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TRANS-SASAKIAN MANIFOLDS SATISFYING CERTAIN CONDITIONS

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Abstract. The objective of the present paper is to study the properties of special weakly Ricci symmetric and generalized Ricci recurrent trans Sasakian manifolds.

Keywords: Trans Sasakian manifolds, $m-$projective curvature tensor, generalized $\phi-$recurrent, special weakly Ricci symmetric and generalized Ricci recurrent manifolds.

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1. Introduction

The class of co Kähler, Sasakian and Kenmotsu manifolds are precisely the three classes which occur in a classification theorem of connected almost Hermitian manifolds M^{2m+1} for the automorphism group have maximum dimension $(m+1)^2$ [3]. Tanno [16] classified connected almost contact metric manifold whose automorphism group possess the maximum dimension. The sectional curvature of plane sections containing ξ is constant, say c, in such a manifold. He also proved that it can be divided into three classes:
(i) homogeneous normal contact Riemannian manifolds with $c > 0$,
(ii) global Riemannian products of a line or a circle with a Kähler manifold of constant holomorphic sectional curvature, if $c = 0$ and
(iii) a wrapped product space $R \times_f C$, if $c < 0$.
It is well known that the manifold of class (i) is characterized by admitting a Sasakian structure. The manifold of class (ii) is characterized by a tensorial relation admitting a cosymplectic structure. In [18], Kenmotsu characterized the differential geometric properties of the manifolds of class (iii); the structure so obtained is now called a Kenmotsu structure. These structure are not Sasakian [18] in general. In the Grey Hervella classification of almost Hermitian manifolds [17], there appears a class, W_4 of Hermitian manifolds, which are closely related to locally conformal Kaehler manifolds [5]. An almost contact metric structure on a manifold is called a trans Sasakian structure [19] if the product manifold $M \times R$ belongs to the class W_4. In [22], the class $C_6 \bigotimes C_5$ coincides with the class of trans Sasakian structure of type (α, β). A trans Sasakian manifold of type $(0, 0)$, $(0, \beta)$ and $(\alpha, 0)$ are known as cosymplectic [1], $\beta-$Kenmotsu and $\alpha-$Sasakian manifolds [4]

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respectively. The different geometrical properties of trans Sasakian manifolds have studied by De and Tripathi [23], Bagewadi and Venkatesha [24], Bagewadi and Girish [25], De and Sarkar [21], China and Gonzales [20], Kim, Prasad and Tripathi [26] and many others.

In this paper, author studies the properties of trans Sasakian manifolds and establish some geometrical properties in the form of theorems and lemma.

2. Preliminaries

If on an $n-$dimensional differentiable manifold M_n, $(n = 2m + 1)$, of differentiability class C^{r+1}, there exists a vector valued real linear function ϕ, a $1-$form η, the associated vector field ξ and the Riemannian metric g satisfying

$$(a) \quad \phi^2 X = -X + \eta(X)\xi, \quad (b) \quad \eta(\phi X) = 0, \tag{1}$$

and

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2}$$

for arbitrary vector fields X and Y, then (M_n, g) is said to be an almost contact metric manifold and the structure $\{\phi, \eta, \xi, g\}$ is called an almost contact metric structure to M_n [1].

In view of (1) (a), (1) (b) and (2), we conclude

$$\eta(\xi) = 1, \quad g(X, \xi) = \eta(X), \quad \phi(\xi) = 0 \tag{3}$$

and

$$g(X, \phi Y) + g(\phi X, Y) = 0. \tag{4}$$

An almost contact metric manifold $(M_n, \phi, \xi, \eta, g)$ is called a trans Sasakian structure [19] if $(M_n \bigotimes R, J, G)$ belongs to the class W_4 of the Gray Hervella classification of almost Hermitian manifolds [17], where J is the complex structure and G be the Hermitian metric on $(M_n \bigotimes R)$. This leads to the following expression

$$(D_X\phi)(Y) = \alpha\{g(X, Y)\xi - \eta(Y)X\} + \beta\{g(\phi X, Y)\xi - \eta(Y)\phi X\}, \tag{5}$$

for smooth functions α and β on M_n, and we say that the trans Sasakian structure of type (α, β) [14]. In particular, if $\alpha = 0$ and $\beta = 0$, then (5) gives

$$(D_X\phi)(Y) = 0,$$

which is characterized by cosymplectic structure [1]. Also if $\alpha = 0$ and $\beta = 1$; $\alpha = 1$ and $\beta = 0$, then (5) becomes

$$(D_X\phi)(Y) = g(\phi X, Y)\xi - \eta(Y)\phi X;$$
$$(D_X\phi)(Y) = g(X, Y)\xi - \eta(Y)X,$$

which are characterized by Kenmotsu and Sasakian structures [14] respectively. From (5), it follows that

$$D_X\xi = -\alpha\phi X + \beta(X - \eta(X)\xi). \tag{6}$$

Also the following relations hold in a trans Sasakian manifold

$$(D_X\eta)(Y) = -\alpha g(\phi X, Y) + \beta(\phi X, \phi Y), \tag{7}$$

$$\begin{aligned} R(X, Y)\xi &= (\alpha^2 - \beta^2)(\eta(Y)X - \eta(X)Y) + 2\alpha\beta(\eta(Y)\phi X - \eta(X)\phi Y) \\ &\quad +(Y\alpha)\phi X - (X\alpha)\phi Y + (Y\beta)\phi^2 X - (X\beta)\phi^2 Y, \end{aligned} \tag{8}$$

$$R(\xi, X)\xi = (\alpha^2 - \beta^2 - \xi\beta)(\eta(X)\xi - X), \tag{9}$$

$$\eta(R(X, Y)Z) = (\alpha^2 - \beta^2)\{\eta(X)g(Y, Z) - \eta(Y)g(X, Z)\}\xi, \tag{10}$$

$$2\alpha\beta + \xi\alpha = 0, \tag{11}$$

$$S(X,\xi) = (2m(\alpha^2 - \beta^2) - \xi\beta)\eta(X) - (2m-1)X\beta - (\phi X)\alpha, \tag{12}$$

$$Q\xi = (2m(\alpha^2 - \beta^2) - \xi\beta)\xi - (2m-1)grad\beta + \phi(grad\alpha), \tag{13}$$

for arbitrary vector fields X, Y and Z. If

$$(2m-1)grad\beta = \phi(grad\alpha), \tag{14}$$

then (12) and (13) become

$$S(X,\xi) = 2m(\alpha^2 - \beta^2)\eta(X), \tag{15}$$

$$Q\xi = 2m(\alpha^2 - \beta^2)\xi. \tag{16}$$

A Riemannian manifold M_n is said to be $\eta-$Einstein if its Ricci tensor S assume the form

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{17}$$

for arbitrary vector fields X and Y, where a and b are smooth functions on (M_n, g) [1]. In 1971, Pokhariyal and Mishra [9] defined a tensor W^* of type $(1,3)$ on a Riemannian manifold as

$$W^*(X,Y)Z = R(X,Y)Z - \frac{1}{4m}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \tag{18}$$

so that $'W^*(X,Y,Z,U) \overset{\text{def}}{=} g(W^*(X,Y)Z,U) =' W^*(Z,U,X,Y)$ and $'W^*_{ijkl}w^{ij}w^{kl} = 'W_{ijkl}w^{ij}w^{kl}$, where $'W^*_{ijkl}$ and $'W_{ijkl}$ are components of $'W^*$ and $'W$, w^{kl} is a skew-symmetric tensor [7], [15], Q is the Ricci operator, defined by $S(X,Y) \overset{\text{def}}{=} g(QX,Y)$ and S is the Ricci tensor for arbitrary vector fields X, Y, Z. Such a tensor field W^* is known as $m-$projective curvature tensor. Ojha [7], [8] studied the properties of $m-$projective curvature tensor in Sasakian and Kähler manifolds. He has also shown that it bridges the gap between conformal curvature tensor, conharmonic curvature tensor and concircular curvature tensor on one side and $H-$projective curvature tensor on the other. The properties of $m-$projective curvature tensor studied by Chaubey and Ojha [10], Chaubey, Prakash and Nivas [12], Taleshian and Asghari [29], De and Mallick [28], Chaubey ([11]; [13]; [35]; [36]), Prakash [37] and other geometers.

3. $m-$projectively flat trans Sasakian manifold

In view of $W^* = 0$, (18) becomes

$$R(X,Y)Z = \frac{1}{4m}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY]. \tag{19}$$

Replacing Z by ξ in (19) and then using (3), (8) and (15), we obtain

$$(\alpha^2 - \beta^2)(\eta(Y)X - \eta(X)Y) + 2\alpha\beta(\eta(Y)\phi X - \eta(X)Y) + (Y\alpha)\phi X - (X\alpha)\phi Y$$
$$+(Y\beta)\phi^2 X - (X\beta)\phi^2 Y = \frac{1}{4m}[2m(\alpha^2 - \beta^2)\{\eta(Y)X - \eta(X)Y\} + \eta(Y)QX - \eta(X)QY].$$

Again substituting ξ in place of X in the above relation and using (1), (3), (11) and (16), we have

$$QY = 2m[(\alpha^2 - \beta^2) - 2\xi\beta]Y + 4m(\xi\beta)\eta(Y)\xi, \tag{20}$$

which gives

$$S(Y,Z) = ag(Y,Z) + b\eta(Y)\eta(Z), \tag{21}$$

where

$$a = 2m(\alpha^2 - \beta^2 - 2\xi\beta), \qquad b = 4m(\xi\beta). \tag{22}$$

Contracting (20) with respect to Y and using (3), we get

$$r = (2m+1)(2m(\alpha^2 - \beta^2) - (m+1)\xi\beta). \tag{23}$$

Hence we can state the following:

Theorem 3.1. *If an $m-$projectively flat trans-Sasakian manifold of type (α, β) and of dimension $(2m+1)$, satisfies $\phi(grad\alpha) = (2m-1)grad\beta$, then the manifold is an $\eta-$Einstein manifold and the scalar curvature is given by (23).*

In consequence of (3), (20), (21) and (22), (19) becomes

$$'R(X,Y,Z,U) = a_1\{g(Y,Z)g(X,U) - g(X,Z)g(Y,U)\} + b_1\{\eta(Y)\eta(Z)g(X,U) \\ -\eta(X)\eta(Z)g(Y,U) + \eta(X)\eta(U)g(Y,Z) - \eta(Y)\eta(U)g(X,Z)\}, \tag{24}$$

where $a_1 = \frac{a}{2m}$, $b_1 = \frac{b}{4m}$ and $'R(X,Y,Z,U) = g(R(X,Y)Z,U)$. As a generalization of constant curvature, the notion of the manifold of quasi-constant curvature arose during the study of conformally flat hyper surfaces by Chen and Yano [6]. A Riemannian manifold (M_n, g), $n > 3$, is said to be the manifold of quasi-constant curvature [6] if it is conformally flat and its curvature tensor $'R$ of type $(0,4)$ is of the form

$$'R(X,Y,Z,U) = a_1\{g(Y,Z)g(X,U) - g(X,Z)g(Y,U)\} + a_2\{T(Y)T(Z)g(X,U) \\ -T(X)T(Z)g(Y,U) + T(X)T(U)g(Y,Z) - T(Y)T(U)g(X,Z)\}, \tag{25}$$

where T is nowhere vanishing $1-form$, *i.e.*, $g(X,\lambda) = T(X)$ for unit vector field λ and a_1, a_2 are scalars of which $a_2 \neq 0$. It can be easily seen that the manifold is conformally flat if the curvature tensor $'R$ takes the form (25). Mocanu [34] pointed that the manifolds introduced by Chen and Yano and Vranceanu are same. Thus, a Riemannian manifold is said to be the manifold of quasi-constant curvature if the curvature tensor $'R$ satisfies the relation (25). Hence, by virtue of (25), it follows from (24) that an $m-$projectively flat trans-Sasakian manifold is a manifold of quasi-constant curvature. Hence we state:

Theorem 3.2. *Every $m-$projectively flat trans-Sasakian manifold of type (α, β) and of dimension $(2m+1)$, satisfies $\phi(grade\alpha) = (2m-1)grad\beta$, is a manifold of quasi-constant curvature.*

4. $m-$projectively flat trans-Sasakian manifold satisfying $R(X.Y).S = 0$.

By taking $R(X,Y).S = 0$, we can easily find $S(R(X,Y)Z,U) + S(Z,R(X,Y)U) = 0$ which gives

$$g(Y,Z)S(QX,U) - g(X,Z)S(QY,U) + g(Y,U)S(Z,QX) - g(X,U)S(Z,QY) = 0, \tag{26}$$

after consideration of (19). Substituting $Y = Z = \xi$ in (26) and then using (3), (4), (15) and (16), we get

$$S(QX,U) - 4m^2(\alpha^2-\beta^2)^2\eta(X)\eta(U) + 2m(\alpha^2-\beta^2)\eta(U)\eta(QX) - 4m^2(\alpha^2-\beta^2)^2g(X,U) = 0. \tag{27}$$

Let λ be an eigen value of the endomorphism Q corresponding to an eigen vector X, then $QX = \lambda X$ and hence the equation (27) becomes

$$(\lambda^2 - 4m^2(\alpha^2-\beta^2)^2)g(X,U) + \{2m\lambda(\alpha^2-\beta^2) - 4m^2(\alpha^2-\beta^2)^2\}\eta(X)\eta(U) = 0. \tag{28}$$

Putting $U = \xi$ in (28) and then using (3), we obtain

$$\{\lambda^2 + 2m\lambda(\alpha^2-\beta^2) - 8m^2(\alpha^2-\beta^2)^2\}\eta(X) = 0. \tag{29}$$

Since the vector field X can not be perpendicular to the characteristic vector field ξ, therefore (29) gives

$$\lambda^2 + 2m\lambda(\alpha^2 - \beta^2) - 8m^2(\alpha^2 - \beta^2)^2 = 0, \tag{30}$$

which follows that the symmetric endomorphism Q of the tangent space corresponding to S has two different non-zero eigen values namely $2m(\alpha^2 - \beta^2)$ and $-4m(\alpha^2 - \beta^2)$. Thus we can state:

Theorem 4.1. *In a trans-Sasakian manifold of type (α, β) and dimension $(2m+1)$ satisfying $\phi(grad\alpha) = (2m-1)grad\beta$, which is $m-$projectively flat together with $R(X,Y).S = 0$, the symmetric endomorphism Q of the tangent space corresponding to S has two different non-zero eigen values namely $2m(\alpha^2 - \beta^2)$ and $-4m(\alpha^2 - \beta^2)$.*

Let λ_1 and λ_2 be the eigen values corresponding to the Ricci operator Q and multiplicity of λ_1 be p, then multiplicity of λ_2 is $2m + 1 - p$. Since the scalar curvature is the trace of the Ricci operator Q, therefore

$$r = p\lambda_1 + (2m + 1 - p)\lambda_2 \tag{31}$$

and

$$\lambda_1 = -4m(\alpha^2 - \beta^2); \qquad \lambda_2 = 2m(\alpha^2 - \beta^2). \tag{32}$$

From equation (19), we have

$$4mg(R(X,Y)Z,U) = g(Y,Z)g(QX,U) - g(X,Z)g(QY,U) - S(X,Z)g(Y,U) + S(Y,Z)g(X,U). \tag{33}$$

Let us put $X = U = e_i$, where $\{e_i\}, i = 1, 2, ...(2m + 1)$, is the set of orthonormal basis of the tangent space at each of M_n and then summing for $i = 1, 2, ...(2m + 1)$, we get

$$S(Y,Z) = \frac{r}{2m+1} g(Y,Z). \tag{34}$$

Again replacing Y and Z by ξ in (34) and using (3) and (15), we find

$$r = 2m(2m + 1)(\alpha^2 - \beta^2). \tag{35}$$

By virtue of equations (31), (32) and (35), we get $p = 0$, which shows that the dimension of the manifold remains unaltered. Again, if V_1 and V_2 denote the eigen subspaces corresponding to the eigen values λ_1 and λ_2 respectively of the manifold, then the sectional curvature on V_1 and V_2 for the vector fields X and Y are $\frac{\lambda_1}{2m}$ and $\frac{\lambda_2}{2m}$ respectively. Since $\lambda_1 = -4m(\alpha^2 - \beta^2) \neq 0$, $(\alpha \neq \beta)$, therefore we state the following:

Theorem 4.2. *If an $m-$projectively flat trans-Sasakian manifold of type (α, β) and dimension $(2m + 1)$ satisfying $\phi(grad\alpha) = (2m - 1)grad\beta$ and $R(X,Y).S = 0$, then the manifold has no flat points.*

In consequence of (3) and (15), (27) becomes

$$S^2(X,U) = 4m^2(\alpha^2 - \beta^2)^2 g(X,U), \tag{36}$$

where $S^2(X,U) \overset{\text{def}}{=} S(QX,U)$. It is well known that

Lemma 4.1. *If $\theta = g\bar{\wedge}A$ be the Kulkarni-Nomizu product of g and A, where g being Riemannian metric and A be a symmetric tensor of type $(0,2)$ at point x of a semi-Riemannian manifold (M_n, g). Then the relation $\theta.\theta = \alpha Q(g,\theta)$, $\alpha \in R$, is satisfied at x if and only if the condition $A^2 = \alpha A + \lambda g$, $\lambda \in R$ holds at x* [32].

In consequence of (36) and lemma (4.1), we state the following theorem:

Theorem 4.3. *If an $m-$ projectively flat trans-Sasakian manifold of type (α,β) and dimension $(2m+1)$ satisfying $\phi(grad\alpha) = (2m-1)grad\beta$ and $R(X,Y).S = 0$, then $\theta.\theta = 0$, where $\theta = g\bar{\wedge}A$ and $\alpha = 0$.*

5. Trans-Sasakian manifold satisfying $R(X,Y).W^* = 0$.

In consequence of (3) and (10), (18) becomes

$$\eta(W^*(X,Y)Z) = (\alpha^2-\beta^2)\{\eta(X)g(Y,Z)-\eta(Y)g(X,Z)\}$$
$$-\frac{1}{4m}\{\eta(X)S(Y,Z)-\eta(Y)S(X,Z)+g(Y,Z)\eta(QX)-g(X,Z)\eta(QY). \quad (37)$$

Replacing Z by ξ in (37) and using (1), (3) and (15), we obtain

$$\eta(W^*(X,Y)\xi) = 0. \quad (38)$$

Again substituting $X=\xi$ in (37) and using (3), (15) and (16), we find

$$\eta(W^*(\xi,Y)Z) = \frac{1}{2}(\alpha^2-\beta^2)g(Y,Z) - \frac{1}{4m}S(Y,Z). \quad (39)$$

We have,

$$(R(X,Y).W^*)(Z,U)V = R(X,Y)W^*(Z,U)V - W^*(R(X,Y)Z,U)V$$
$$-W^*(Z,R(X,Y)U)V - W^*(Z,U)R(X,Y)V.$$

After considering $R(X,Y).W^*=0$ in above, we can easily find that

$$R(X,Y)W^*(Z,U)V - W^*(R(X,Y)Z,U)V - W^*(Z,R(X,Y)U)V - W^*(Z,U)R(X,Y)V = 0.$$

Taking inner product of above equation with ξ, we obtain

$$g(R(X,Y)W^*(Z,U)V,\xi) - g(W^*(R(X,Y)Z,U)V,\xi)$$
$$-g(W^*(Z,R(X,Y)U)V,\xi) - g(W^*(Z,U)R(X,Y)V,\xi) = 0.$$

Putting X by ξ in the above equation and using (3), (8), (10) and (37), we obtain

$$-'W^*(Z,U,V,Y) + \eta(Y)\eta(W^*(Z,U)V) - \eta(Z)\eta(W^*(Y,U)V) - \eta(U)\eta(W^*(Z,Y)V)$$
$$+g(Y,Z)\eta(W^*(\xi,U)V) + g(Y,U)\eta(W^*(Z,\xi)V) - \eta(V)\eta(W^*(Z,U)Y) = 0. \quad (40)$$

Substituting $Y=Z=e_i$ in (40) and using (38) and (39), where $\{e_i, i=1,2,3,...,(2m+1)\}$ be an orthonormal basis of the tangent space at any point and then taking the sum for i, $1\le i\le (2m+1)$, we obtain

$$S(U,V) = 4m[\{\frac{r}{4m} - m(\alpha^2-\beta^2)\}g(U,V) + \{\frac{r}{4m} - \frac{2m+1}{2}(\alpha^2-\beta^2)\}\eta(U)\eta(V)], \quad (41)$$

which gives

$$QU = 4m[\{\frac{r}{4m} - m(\alpha^2-\beta^2)\}U + \{\frac{r}{4m} - \frac{2m+1}{2}(\alpha^2-\beta^2)\}\eta(U)\xi], \quad (42)$$

and

$$r = 2m(2m+1)(\alpha^2-\beta^2). \quad (43)$$

Theorem 5.1. *A trans-Sasakian manifold of type (α,β) and dimension $(2m+1)$ satisfying $R(X,Y).W^* = 0$ is an $\eta-$Einstein manifold and it is a manifold of constant curvature $2m(2m+1)(\alpha^2-\beta^2)$.*

6. On special weakly Ricci-symmetric trans-Sasakian manifold

An $n-$dimensional trans-Sasakian manifold (M_n, g) is called a special weakly Ricci-symmetric manifold $(SWRS)_n$ if

$$(D_X S)(Y, Z) = 2\pi(X)S(Y, Z) + \pi(Y)S(X, Z) + \pi(Z)S(X, Y), \tag{44}$$

where π is a $1-$form and is defined by $\pi(X) = g(X, \rho)$ for associated vector field ρ ([30];[31]). Taking $Z = \xi$ in (44) and using (3) and (15), we get

$$(D_X S)(Y, Z) = 2m(\alpha^2 - \beta^2)\{2\pi(X)\eta(Y) + \pi(Y)\eta(X)\} + \pi(\xi)S(X, Y). \tag{45}$$

We also know that,

$$(D_X S)(Y, \xi) = D_X S(Y, \xi) - S(D_X Y, \xi) - S(Y, D_X \xi). \tag{46}$$

In consequence of (6) and (15), (46) becomes

$$\begin{aligned}(D_X S)(Y, Z) &= D_X\{2m(\alpha^2 - \beta^2)\eta(Y)\} - 2m(\alpha^2 - \beta^2)\eta(D_X Y) \\ &\quad -S(Y, -\alpha\phi X + \beta(X - \eta(X)\xi)).\end{aligned} \tag{47}$$

Equation (47) with equations (3), (7), (15), (45) and $X = \xi$ becomes

$$6m(\alpha^2 - \beta^2)\eta(Y)\pi(\xi) + 2m(\alpha^2 - \beta^2)\pi(Y) = 0. \tag{48}$$

Finally taking $Y = \xi$ in (48) and using (3), we get

$$8m(\alpha^2 - \beta^2)\pi(\xi) = 0, \tag{49}$$

which implies

$$\pi(\xi) = 0. \tag{50}$$

In view of (50), (48) gives

$$\pi(Y) = 0,$$

which is inadmissible. Thus, we state the following theorem:

Theorem 6.1. *A trans-Sasakian manifold (M_n, g) of type (α, β) and dimension n satisfying $\phi(grade\alpha) = (2m - 1)grad\beta$ can not be a special weakly Ricci-symmetric manifold $(SWRS)_n$.*

Now taking cyclic sum of (44), we get

$$\begin{aligned}&(D_X S)(Y, Z) + (D_Y S)(Z, X) + (D_Z S)(X, Y) \\ &= 4[\pi(X)S(Y, Z) + \pi(Y)S(Z, X) + \pi(Z)S(X, Y)].\end{aligned} \tag{51}$$

If (M_n, g) admits a cyclic Ricci tensor, i. e. $(D_X S)(Y, Z)+(D_Y S)(Z, X)+(D_Z S)(X, Y) = 0$, then (51) reduces to

$$\pi(X)S(Y, Z) + \pi(Y)S(Z, X) + \pi(Z)S(X, Y) = 0. \tag{52}$$

Taking $Z = \xi$ in (52) and using (15), we get

$$2m(\alpha^2 - \beta^2)\{\pi(X)\eta(Y) + \pi(Y)\eta(X)\} + \pi(\xi)S(X, Y) = 0. \tag{53}$$

Again replacing X by ξ in (53) and using (3) and (15), we obtain

$$2m(\alpha^2 - \beta^2)\{2\pi(\xi)\eta(Y) + \pi(Y)\} = 0. \tag{54}$$

Also taking $Y = \xi$ in (54) and using (3), we have

$$\pi(\xi) = 0. \tag{55}$$

In consequence of (55), (54) gives $\pi(Y) = 0$, which is a contradiction. Thus we state the following:

Theorem 6.2. *If a trans-Sasakian manifold (M_n, g) of type (α, β) and dimension n satisfying $\phi(grad\alpha) = (2m-1)grad\beta$ with a cyclic Ricci tensor, then (M_n, g) can not be a special weakly Ricci-symmetric manifold $(SWRS)_n$.*

7. Generalized Ricci-recurrent trans-Sasakian manifold

A non-flat Riemannian manifold M_n of dimension greater than two is called a generalized Ricci-recurrent manifold [27] if its Ricci tensor S satisfies the condition

$$(D_X S)(Y, Z) = A(X)S(Y, Z) + B(X)S(Y, Z), \tag{56}$$

where D is the Riemannian connection of the Riemannian metric g and A, B are $1-$forms associated with the vector fields P_1, P_2 respectively on M, *i.e.*

$$A(X) = g(X, P_1); B(X) = g(X, P_2), \tag{57}$$

for arbitrary vector fields X, Y and Z. If the $1-$form B vanishes identically, the manifold M_n reduces to the well know Ricci-recurrent manifold [2].

Let M_n be a generalized Ricci-recurrent trans-Sasakian manifold. It is known that

$$(D_X S)(Y, Z) = XS(Y, Z) - S(D_X Y, Z) - S(Y, D_X Z) \tag{58}$$

for arbitrary vector fields X, Y and Z. From equations (56) and (58), we get

$$A(X)S(Y, Z) + B(X)S(Y, Z) = XS(Y, Z) - S(D_X Y, Z) - S(Y, D_X Z).$$

Replacing Z by ξ in above equation and using (1), (3), (6) and (15), we find

$$\begin{aligned} &\{2m(\alpha^2 - \beta^2)A(X) + B(X)\}\eta(Y) \\ &= 2m(\alpha^2 - \beta^2)(D_X\eta)(Y) + \alpha S(Y, \phi X) + \beta S(Y, \phi^2 X). \end{aligned} \tag{59}$$

In consequence of (7), (59) becomes

$$\begin{aligned} &\{2m(\alpha^2 - \beta^2)A(X) + B(X)\}\eta(Y) - \beta S(Y, \phi^2 X) \\ &= -2m(\alpha^2 - \beta^2)\{\alpha g(\phi X, Y) + \beta g(Y, \phi^2 X)\} + \alpha S(Y, \phi X). \end{aligned} \tag{60}$$

Putting $Y = \xi$ in (60) and using (3), we obtain

$$2m(\alpha^2 - \beta^2)A(X) + B(X) = 0. \tag{61}$$

Hence we can state the following theorem:

Theorem 7.1. *If a generalized Ricci-recurrent trans-Sasakian manifold of type (α, β) and dimension $(2m+1)$, satisfies $\phi(grad\alpha) = (2m-1)grad\beta$, then the associated vector fields of the $1-$forms A and B are in the opposite direction.*

Let us consider that a generalized Ricci-recurrent trans-Sasakian manifold M_n admits a cyclic Ricci tensor S, i. e.,

$$(D_X S)(Y, Z) + (D_Y S)(Z, X) + (D_Z S)(X, Y) = 0, \tag{62}$$

for arbitrary vector fields X, Y and Z. In view of (56), (62) follows that

$$\begin{aligned} &A(X)S(Y, Z) + B(X)g(Y, Z) + A(Y)S(Z, X) \\ &+B(Y)g(Z, X) + A(Z)S(X, Y) + B(Z)g(X, Y) = 0. \end{aligned} \tag{63}$$

Replacing Z by ξ in (63) and using (2) and (16), we find

$$\begin{aligned} &\{2m(\alpha^2 - \beta^2)A(X) + B(X)\}\eta(Y) + A(\xi)S(X, Y) \\ &= \{2m(\alpha^2 - \beta^2)A(Y) + B(Y)\}\eta(X) + B(\xi)g(X, Y). \end{aligned} \tag{64}$$

In view of (61), (64) gives

$$A(\xi)S(X,Y) = -B(\xi)g(X,Y), \tag{65}$$

where $B(\xi) = -2m(\alpha^2 - \beta^2)A(\xi)$. Thus we state:

Theorem 7.2. *If a generalized Ricci-recurrent trans-Sasakian manifold of type (α, β) and dimension $(2m+1)$, admits a cyclic Ricci tensor and $\phi(grade\alpha) = (2m-1)grad\beta$, then manifold is an Einstein manifold, provided $A(\xi) \neq 0$.*

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