of 1930 [7] there has been a huge intrest in Ramsey theory, leading to many results in various application field of mathematics (refer [5] and [6]). A graph G on n vertices is said to be star, a vertex in $V(G)$ is adjacent to the remaining vertices of G. For every pair of positive integers i and j, *Ramsey number* $R(i,j)$ is the smallest positive integer such that every graph on at least $R(i,j)$ vertices contains the clique of size i or an independent set of size j. Ramsey's Theorem [7], in its graph-theoritic version, states that the number $R(i,j)$ exists for every pair of positive integers i and j. The suprising thing in this area is very hard to find out the value of Ramsey numbers for eventually smaller numbers. Eventhough they work this subject for past eight decades, we still don't know the value of $R(4,6)$ and $R(3,10)$. This is most adequately addressed by the following quote, attributed by Paul Erdos [8]. *"Imagine an alien force, vastly more powerful than us, landing on Earth and demanding the value of $R(5,5)$ or they will destroy our planet. In that case, we should marshall all our computers and all our mathematicians and attempt to find the value. But*

[1] Assistant Professor, Ramanujan Institute for Advanced Study in Mathematics, University of Madras, Chepauk, Chennai-600 005, Tamil Nadu, India.
e-mail: sk.kaliraj@gmail.com; ORCID: https://orcid.org/0000-0003-0902-3842.
[2] M. Phil Scholar, Ramanujan Institute for Advanced Study in Mathematics, University of Madras, Chepauk, Chennai-600 005, Tamil Nadu, India.
[2] Current Designation: Research Scholar, Department of Mathematics, SASTRA University, Thanjavur 613401, Tamil Nadu, India.
e-mail: nareshhari1403@gmail.com; ORCID: 0000-0002-1717-239X.
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suppose instead, that they ask for $R(6,6)$*. In that case, we should attempted to destroy aliens".* During last two decades, with the use of computers, lower bound and upper bound has been established for more Ramsey numbers. However, no more than 16 non trivial Ramsey numbers has been determined for the pair of integers $i, j \geq 3$.

2. Preliminaries

All graphs we consider are simple and finite. A subset S of vertices of a graph is a *clique* if all the vertices in S are pairwise adjacent, and S is an *independent set* if no two vertices of S are adjacent. For complete graph and edgeless graph the vertex subset S is replaced by $V(G)$ from clique and independent set respectively. For any graph class $\mathcal{G}$ any two positive integers i and j, the Ramsey number $R_{\mathcal{G}}(i,j)$ is the smallest integer such that every graph in $\mathcal{G}$ on atleast $R_{\mathcal{G}}(i,j)$ vertices has a clique of size i or an independent set of size j. It is well-known that Ramsey numbers for general graphs are symmetric, i.e., that $R(i,j) = R(j,i)$ for all $i, j \geq 1$. More generally, $R_{\mathcal{G}}(i,j) = R_{\mathcal{G}}(j,i)$ for all $i, j \geq 1$ for every class $\mathcal{G}$ that is closed under taking complements, i.e., if for any graph G in $\mathcal{G}$, its complement $\overline{G}$ also belongs to $\mathcal{G}$. If $\mathcal{G}$ is not closed under complements, then the Ramsey numbers for $\mathcal{G}$ are typically not symmetric. For any two graph class $\mathcal{G}$ and $\mathcal{G}'$ such that $\mathcal{G} \subseteq \mathcal{G}'$, we clearly have that $R_{\mathcal{G}}(i,j) \leq R_{\mathcal{G}'}(i,j)$ for all $i, j \geq 1$. In particular, it holds that $R_{\mathcal{G}}(i,j) \leq R(i,j)$ for any graph class $\mathcal{G}$ and for all $i, j \geq 1$, which implies that all such number $R_{\mathcal{G}}(i,j)$ exists.

The following results are the Ramsey numbers of few graph classes.

Observation 2.1. [2] *For any graph class* $\mathcal{G}$*,* $R_{\mathcal{G}}(1,j) = 1$ *for all* $j \geq 1$ *and* $R_{\mathcal{G}}(i,1) = 1$ *for all* $i \geq 1$*.*

Observation 2.2. [2] *If* $\mathcal{G}$ *contains all Edgeless graphs, then* $R_{\mathcal{G}}(2,j) = j$ *for all* $j \geq 1$*. Moreover, If* $\mathcal{G}$ *contains all Complete graphs, then* $R_{\mathcal{G}}(i,2) = i$ *for all* $i \geq 1$*.*

Theorem 2.1. [9] *Let* $\mathcal{P}$ *be the class of planar graphs. Then,*

- $R_{\mathcal{P}}(2,j) = j$ *for all* $j \geq 2$*.*
- $R_{\mathcal{P}}(3,j) = 3j - 3$ *for all* $j \geq 2$*.*
- $R_{\mathcal{P}}(i,j) = 4j - 3$ *for all* $i \geq 4$ *and* $j \geq 2$ *such that* $(i,j) \neq (4,2)$*.*
- $R_{\mathcal{P}}(4,2) = 4$*.*

Theorem 2.2. [1] *Let* $\mathcal{L}$ *be the class of line graphs. Then,*

- $R_{\mathcal{L}}(2,j) = j$ *for all* $j \geq 1$*.*
- *For every integer* $j \geq 1$*,* $R_{\mathcal{L}}(3,j) = \begin{cases} \frac{5(j-1)-1}{2} + 1 & \text{if } j \text{ is even} \\ \frac{5(j-1)}{2} + 1 & \text{if } j \text{ is odd} \end{cases}$.
- *For every pair of integers* $i \geq 4$ *and* $j \geq 1$*,*

$$R_{\mathcal{L}}(i,j) = \begin{cases} 1(j-1) - (t+r) + 2 & i = 2k \\ 1(j-1) - r + 2 & i = 2k+1 \end{cases}$$

where $j = 2k + r$*,* $t \geq 0$ *and* $1 \leq r \leq k$*.*

Theorem 2.3. [2] *Let* $\mathcal{G}$ *be the class of split graphs. Then* $R_{\mathcal{G}}(i,j) = i + j - 1$*.*

Theorem 2.4. [2] *Let* $\mathcal{G}$ *be the class of threshold graphs. Then* $R_{\mathcal{G}}(i,j) = i + j - 2$*.*

Theorem 2.5. [2] *Let* j *be a positive integer and* $\mathcal{G}(j)$ *be the class of* $\overline{K_j}$*-free graphs. Then* $R_{\mathcal{G}(j)}(i,j) = R(i,j)$ *for every positive integer* i*.*

3. Edgeless and complete graphs

In this section, we determine all Ramsey numbers for class of all edgeless and complete graphs.

Proposition 3.1. *Let $\mathcal{G}$ be the class of all edgeless graphs, then $R_{\mathcal{G}}(i,j) = j$ for all $i, j \geq 2$.*

Proof. The proof for $j = 1, 2$ was follows from the Observations 2.1 and 2.2 respectively. Since $\mathcal{G}$ is class of all edgeless graphs i.e. it is only collections of independent set of size j where $j = 1, 2, \ldots$. Then clearly by definition $R_{\mathcal{G}}(i,j) = j$ for all $i, j \geq 2$. □

Proposition 3.2. *Let $\mathcal{G}$ be the class of all complete graphs, then $R_{\mathcal{G}}(i,j) = i$ for all $i, j \geq 2$.*

Proof. The proof of this theorem is similar way of Proposition 3.1, but i is used instead of j and the collection is only cliques. □

Theorem 3.1. *Let $\mathcal{G}$ be the class of all edgeless and Complete graphs, then $R_{\mathcal{G}}(i,j) = \max\{i,j\}$ for all $i, j \geq 2$.*

Proof. We split the proof of the theorem in three cases based on the values of i and j.
Case (i) $i > j$

From the theorem we have $R_{\mathcal{G}}(i,j) = i$. The graphs on i vertices in the graph class $\mathcal{G}$ are K_i and $\overline{K_i}$. We can easily see that, in one graph has a clique on i vertices i.e. K_i and on another i.e. $\overline{K_i}$ it contains induced subgraph as independent set on j vertices. Suppose on $i-1$ vertices graphs in the graph class $\mathcal{G}$ one graph say K_{i-1} doesn't have clique on i vertices and independent set on j vertices, so i is the minimum value for this case and the above result holds.
Case (ii) $i < j$

The proof follows from the case (i) based on the value j.
Case (iii) $i = j = k$(say)

Then $R_{\mathcal{G}}(k,k) = \max k, k = k$. The graphs on k vertices in the graph class $\mathcal{G}$ are K_k and $\overline{K_k}$ and by definition one is clique of size k and another is independent set on size k. So, the result is also true for this case. □

4. Star graphs

In this section, we determine all Ramsey number for class of all star graphs and with other graph classes.

Theorem 4.1. *Let $\mathcal{G}$ be the class of all star graphs. Then*

$$R_{\mathcal{G}}(i,j) = \begin{cases} 2 & \text{if} \quad i = 2 \\ j+1 & \text{if} \quad i \geq 3 \end{cases}$$

for every pair of integers $i \geq 2$ and $j \geq 2$.

Proof. The proof of this theorem contains two cases.
Case (i) $i = 2$

Since the graph class $\mathcal{G}$ contains K_2 and by the definition $R_{\mathcal{G}}(2,j) = 2$ for all $j \geq 2$.

Case (ii) $i \geq 3$

Let G be a star graph on atleast $j+1$ vertices. Then, clearly G contains independent set on size j. Hence $R_{\mathcal{G}}(i,j) \leq j+1$ for all $i \geq 3$ and $j \geq 2$. For the lower bound, consider

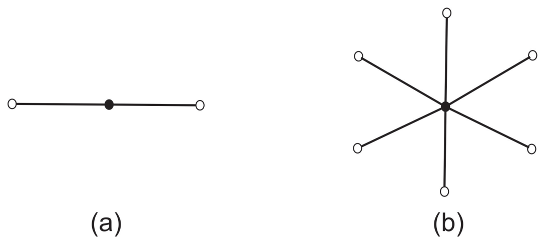

FIGURE 1. Graph of (a)$R_{\mathcal{G}}(i, 2)$ for all $i \geq 3$ (b)$R_{\mathcal{G}}(i, 6)$ for all $i \geq 3$, where $\mathcal{G}$ is the class of all star graphs.

a graph on j vertices, which contains independent set on j vertices. Clearly, the graph must be edgeless, which is contradiction to our assumption on graph class $\mathcal{G}$ and hence $R_{\mathcal{G}}(i, j) \geq j + 1$ for all $i \geq 3$ and $j \geq 2$. □

The graph of $R_{\mathcal{G}}(i, j)$ vertices for the class of all star graphs showed in Figure 1, where the collection of unshaded vertices represents the independent set for the graph on $R_{\mathcal{G}}(i, j)$ vertices.

Corollary 4.1. *Let $\mathcal{G}$ be the class of all star and edgeless graphs. Then*

$$R_{\mathcal{G}}(i, j) = \begin{cases} j & \text{if} \quad i = 2 \\ j + 1 & \text{if} \quad i \geq 3 \end{cases}$$

for every pair of integers $i \geq 2$ and $j \geq 2$.

Corollary 4.2. *Let $\mathcal{G}$ be the class of all star and complete graphs. Then*

$$R_{\mathcal{G}}(i, j) = \begin{cases} 2 & \text{if} \quad i = 2 \\ i & \text{if} \quad j = 2 \\ j + 1 & \text{if} \quad i, j \geq 3 \end{cases}$$

for every pair of integers $i \geq 2$ and $j \geq 2$.

Corollary 4.3. *Let $\mathcal{G}$ be the class of all star, complete and edgeless graphs. Then*

$$R_{\mathcal{G}}(i, j) = \begin{cases} \max\{i, j\} & \text{if} \quad i \text{ or } j = 2 \\ j + 1 & \text{if} \quad i \leq j \text{ and } i, j \geq 3 \\ i & \text{if} \quad i > j \text{ and } i, j \geq 3 \end{cases}$$

for every pair of integers $i \geq 2$ and $j \geq 2$.

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Kaliraj K. is working as an assistant professor in Ramanujan Institute for Advanced Study in Mathematics, University of Madras, Chepauk, Chennai-600 005, Tamil Nadu, India. He has 6 years of teaching experience. He completed his doctoral degree in the field of Graph Theory, particularly in Graph Colorings under Anna University, Chennai.

Naresh Kumar H. is a m. phil scholar in Ramanujan Institute for Advanced Study in Mathematics, University of Madras, Chepauk, Chennai-600 005, Tamil Nadu, India. Currently, he is a Ph.D. scholar in Pure Mathematics at SASTRA University, Thanjavur. His research interests are in the area of graph theory and combinatorics.