TWMS J. App. Eng. Math. V.9, N.2, 2019, pp. 257-266

MUTANT FUZZY SETS

MUSTAFA BURÇ KANDEMİR[1], §

ABSTRACT. In this paper, the notion of mutant fuzzy sets that by adhering to the classical sense in any semigroups has been introduced and its some of structural properties have been studied. In addition to this, the concept of t-norm based mutation for fuzzy sets on any crisp set has been given, and some of results have been investigated.

Keywords: Fuzzy sets, Mutant sets, Semigroup, t-norm

AMS Subject Classification: 03E72, 20Mxx, 08Axx

1. Introduction

In biology, a *mutation* is defined as the permanent alteration of the nucleotide sequences of the genetic elements. In a colloquial manner, the progeny formed by the mating of any two individuals can be expressed as being different from expected. Mullin described the mutation process in algebraic sense in mathematics and he posed research problems and defined a mutation in a grupoid in [3]. In [4], he assumed that the set S is a population of individuals and the mating of any two individuals of the population is a binary operation on S. Thus, the progeny of the two individuals a and b both in S have written as $a * b$ which is also in S. According to [4], it could be divided S into equivalence classes according to some properties of elements of S (e.g. eye color, blood type). Sometimes, it may be ensued an offspring which is not in M from the mating of any two individuals that is in M. In that case, a *mutation* has taken place in the set M. So, in his related article [4], Mullin algebraically stated the definition of mutation such as $M * M \subseteq M^c$ where M is a subset of the algebraic system $(S, *)$ and M^c is a complement of M. He said that the set M is a *mutant set* in $(S, *)$. Afterwards, Mullin generalized and applied the results of mutant sets for group theory and ring theory in [5]. Iseki [6] introduced the definition of mutation in a semigroup as generalized sense. In [7], Iseki gave some results of the Cartesian product of mutant sets. In [8], Kim discussed some properties of mutant sets and gave some results in topological semigroups and algebraic semigroups.

The notion of fuzzy set was introduced by Zadeh in his paper [1] in 1965. This concept provides a natural foundation for modeling mathematically the fuzzy phenomena, which exist pervasively in our real world, and for building new branches of fuzzy mathematics.

[1] Department of Mathematics, Faculty of Science, Mugla Sıtkı Koçman University, 48000, Mugla, Turkey.
e-mail: mbkandemir@mu.edu.tr; ORCID: http://orcid.org/0000-0002-0159-5670.

§ Manuscript received: May 15, 2017; accepted: December 11, 2018.
TWMS Journal of Applied and Engineering Mathematics Vol.9, No.2

In his pioneer paper, he established set-theoretical operations for fuzzy sets. Fuzzy set theory has been applied to many areas of the day.

In this paper, mutant fuzzy sets in semigroups with respect to *product* of two fuzzy sets given any semigroup are established and its basic properties are studied. In Section 3, t-norm based mutation for fuzzy sets on any crisp set are discussed. Prominent t-norm based operations among fuzzy sets such as *algebraic product*, *bounded product*, *drastic product* are used, and some results are given.

2. Preliminaries

As the preliminary information, which is necessary to study, give some definitions and properties.

For a non-empty set S, a function $*$ from $S \times S$ to S is called a *binary operation* on S, and the ordered pair $(S, *)$ is called an *algebraic structure*. A *semigroup* is an algebraic structure consisting of a set together with an associative binary operation, that is, $\forall a, b, c \in S$, $(a * b) * c = a * (b * c)$. We use the multiplication notation for a binary operation in this paper. Where the semigroup is written multiplicatively and where the nature of the multiplication is clear from the context, we shall write simply S rather than $(S, \cdot)$. Expressions such as abc and $a_1a_2 \dots a_n$ where $a, b, c, a_1, a_2, \dots, a_n$ are elements of S, then have unambiguous meaning, and we can use the notation a^n ($n \in \mathbb{N}$) to mean the product of n elements each equal to a. If a semigroup S has the property that, for all $a, b \in S$, $ab = ba$, we call that S is a *commutative semigroup.* Let f be a mapping from the semigroup S to the semigroup T. We call that f is a *homomorphism* if $f(ab) = f(a)f(b)$ for all $a, b \in S$ [9, 11].

Besides, Zadeh [1] is described, a fuzzy set whose definition as follows is a class of objects with a continuum of grades of membership.

Definition 2.1. [1] *Let S be a non-empty set. A* fuzzy set *A in S is defined by a membership function $\mu_A : S \to [0, 1]$ whose membership value $\mu_A(a)$ specifies the degree to which $a \in S$ belongs to the fuzzy set A, for each $a \in S$.*

The family of all fuzzy sets in S will be denoted by $\mathcal{F}(S)$. Let $A \in \mathcal{F}(S)$. If $\mu_A(a) = 0$ for all $a \in S$ then we call that A is *empty (or null) fuzzy set* and denoted by Φ. In a similar manner, if $\mu_A(a) = 1$ for all $a \in S$, then it is called that A is *universal fuzzy set* and denoted by $\mathcal{S}$. Besides, if $\mu_A(a) = \alpha$ for all $a \in S$ and $\alpha \in [0, 1]$, then A is called *α-universal fuzzy set* and denoted by $\mathcal{S}^\alpha$.

If $A, B \in \mathcal{F}(S)$ then some basic operations and definitions are given componentwise proposed by Zadeh [1] as follows:

(1) $A \subseteq B \Leftrightarrow \mu_A(a) \leq \mu_B(a)$, for all $a \in S$.
(2) $A = B \Leftrightarrow \mu_A(a) = \mu_B(a)$, for all $a \in S$.
(3) $C = A \cup B \Leftrightarrow \mu_C(a) = \mu_A(a) \vee \mu_B(a)$, for all $a \in S$.
(4) $D = A \cap B \Leftrightarrow \mu_D(a) = \mu_A(a) \wedge \mu_B(a)$, for all $a \in S$.
(5) $E = A^c \Leftrightarrow \mu_E(a) = 1 - \mu_A(a)$, for all $a \in S$.
(6) Let $\alpha \in [0, 1]$. αA is called *α-layer* of A such that its membership function is defined by $\alpha\mu_A(a) = \alpha \wedge \mu_A(a)$ for all $a \in S$.
(7) The *core* of a fuzzy set A, denoted by $\mathfrak{c}(A)$, is the crisp set of all $x \in X$ such that $\mu_A(x) = 1$ [12].
(8) The *support* of a fuzzy set A, denoted by $\mathfrak{s}(A)$, is the crisp set of all $x \in X$ such that $\mu_A(x) > 0$ [12].

(9) The set of elements that belong to the fuzzy set A at least to the degree α is called the *α-level set*:

$$A_\alpha = \{x \in S \mid \mu_A(x) \geq \alpha\}$$

$A'_\alpha = \{x \in X \mid \mu_A(x) > \alpha\}$ is called *strong α-level set* [12].

(10) The *cartesian product* $A \otimes B$ of A and B is a fuzzy set in the product space $S \times S$ whose membership function is defined as

$$\mu_{A\otimes B}(a, b) = \mu_A(a) \wedge \mu_B(b).$$

Definition 2.2. [1] *Let S and T be non-empty sets, f be a mapping from S into T, and let A be a fuzzy set on S and B be a fuzzy set on T. The fuzzy sets $f[A]$ on T and $f^{-1}[B]$ on S, defined by*

$$f(\mu_A)(b) = \begin{cases} \bigvee\{\mu_A(a) \mid a \in S, f(a) = b\} & , f^{-1}[\{b\}] \neq \varnothing \\ 0 & , \textit{otherwise} \end{cases}$$

for all $b \in T$, and

$$f^{-1}(\mu_B)(a) = \mu_B(f(a))$$

for all $a \in S$.

Intrinsically, the definitions of fuzzy algebraic structures have been defined in time. [9] and [10] for detailed information about fuzzy algebraic structures are recommended.

The notion of product of any two fuzzy sets on a semigroup is given as follows.

Definition 2.3. [9, 10] *Let S be a semigroup and A and B be fuzzy subsets of S such that its membership functions are $\mu_A : S \to [0, 1]$ and $\mu_B : S \to [0, 1]$, respectively, and $a, b, c \in S$. Then the* product *of A and B, denoted by $A \circ B$, defined by*

$$\mu_{A\circ B}(c) = (\mu_A \circ \mu_B)(c) = \bigvee\{\mu_A(a) \wedge \mu_B(b) \mid a, b \in S, ab = c\}$$

for all $c \in S$.

As is well-known, the operation $\circ$ is associative.

Note that, the m times multiplication of the fuzzy set A can be defined as $A^m = A \circ A \circ \cdots \circ A$ and its membership function is

$$\mu_{A^m}(a) = \bigvee\left\{\bigwedge_{i\in\{1,2,\ldots,m\}} \mu_A(a_i) \mid a_1a_2\ldots a_m = a\right\}.$$

from Definition 2.3.

Lemma 2.1. [15, 16] *Let S and T be semigroups and $f : S \to T$ be a epimorphism. Let A, B be fuzzy sets on S. Then $f[A \circ B] = f[A] \circ f[B]$.*

Note that, Lemma 2.1 is valid finite family of fuzzy sets.

Lemma 2.2. [15, 16] *Let S and T be semigroups and $f : S \to T$ be a epimorphism. Let A, B be fuzzy sets on T. Then $f^{-1}[A \circ B] = f^{-1}[A] \circ f^{-1}[B]$.*

Lemma 2.3. *Let S be a semigroup and A and B fuzzy set on S. If $A \subseteq B$, then $A^m \subseteq B^m$.*

Proof. Since $A \subseteq B$, $\mu_A(a) \leq \mu_B(a)$ for all $a \in S$. So, for all $a = a_1a_2 \ldots a_m$,

$$\bigvee\left\{\bigwedge_{i\in\{1,2,\ldots,m\}} \mu_A(a_i)\right\} \leq \bigvee\left\{\bigwedge_{i\in\{1,2,\ldots,m\}} \mu_B(a_i)\right\}.$$

Thus $A^m \subseteq B^m$. $\square$

Lemma 2.4. *Let S and T be semigroups and $A \in \mathcal{F}(S)$ and $B \in \mathcal{F}(T)$. Then $(A \otimes B)^m = A^m \otimes B^m$.*

Proof. For any $(a,b) \in S \times T$,

$$\begin{aligned}
\mu_{(A\otimes B)^m}(a,b) &= \bigvee\left\{\bigwedge_{i\in\{1,2,\ldots,m\}} \mu_{A\otimes B}(a_i,b_i) \mid \prod_{i=1}^{m}(a_i,b_i)=(a,b), a_i \in S, b_i \in T\right\} \\
&= \bigvee\left\{\bigwedge_{i\in\{1,2,\ldots,m\}} (\mu_A(a_i)\wedge\mu_B(b_i)) \mid \prod_{i=1}^{m}(a_i,b_i)=(a,b), a_i \in S, b_i \in T\right\} \\
&= \bigvee\left\{\left(\bigwedge_{i\in\{1,2,\ldots,m\}} \mu_A(a_i)\right)\wedge\left(\bigwedge_{i\in\{1,2,\ldots,m\}} \mu_B(b_i)\right) \mid \prod_{i=1}^{m}a_i=a, \prod_{i=1}^{m}b_i=b, a_i \in S, b_i \in T\right\} \\
&= \bigvee\left\{\bigwedge_{i\in\{1,2,\ldots,m\}} \mu_A(a_i) \mid a_i \in S, \prod_{i=1}^{m}a_i=a\right\}\wedge\bigvee\left\{\bigwedge_{i\in\{1,2,\ldots,m\}} \mu_B(b_i) \mid b_i \in T, \prod_{i=1}^{m}b_i=b\right\} \\
&= \mu_{A^m}(a)\wedge\mu_{B^m}(b) \\
&= \mu_{A^m\otimes B^m}(a,b)
\end{aligned}$$

Hence, it is obtained that $(A \otimes B)^m = A^m \otimes B^m$. □

3. Fuzzy Mutants in Semigroups

In this section, the concept of a mutant fuzzy set in any semigroups can be defined, analogously.

Definition 3.1. *Let S be a semigroup and M be a fuzzy set on S. M is called* mutant fuzzy set *over S if $M \circ M \subseteq M^c$.*

Definition 3.2. *Let S be a semigroup and M be a fuzzy set on S. M is called* (m,n)-mutant *in S if $M^m \subseteq (M^n)^c$.*

It can be easily seen that Definition 3.2 is a generalized form of Definition 3.1. It means that the $(2,1)$-mutant is a mutant in the sense of Definition 3.1.

Let M be a fuzzy mutant set on S, and $a \in M$. $\mu_M(a)$ is called *the degree of mutation* of the element a in M.

Example 3.1. *Let $S = \{0,1\}$. If the Cayley table is constructed as follows,*

$*$	0	1
0	0	1
1	1	1

$(S,*)$ is a semigroup. It is called that it is a two-element semigroup*. Let*

$$M = \{(0,0.4),(1,0.3)\} \in \mathcal{F}(S)$$

i.e. M is a fuzzy set on S. Then, we have its complement

$$M^c = \{(0,0.6),(1,0.7)\} \in \mathcal{F}(S).$$

So, let us calculate $M \circ M$. For $0 \in S$,

$$\begin{aligned}
\mu_{M\circ M}(0) &= \bigvee\{\mu_M(a)\wedge\mu_M(b) \mid a*b=0\} \\
&= \mu_M(0)\wedge\mu_M(0) \\
&= \mu_M(0) = 0.4,
\end{aligned}$$

For $1 \in S$,

$$\begin{aligned}\mu_{M\circ M}(1) &= \bigvee\{\mu_M(a)\wedge\mu_M(b) \mid a*b=1\}\\ &= (\mu_M(0)\wedge\mu_M(1))\vee(\mu_M(1)\wedge\mu_M(0))\vee(\mu_M(1)\wedge\mu_M(1))\\ &= 0.4\vee 0.4\vee 0.3 = 0.4.\end{aligned}$$

Thus, it is obtained that

$$M\circ M = \{(0,0.4),(1,0.4)\}\subset M^c.$$

Hence, the fuzzy set M is a mutant fuzzy set in S.

Moreover, for arbitrary natural numbers $m,n\in\mathbb{N}$, it is obtained that

$$M^m = M^n = \{(0,0.4),(1,0.4)\}$$

and

$$(M^n)^c = \{(0,0.6),(1,0.6)\}.$$

So, M is an (m,n)-mutant fuzzy set on S.

An example for non-mutant fuzzy sets can be given as follows:

Example 3.2. *Let $(S,*)$ be semigroup as in Example 3.1. Consider the fuzzy set $A = \{(0,0.7),(1,1)\}\in\mathcal{F}(S)$. So, $A^c = \{(0,0.3),(1,0)\}\in\mathcal{F}(S)$. If we calculate the fuzzy set $A\circ A\in\mathcal{F}(S)$, we have $A\circ A = \{(0,0.7),(1,1)\}$. Thus $A\circ A \not\subseteq A^c$. Hence A is not a mutant fuzzy set in S.*

Theorem 3.1. *Every subset of an (m,n)-mutant fuzzy set of S is an (m,n)-mutant of S.*

Proof. Let A be a fuzzy subset of the (m,n)-mutant fuzzy set M. If $A\subseteq M$, $A^m\subseteq M^m$ from Lemma 2.3. Since M is a mutant,

$$A^m\subseteq M^m\subseteq (M^n)^c\subseteq (A^n)^c$$

is obtained. Thus A is an (m,n)-mutant on S. □

Theorem 3.2. *Let M_1 and M_2 be (m,n)-mutant fuzzy sets on S. Then $M_1\cap M_2$ is an (m,n)-mutant fuzzy set on S.*

Proof. It is known that $M_1\cap M_2\subseteq M_1$ and $M_1\cap M_2\subseteq M_2$. From Lemma 2.3, $(M_1\cap M_2)^m\subseteq M_1^m$ and $(M_1\cap M_2)^m\subseteq M_2^m$ is obtained. Since M_1 and M_2 are (m,n)-mutant, so

$$(M_1\cap M_2)^m\subseteq M_1^m\subseteq (M_1^n)^c$$

and

$$(M_1\cap M_2)^m\subseteq M_2^m\subseteq (M_2^n)^c.$$

Thus, it is obtained that

$$\begin{aligned}(M_1\cap M_2)^m &\subseteq M_1^m\cup M_2^m\\ &\subseteq (M_1^n)^c\cup(M_2^n)^c\\ &= (M_1^n\cap M_2^n)^c\\ &= ((M_1\cap M_2)^n)^c.\end{aligned}$$

Hence $M_1\cap M_2$ is (m,n)-mutant. □

Corollary 3.1. *Let M_i be (m,n)-mutant fuzzy sets on S for $i\in I$. Then $\bigcap_{i\in I} M_i$ is an (m,n)-mutant fuzzy set on S, where $\bigcap_{i\in I} M_i$ is non-null.*

Theorem 3.3. *Let M be an (m,n)-mutant fuzzy set on S. αM is an (m,n)-mutant on S.*

Proof. Since $M^m \subseteq (M^n)^c$. It is obtained that

$$\begin{aligned} \alpha\mu_{M^m}(x) &= \alpha \wedge \mu_{M^m}(x) \\ &\leq \alpha \wedge (1-\mu_{M^n}(x)) \\ &= \alpha\mu_{(M^n)^c}(x), \forall x \in S. \end{aligned}$$

Then $\alpha M^m \subseteq \alpha(M^n)^c$. Thus αM is an (m,n)-mutant. □

Theorem 3.4. *S and T be semigroups, $f : S \to T$ be an epimorphism. If M is an (m,n)-mutant fuzzy set on S and $f[(M^n)^c] \subseteq (f[M^n])^c$, then $f[M]$ is an (m,n)-mutant fuzzy set on T.*

Proof. From Lemma 2.1, it is obtained that

$$(f[M])^m = f[M^m] \subseteq f[(M^n)^c] \subseteq (f[M^n])^c = ((f[M])^n)^c.$$

□

Theorem 3.5. *S and T be semigroups, $f : S \to T$ be an epimorphism. The inverse image under a epimorphism f of an (m,n)-mutant fuzzy set is an (m,n)-mutant fuzzy set.*

Proof. Let M be a (m,n)-mutant fuzzy set in T. From Lemma 2.2, it is obtained that

$$\begin{aligned} \left(f^{-1}[M]\right)^m &= f^{-1}[M^m] \\ &\subseteq f^{-1}\left[(M^n)^c\right] \\ &= \left(f^{-1}[M^n]\right)^c \\ &= \left(\left(f^{-1}[M]\right)^n\right)^c \end{aligned}$$

Hence, $f^{-1}[M]$ is (m,n)-mutant. □

Theorem 3.6. *Let S and T be semigroups, M and N be (m,n)-mutant fuzzy sets on S and T, respectively. The cartesian product of $M \otimes N$ is an (m,n)-mutant fuzzy set on $S \times T$.*

Proof. Since M and N are (m,n)-mutant fuzzy set in S and T, respectively, we have $M^m \subseteq (M^n)^c$ and $N^m \subseteq (N^n)^c$. From Lemma 2.4, we have

$$(A \otimes B)^m = A^m \otimes B^m \subseteq (A^n)^c \otimes (B^n)^c \subseteq (A^n \otimes B^n)^c = ((A \otimes B)^n)^c.$$

□

Theorem 3.7. *Let M be a mutant fuzzy set on S. $\mathfrak{c}(M)$, the core of M, is a mutant set on S.*

Proof. Let $a \in \mathfrak{c}(M)\mathfrak{c}(M)$. Then, there exists $a_1, a_2 \in \mathfrak{c}(M)$ such that $a = a_1a_2$. Thus, we have $\mu_M(a_1) = 1$ and $\mu_M(a_2) = 1$ for $a_1a_2 = a$. Then $\mu_M(a_1) \wedge \mu_M(a_2) = 1$ for $a_1a_2 = a$, and $\bigvee\{\mu_M(a_1) \wedge \mu_M(a_2) = 1 \mid a_1a_2 = a\} = 1$, so that, $\mu_{M^2}(a) = 1$ and $a \in \mathfrak{c}(M^2)$. Since M is mutant, i.e. $M^2 \subseteq M^c$, $\mu_{M^c}(a) > \mu_{M^2}(a) = 1$ is obtained. Thus, $\mu_M(a) = 0$, i.e. $a \notin \mathfrak{c}(M)$. Hence $a \in (\mathfrak{c}(M))^c$ is obtained. The proof is completed. □

Lemma 3.1. *Let M be a fuzzy set on S. If $\alpha > 0.5$, then $(M^c)_\alpha \subseteq (M_\alpha)^c$.*

Proof. Suppose that $a \in (M^c)_\alpha$, then $\mu_{M^c}(a) \geq \alpha$. So, $1 - \mu_M(a) \geq \alpha$, and $\mu_M(a) \leq 1-\alpha$ are obtained. Since $\alpha > 0.5$, then $\mu_M(a) < \alpha$. Herefrom, we have $a \notin M_\alpha$. Hence, $a \in (M_\alpha)^c$. □

Theorem 3.8. *Let M be a mutant fuzzy set on S. M_α, the α-level set of M, is a mutant set on S for $\alpha > 0.5$.*

Proof. Suppose that $a \in M_\alpha M_\alpha$. Then there exist $a_1, a_2 \in M_\alpha$ such that $a = a_1a_2$. Thus, $\mu_M(a_1) \geq \alpha$ and $\mu_M(a_2) \geq \alpha$ for $a_1a_2 = a$ are obtained. So, $\mu_M(a_1) \wedge \mu_M(a_2) \geq \alpha$ for $a_1a_2 = a$, and then $\bigvee\{\mu_M(a_1) \wedge \mu_M(a_2) \mid a_1a_2 = a\} \geq \alpha$, so that, $\mu_{M^2}(a) \geq \alpha$ and $a \in (M_\alpha)^2$. Since M is mutant i.e. $M^2 \subseteq M^c$, $\mu_{M^c}(a) > \mu_{M^2}(a) \geq \alpha$ is obtained. So, we have $a \in (M^c)_\alpha$. Since $\alpha > 0.5$, $(M^c)_\alpha \subseteq (M_\alpha)^c$ from Lemma 3.1, it is obtained that $a \in (M_\alpha)^c$. Hence we have $(M_\alpha)^2 \subseteq (M_\alpha)^c$. Thus, M_α is mutant subset of S for $\alpha > 0.5$. □

Theorem 3.9. *Let $\mathcal{S}^\alpha$ be an α-universal fuzzy set. If $\alpha \leq 0.5$, then $\mathcal{S}^\alpha$ is a mutant fuzzy set.*

Proof. It is needed to show that $(\mathcal{S}^\alpha)^2 \subseteq (\mathcal{S}^\alpha)^c$ for $\alpha \leq 0.5$. It is known that if $\alpha \leq 0.5$, then $1 - \alpha \geq 0.5$. So, it is obtained that

$$\begin{aligned}
\mu_{\mathcal{S}^\alpha \circ \mathcal{S}^\alpha}(a) &= \bigvee\{\mu_{\mathcal{S}^\alpha}(a_1) \wedge \mu_{\mathcal{S}^\alpha}(a_2) \mid a_1, a_2 \in S, a_1a_2 = a\} \\
&= \bigvee\{\alpha \wedge \alpha \mid a_1, a_2 \in S, a_1a_2 = a, \alpha \leq 0.5\} \\
&= \alpha \\
&\leq 1 - \alpha \\
&= 1 - \mu_{\mathcal{S}^\alpha}(a) \\
&= \mu_{(\mathcal{S}^\alpha)^c}(a)
\end{aligned}$$

for each $a \in S$. Hence, $\mathcal{S}^\alpha$ is a mutant fuzzy set for $\alpha \leq 0.5$. □

4. t-norm Based Mutation

In fuzzy set theory, there are t-norm based operations between fuzzy sets. Definition of t-norm is as follows:

Definition 4.1. [13] *A* t-norm *is a binary operation $\top$ on the unit interval $[0, 1]$ which is commutative, associative, monotone and has 1 as neutral element, i.e., it is a function $\top : [0, 1] \times [0, 1] \to [0, 1]$ such that for all $\alpha, \beta, \gamma \in [0, 1]$:*

(T1) $\alpha \top \beta = \beta \top \alpha$,
(T2) $(\alpha \top \beta) \top \gamma = \alpha \top (\beta \top \gamma)$,
(T3) $\alpha \leq \beta \Rightarrow \alpha \top \gamma \leq \beta \top \gamma$,
(T4) $\alpha \top 1 = x$.

As known, a t-norm is called *Archimedean* if for each $\alpha, \beta \in (0, 1)$ there is a natural number n such that $\underbrace{\alpha \top \cdots \top \alpha}_{n \text{ times}} \leq \beta$.

The most known of t-norms can be listed as follows;

- For each $\alpha, \beta \in [0, 1]$, $\alpha \top_0 \beta = \min\{\alpha, \beta\}$. It is known as *Gödel t-norm.*
- For each $\alpha, \beta \in [0, 1]$, $\alpha \top_1 \beta = \alpha \cdot \beta$. It is called *product t-norm.*
- $\alpha, \beta \in [0, 1]$, $\alpha \top_2 \beta = \max\{0, \alpha + \beta - 1\}$. It is known as *Łukasiewicz t-norm.*
- $\alpha, \beta \in [0, 1]$, $\alpha \top_3 \beta = \begin{cases} \min\{\alpha, \beta\} & , \text{ if } \alpha = 1 \text{ or } \beta = 1 \\ 0 & , \text{ otherwise} \end{cases}$. It is called *drastic t-norm.*

As stated in [14], further fuzzy set-theoretic operations can be expressed using the concept of t-norm. Prominent operations can be given as follows:

Definition 4.2. [14] *Let A and B be fuzzy sets on non-empty set S.*

(a) *The* algebraic product *of A and B, denoted by $A \bullet B$, is a fuzzy set on S, and its membership function defined as*

$$\mu_{A\bullet B}(x) = \mu_A(x)\top_1\mu_B(x), \forall x \in S.$$

(b) *The* bounded product *of A and B, denoted by $A \odot B$, is a fuzzy set on S, and its membership function defined as*

$$\mu_{A\odot B}(x) = \mu_A(x)\top_2\mu_B(x), \forall x \in S.$$

(c) *The* drastic product *of A and B, denoted by $A \star B$, is a fuzzy set on S, and its membership function defined as*

$$\mu_{A\star B}(x) = \mu_A(x)\top_3\mu_B(x), \forall x \in S.$$

These definitions are regarded as generalization of intersection between fuzzy sets.

Theorem 4.1. *Let A be a fuzzy set on the set S. If $\mu_A(x) < 0.5$ for each $x \in S$, then A is a mutant fuzzy set with respect to the t-norm $\top_0$ i.e. $A \cap A \subset A^c$.*

Proof. Let A be a fuzzy set on S. Then $A\top_0 A$ is a fuzzy set on S. For each $x \in S$, we compute that

$$\mu_A(x)\top_0\mu_A(x) = \min\{\mu_A(x), \mu_A(x)\} = \mu_A(x).$$

Since $\mu_A(x) < 0.5$ for each $x \in S$, then $1 - \mu_A(x) \geq 0.5$, i.e. $\mu_{A^c}(x) \geq 0.5$. Hence it is obtained that $A\top_0 A \subset A^c$. Thus, A is a mutant fuzzy set with respect to $\top_0$. □

Theorem 4.2. *Let A be a fuzzy set on S. If $\mu_A(x) \leq 0.6$ for each $x \in S$, then A is a mutant fuzzy set with respect to the t-norm $\top_1$, i.e. $A \bullet A \subset A^c$.*

Proof. From Definition 4.2 (a), $A \bullet A$ is a fuzzy set on S. Then, for each $x \in S$,

$$\mu_{A\bullet A}(x) = \mu_A(x)\top_1\mu_A(x) = (\mu_A(x))^2 < \mu_A(x).$$

Since $\mu_A(x) \leq 0.6$ for each $x \in S$, then $1 - \mu_A(x) > 0.4$, i.e. $\mu_{A^c}(x) > 0.4$. We have $\mu_{A^c}(x) > (\mu_A(x))^2 = \mu_{A\bullet A}(x)$ for each $x \in S$. Hence it is obtained that $A \bullet A \subset A^c$. Thus, A is a mutant fuzzy set with respect to $\top_1$. □

Theorem 4.3. *Let A be a fuzzy set on S. If $\mu_A(x) \leq 0.6$ for each $x \in S$, then A is a mutant fuzzy set with respect to the t-norm $\top_2$, i.e. $A \odot A \subset A^c$.*

Proof. From Definition 4.2 (b), $A \odot A$ is a fuzzy on S. It is computed that

$$\mu_{A\odot A}(x) = \mu_A(x)\top_2\mu_A(x) = \max\{0, 2\mu_A(x) - 1\}, \forall x \in S.$$

Since $0 \leq \mu_A(x) \leq 0.6$, we have $0 \leq 2\mu_A(x) - 1 \leq 0.2$, and if $\mu_A(x) \leq 0.6$, $\mu_{A^c}(x) > 0.4$. Hence $\mu_{A\odot A}(x) < \mu_{A^c}(x)$ for each $x \in S$. Thus, $A \odot A \subset A^c$, i.e. A is a mutant fuzzy set with respect to $\top_2$. □

Theorem 4.4. *Let A be a fuzzy set on S. If $\mu_A(x) \in [0, 1)$ for each $x \in S$, then A is a mutant fuzzy set with respect to the t-norm $\top_3$, i.e. $A \star A \subset A^c$.*

Proof. Form Definition 4.2 (c), $A \star A$ is a fuzzy set on S. Since $\mu_A(x) \in [0, 1)$ i.e. $\mu_A(x) \neq 1$ for all $x \in S$, and

$$\mu_{A\star A}(x) = \begin{cases} \min\{\mu_A(x), \mu_A(x)\} & , \text{ if } \mu_A(x) = 1 \\ 0 & , \text{ otherwise} \end{cases} = \begin{cases} \mu_A(x) & , \text{ if } \mu_A(x) = 1 \\ 0 & , \text{ otherwise} \end{cases}$$

we have $\mu_{A\star A}(x) = 0$ for all $x \in S$. At the same time, since $\mu_A(x) \in [0, 1)$, then $\mu_{A^c}(x) \in (0, 1]$. So, we have $\mu_{A\star A}(x) \leq \mu_{A^c}(x)$, for all $x \in S$. Thus $A \star A \subseteq A^c$, i.e. A is a mutant fuzzy set with respect to t-norm $\top_3$. □

Theorem 4.5. *Let A be a fuzzy set on S and $\top$ be a t-norm. If $\top$ is an Archimedean and $\mu_A(x) \leq 0.5$ for all $x \in S$, then A is an $(n,1)$-mutant fuzzy set.*

Proof. Let A be a fuzzy set on S and $\top$ be a t-norm. Since $\top$ is an Archimedean and $\mu_A(x) < 0.5$ for each $x \in S$, then we have

$$\mu_{A^n}(x) = \underbrace{\mu_A(x) \top \mu_A(x) \top \cdots \top \mu_A(x)}_{n \text{ times}} \leq \mu_A(x) < \mu_{A^c}(x).$$

Hence, it is obtained that $A^n \subset A^c$. Therefore, A is an $(n,1)$-mutant fuzzy set. □

5. Conclusion

In Mullin' s article [4], he stated that difficulties will arise in the mathematical formalization of biological mutation. He said that the most important of these problems is that offspring of any pair of individuals that is in the mutant set in the algebraic model is always a biological mutant. It is inconsistent with known biological facts. On the other hand, the fuzzy set theory is one of the most important mathematical tools which model phenomena that classical mathematical methods can not model. In this article, the notion of mutant fuzzy sets on an algebraic structure have been introduced, and we have tried to overcome above the problems encountered in classical theory of modeling for mutation. Naturally, mutant fuzzy sets are a generalization of mutant sets. So, we have a grade of mutations of the offspring of the individuals who mate in all the individual' s set. In this way, we are getting closer to the biological realities by grading the mutation. This is one of our main purposes. The author believes that the grade of mutation is even more useful in biological applications. Therefore, this article can be a useful tool for those who work in this field.

Besides these, the t-norm based mutation for fuzzy sets on any set without binary operation (i.e. this is not an algebraic structure) is also stated, and some results have been given. Because this situation is independent of any algebraic structure, it may be more appropriate to model the biological events of the mutation.

The author hopes that this article is shed light on to working scientists in these areas.

Acknowledgement The author would like to extend their gratitude to Assoc. Prof. Dr. Ummahan ACAR for her constructive criticism and advices. Furthermore, the author thank the anonymous referees for their constructive comments.

References

[1] Zadeh, L. A., (1965), Fuzzy sets, Inform. Control, 8, pp. 338–353.
[2] Rosenfeld, A., (1971), Fuzzy groups, J. Math. Anal. Appl., 35, pp. 512–517.
[3] Mullin, A. A., (1961), Properties of mutants, Bull. Amer. Math. Soc., 67, pp. 82.
[4] Mullin, A. A., (1962), On mutant sets, Bull. Math. Biol., 24, pp. 209–215.
[5] Mullin, A. A., (1962), Some theorems on the structure of mutant sets and their applications to group and ring theory, Notre Dame J. Form. Log., 3 (3), pp. 148–151.
[6] Iseki, K., (1962), On $(m, n)-$mutant sets in semigroups, Proc. Japan Acad., 38 (6), pp. 269–270.
[7] Iseki, K., (1962), On mutant sets in semigroup, Proc. Japan Acad., 38 (8), pp. 478–479.
[8] Kim, J. B., (1969), Mutants in semigroups, Czech. Math. Jour., 19, pp. 86–90.
[9] Mordeson, J. N., Malik, D.S. and Kuroki N., (2003), Fuzzy semigroups, Springer-Verlag, pp. 303.
[10] Mordeson, J. N., Bhutani, K. R. and Rosenfeld, A., (2005), Fuzzy Group Theory, Springer-Verlag, pp. 300.
[11] Howie, J. M., (1995), Fundamentals of Semigroup Theory, Oxford University Press Inc., pp. 364.
[12] Zimmermann, H. J., (1996), Fuzzy Set Theory and Its Applications 3rd Ed., Kluwer Academic Pub., pp. 435.
[13] Klement, E. P., Mesiar, R. and Pap, E., (2000), Triangular Norms, Springer, pp. 385.

[14] Bandemer, H. and Gottwald, S., (1995), Fuzzy Sets, Fuzzy Logic, Fuzzy Methods with Applications, Wiley and Sons Ltd., pp. 239.
[15] Liu, W. J., (1982), Fuzzy invariant subgroups and fuzzy ideals, Fuzzy Sets and Systems, 8, pp. 133–139.
[16] Zahedi, M. M., (1991), A characterization of L-fuzzy prime ideals, Fuzzy Sets and Systems, 44, pp. 147–160.

Mustafa Burç Kandemir received his BSc, MSc and PhD degrees from Department of Mathematics at Mugla Sıtkı Koçman University in 2005, 2008 and 2014, respectively. He is an assistant professor in Department of Mathematics at Mugla Sıtkı Koçman University. His main research interests are fuzzy set theory, soft set theory, fuzzy soft set theory and their relationships with topology and algebra.