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INTUITIONISTIC FUZZY BI-IMPLICATOR AND PROPERTIES OF LUKASIEWICZ INTUITIONISTIC FUZZY BI-IMPLICATOR

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ABSTRACT. This paper presents axiomatic as well as constructive definitions of intuitionistic fuzzy bi-implicators based on intuitionistic fuzzy t-norms and their intuitionistic fuzzy residual implicators. The inter-relationship among different proposed classes is presented along with a detailed study of the properties of one of these intuitionistic fuzzy bi-implicators called the intuitionistic fuzzy $\beta-$bi-implicator operator constructed using Lukasiewicz intuitionistic fuzzy t-norm and its R-implicator.

Keywords: Intuitionistic fuzzy set; Intuitionistic fuzzy implicator; Intuitionistic fuzzy t-norm; Intuitionistic fuzzy bi-implicator.

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1. Introduction

The intuitionistic fuzzy sets (IFS's) and interval valued fuzzy sets (IVFS's) appeared independently as appropriate generalizations of fuzzy sets (FS's). The interval valued fuzzy sets reflected the ambiguous situations unanswered by fuzzy sets in the form of closed interval membership function $[\mu_1, \mu_2]$ such that $\mu_1, \mu_2 \in [0, 1]$ and $\mu_1 \leq \mu_2$. The intuitionistic fuzzy sets however, are equipped with a nonmembership degree ν along with the membership degree μ such that $\mu, \nu \in [0, 1]$ and $\mu + \nu \leq 1$. Though the equivalence of these two approaches has been addressed in [11], but each of these generalizations have given rise to an extensive literature covering multiple aspects of their applications and the possible extensions of fuzzy logical operators and set theoretical concepts [2, 3, 5, 9, 10, 11, 12, 13]. Moreover, the vague set (VS) which was proposed by Gau [15], as another extension of fuzzy set, was later proved in [7] to be an intuitionistic fuzzy set.

In fuzzy literature, a bi-implicator operator has been closely linked to the concepts such as fuzzy similarity [16], fuzzy equality [19], T-equivalence [17] and restricted equivalence

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functions [8]. Unlike their fuzzy counterpart the intuitionistic fuzzy bi-implicator operators have not been much worked upon [18].

In this paper we aim to convey to our reader, a comprehensive picture of some new generalized classes of intuitionistic fuzzy bi-implicators having axiomatic or constructive definitions along with their mutual relationship. Such a study is expected to lay a groundwork for the development of new intuitionistic fuzzy logic or algebra having different intuitionistic fuzzy bi-implicators as basic connective operators. Furthermore, we have studied the properties and characteristics of one of the newly defined constructive bi-implicator called intuitionistic fuzzy $\beta-$*bi-implicator* by utilizing the intuitionistic fuzzy Lukasiewicz implicator along with intuitionistic fuzzy Min t-norm [9] in its definition.

Also, taking into consideration the close relation between IFS's and the other generalized fuzzy sets such as IVFS's and the VS's, we are in a position to claim that, all the results on intuitionistic fuzzy set theory and logic produced in this work can be easily modified and adapted to the extended frame works of any of the mentioned higher order fuzzy sets.

The work presented here is organized as follows:

Section 1 will present basic definitions and concepts of intuitionistic fuzzy set theory and logic. In Section 2 we have presented an axiomatic definition of intuitionistic fuzzy bi-implicator operators which can be regarded as intuitionistic fuzzy generalization of *Fodor-Roubens* fuzzy bi-implicator presented in [14]. Furthermore, we have proposed several new constructive approaches for defining an intuitionistic fuzzy bi-implicator using intuitionistic fuzzy t-norms and their residual implicators. We have studied their interrelationships along with their relation with the class of intuitionistic fuzzy bi-implicator having axiomatic definition. Moreover, in Section 3, we have utilized the Lukasiewicz intuitionistic implicator along with the intuitionistic fuzzy t-norm Min [9] to investigate the different aspects of one of the newly defined intuitionistic fuzzy bi-implicator called $\beta-$ bi-implicator.

Definition 1.1 [1] An *intuitionistic fuzzy set* (*IFS*) on a universe X is an object of the form $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}$, where the functions $\mu_A(x)$ and $\nu_A(x) \in [0, 1]$ define respectively the degree of membership and the degree of non membership of x in the set A, while μ_A and ν_A satisfy $(\forall x \in X)(\mu_A(x) + \nu_A(x) \leq 1)$. The class of all intuitionistic fuzzy sets on X is denoted by $IFS(X)$. A fuzzy set in X is then just an intuitionistic fuzzy set for which $\mu_A(x) + \nu_A(x) = 1$ holds for every $x \in X$. The class of all fuzzy sets in X is denoted by $F(X)$.

For an intuitionistic fuzzy set $A = \{(\mu_A(x), \nu_A(x)) \mid x \in X\}$, we define the *complement* of A in X as $A^c = \{(\nu_A(x), \mu_A(x)) \mid x \in X\}$, the *Support* of A in X as a subset of X given by *Supp*$(A) = \{x \in X : \mu_A(x) \neq 0$ or $\nu_A(x) \neq 1\}$, the *Kernel* of A in X as $Ker(A) = \{x \in X : \mu_A(x) = 1$ and $\nu_A(x) = 0\}$, the universe of discourse $\widetilde{1_X} = \{(x, 1, 0) \mid x \in X\}$ and the *empty set* by $\widetilde{0_X} = \{(x, 0, 1) \mid x \in X\}$. As far as the extension of inclusion of IFS is concerned it is defined as: For all $A, B \in IFS(X)$,

$$A \subseteq B \text{ if and only if } (\forall x \in X)(\mu_A(x) \leq \mu_B(x) \text{ and } \nu_A(x) \geq \nu_B(x)).$$

Definition 1.2 [9] The set $L^* = \left\{(x_1, x_2) \in [0, 1]^2 \mid x_1 + x_2 \leq 1\right\}$ is a complete and bounded lattice $(L^*, \leq_{L^*})$ equipped with order $\leq_{L^*}$, which is defined as: $(x_1, x_2) \leq_{L^*} (y_1, y_2)$ if and only if $x_1 \leq y_1$ and $x_2 \geq y_2$. The elements $1_{L^*} = (1, 0)$ and $0_{L^*} = (0, 1)$ are the greatest and the smallest elements of the lattice L^* respectively. An *IFS* A on X can be equivalently defined as a mapping $A : X \longrightarrow L^*$ such that for any $x \in X$, $A(x) = (\mu_A(x), \nu_A(x)) = (a_1, a_2) \in L^*$.

Definition 1.3 [9] An intuitionistic fuzzy *t-norm* is an increasing, commutative, associative $(L^*)^2 \longrightarrow L^*$ mapping $\check{T}$ satisfying $\check{T}(1_{L^*}, x) = x$ for all $x \in L^*$. For instance, for all $x = (x_1, x_2), y = (y_1, y_2) \in L^*$ the greatest t-norm with respect to ordering $\leq_{L^*}$ is $\check{T}_M(x, y) = x \wedge y =$
$(\min(x_1, y_1), \max(x_2, y_2))$ which is an extension of Min t-norm on $[0, 1]$ to L^*. Moreover, $\check{T}_P(x, y) = (x_1y_1, x_2 + y_2 - x_2y_2)$ is an extension of product t-norm and $\check{T}_L(x, y) = (\max(0, x_1 + y_1 - 1), \min(1, x_2 + 1 - y_1, y_2 + 1 - x_1))$ is one of the extensions of Lukasiewicz t-norm on $[0, 1]$ to L^*. The t-norm $\check{T}_M$ has the property that if $z \leq_{L^*} x$ and $z \leq_{L^*} y$ then $z \leq_{L^*} \check{T}_M(x, y)$ for all $x, y, z \in L^*$.

Definition 1.4 [9] An intuitionistic fuzzy *t-conorm* is an increasing, commutative, associative $(L^*)^2 \longrightarrow L^*$ mapping $\check{S}$ satisfying $\check{S}(0_{L^*}, x) = x$ for all $x \in L^*$. For instance, for all $x = (x_1, x_2), y = (y_1, y_2) \in L^*$ the smallest t-conorm with respect to ordering $\leq_{L^*}$ is $\check{S}_M(x, y) = x \vee y = (\max(x_1, y_1), \min(x_2, y_2))$ which is an extension of Max t-conorm on $[0, 1]$ to L^*. Moreover, $\check{S}_P(x, y) = (x_1 + y_1 - x_1y_1, x_2y_2)$ is an extension of probabilistic sum and, $\check{S}_L(x, y) = (\min(1, x_1 + 1 - y_2, y_1 + 1 - x_2), \max(0, x_2 + y_2 - 1))$ is an extension of Lukasiewicz conorm to L^*. It must be noted that $\check{S}_M$,$\check{S}_P$ and $\check{S}_L$ conorms are the duals of intuitionistic fuzzy t-norms $\check{T}_M$, $\check{T}_P$ and $\check{T}_L$ respectively. It is interesting to note that for all $x, y \in L^*, \check{S}_L(x, 1_{L^*}) = \check{S}_L(1_{L^*}, y) = 1_{L^*}$.

Theorem 1.5 [12] Let $\check{T}$ be an intuitionistic fuzzy t-norm. If $\sup\limits_{z \in Z} \check{T}(x, z) = \check{T}(x, \sup\limits_{z \in Z} z)$, for all non-empty subsets Z of L^*, then $\check{T}$ is intuitionistic fuzzy left continuous t-norm.

Definition 1.6 [9] A *negator* on L^* is a decreasing $L^* \longrightarrow L^*$ mapping $\check{N}$ that satisfies $\check{N}(0_{L^*}) = 1_{L^*}$ and $\check{N}(1_{L^*}) = 0_{L^*}$. If $\check{N}(\check{N}(x)) = x$, $\forall x \in L^*, \check{N}$ is called an involutive negator. The mapping $\check{N}_s$ defined as: $\check{N}_s(x_1, x_2) = (x_2, x_1)$ $\forall (x_1, x_2) \in L^*$ will be called the standard negator. An involutive negator on L^* can always be related to an involutive negator on $[0, 1]$.

Definition 1.7 [9] An intuitionistic fuzzy *implicator* is an $(L^*)^2 \longrightarrow L^*$mapping $\check{I}$ satisfying $\check{I}(0_{L^*}, 0_{L^*}) = 1_{L^*}, \check{I}(0_{L^*}, 1_{L^*}) = 1_{L^*}, \check{I}(1_{L^*}, 0_{L^*}) = 0_{L^*}, \check{I}(1_{L^*}, 1_{L^*}) = 1_{L^*}$. Moreover, we require $\check{I}$ to be decreasing in its first and increasing in its second component.

Definition 1.8 [9] The intuitionistic fuzzy implicator $\check{I}$ is said to satisfy the left ordering property (LOP), if $x \leq_{L^*} y$, then $\check{I}(x, y) = 1_{L^*}$ for all $x, y \in L^*$.

Definition 1.9 [9] Let $\check{S}$ be a t-conorm and $\check{N}$ a negator on L^*. The *S-implicator* generated by $\check{S}$ and $\check{N}$ is the mapping $\check{I}_{\check{S},\check{N}} : (L^*)^2 \longrightarrow L^*$ defined as, for all $x, y \in L^*$

$$\check{I}_{\check{S},\check{N}}(x, y) = \check{S}(\check{N}(x), y). \tag{1}$$

Definition 1.10 [9] Let $\check{T}$ be a t-norm on L^*. The *R-implicator* generated by $\check{T}$ is the mapping $\check{I}_{\check{T}}$ defined as, for all $x, y \in L^*$:

$$\check{I}_{\check{T}}(x, y) = \sup\{\gamma \in L^* \mid \check{T}(x, \gamma) \leq_{L^*} y\}. \tag{2}$$

Remark 1.11 [9] If we take $\check{S} = \check{S}_L$ and $\check{N} = \check{N}_s$ in (1), then, $\check{I}_{\check{S}_L,\check{N}_s}(x, y) = (\min(1, y_1 + 1 - x_1, x_2 + 1 - y_2), \max(0, x_1 + y_2 - 1))$ is an extension of Lukasiewicz implicator on $[0, 1]$ to L^* and is an S-implicator on L^*. Also this extension can be obtained by taking $\check{T} = \check{T}_L$ in (2) which makes it an R-implicator extension on L^*. Thus we have $\check{I}_{\check{S}_L,\check{N}_s}(x, y) = \check{I}_{\check{T}_L}(x, y) = (\min(1, y_1 + 1 - x_1, x_2 + 1 - y_2), \max(0, x_1 + y_2 - 1))$. It is a contrapositive intuitionistic fuzzy extension of Lukasiewicz implicator to L^*.

2. Intuitionistic Fuzzy Bi-Implicators

In this section, we shall firstly present an axiomatic definition of an intuitionistic fuzzy bi-implicator and then will relate it to different new classes of intuitionistic fuzzy bi-implicators having constructive approaches.

Definition 2.1 An *intuitionistic fuzzy bi-implicator* is an $(L^*)^2 \longrightarrow L^*$mapping IBI satisfying for all $w, x, y, z \in L^*$:
$(b1)$. $IBI(x, y) = IBI(y, x)$;
$(b2)$. $IBI(0_{L^*}, 1_{L^*}) = 0_{L^*}$;
$(b3)$. $IBI(x, x) = 1_{L^*}$;
$(b4)$. If $w \le_{L^*} x \le_{L^*} y \le_{L^*} z$, then $IBI(w, z) \le_{L^*} IBI(x, y)$.

Example 2.2 Let for all $w, x, y, z \in L^*$ such that $w = (w_1, w_2), x = (x_1, x_2), y = (y_1, y_2)$ and $z = (z_1, z_2)$. Then the operator defined as:

$$IBI(x, y) = \begin{Bmatrix} 1_{L^*} \text{ if } x = y \\ (\min(1 - x_2, 1 - y_2), \max(x_2, y_2)) \text{ if } x \neq y \end{Bmatrix}$$

is an intuitionistic fuzzy bi-implicator.
Indeed we will show that $IBI(x, y)$ satisfies the four axioms of Definition 2.1:
$(b1)$. $IBI(x, y) = IBI(y, x)$. Straightforward.
$(b2)$. $IBI(0_{L^*}, 1_{L^*}) = IBI((0, 1), (1, 0)) = (\min(1 - 1, 1 - 0), \max(1, 0)) = (0, 1) = 0_{L^*}$.
$(b3)$. $IBI(x, x) = 1_{L^*}$, by its definition.
$(b4)$. Let $w \le_{L^*} x \le_{L^*} y \le_{L^*} z$. This
implies $w_1 \le x_1 \le y_1 \le z_1$ and $w_2 \ge x_2 \ge y_2 \ge z_2$
implies $1 - w_2 \le 1 - x_2 \le 1 - y_2 \le 1 - z_2$
implies $IBI(w, z) = (\min(1 - w_2, 1 - z_2), \max(w_2, z_2)) = (1 - w_2, w_2)$
implies $\min(1 - w_2, 1 - z_2) \le \min(1 - y_2, 1 - x_2)$ and $\max(w_2, z_2) \ge \max(y_2, x_2)$
implies $(\min(1 - w_2, 1 - z_2), \max(w_2, z_2)) \le_{L^*} (\min(1 - y_2, 1 - x_2), \max(y_2, x_2))$
and hence $IBI(w, z) \le_{L^*} IBI(x, y)$.

Remark 2.3 If we take $w = x$ in axiom $(b4)$ of definition 2.1 then axiom $(b4)$ can be equivalently replaced by axiom $(b^\diamond 4)$ provided IBI satisfies $(b1)$:
$(b^\diamond 4)$. If $x \le_{L^*} y \le_{L^*} z$, then $IBI(x, z) \le_{L^*} IBI(x, y)$ and $IBI(x, z) \le_{L^*} IBI(y, z)$.

Proof
(i). $(b4) \Longrightarrow (b^\diamond 4)$
Indeed putting $w = x$ in $(b4)$ implies
$IBI(x, z) \le_{L^*} IBI(x, y)$ i.e. $IBI(x, .)$ is decreasing for all $x \in L^*$.
From $x \le_{L^*} y \le_{L^*} z \le_{L^*} z$ it follows with $(b4)$
$IBI(x, z) \le_{L^*} IBI(y, z)$ i.e. $IBI(., z)$ is increasing for all $z \in L^*$.
(ii). $(b^\diamond 4) \Longrightarrow (b4)$
Suppose $w \le_{L^*} x \le_{L^*} y \le_{L^*} z$.
From $IBI(w, .)$ being decreasing and $x \le_{L^*} z$ we get:

$$IBI(w, z) \le_{L^*} IBI(w, x). \tag{3}$$

From $IBI(., x)$ being increasing and $w \le_{L^*} y$ we get:

$$IBI(w, x) \le_{L^*} IBI(y, x). \tag{4}$$

From (3) and (4) we get: $IBI(w, z) \le_{L^*} IBI(y, x)$ and hence if IBI satisfies $(b1)$: $IBI(w, z) \le_{L^*} IBI(x, y)$.

Definition 2.4 Let $\check{T}$ be a left continuous intuitionistic fuzzy t-norm and $\check{I}_{\check{T}}$ be the corresponding intuitionistic fuzzy R-implicator. Then the *intuitionistic fuzzy $\kappa-$bi-implicator* is the $(L^*)^2 \longrightarrow L^*$ mapping IBI_κ defined as:

$$IBI_\kappa(x, y) = \check{T}(\check{I}_{\check{T}}(x, y), \check{I}_{\check{T}}(y, x)).$$

Definition 2.5 Let $\breve{T}'$ be an intuitionistic fuzzy t-norm, $\breve{T}$ a left continuous intuitionistic fuzzy t-norm and $\breve{I}_{\breve{T}}$ be the corresponding intuitionistic fuzzy R-implicator. Then the *intuitionistic fuzzy $\beta-$bi-implicator* is the $(L^*)^2 \longrightarrow L^*$ mapping IBI_β defined as:

$$IBI_\beta(x,y) = \breve{T}'(\breve{I}_{\breve{T}}(x,y), \breve{I}_{\breve{T}}(y,x)).$$

Definition 2.6 Let $\breve{T}'$ be an intuitionistic fuzzy t-norm, $\breve{T}$ a left continuous intuitionistic fuzzy t-norm, $\breve{I}_{\breve{T}}$ be the corresponding intuitionistic fuzzy R-implicator and $\breve{S}'$ be an intuitionistic fuzzy conorm. Then the *intuitionistic fuzzy $\breve{T}'\breve{S}'-$bi-implicator* is the $(L^*)^2 \longrightarrow L^*$ mapping $IBI_{\breve{T}'\breve{S}'}$ defined as:

$$IBI_{\breve{T}'\breve{S}'}(x,y) = \breve{I}_{\breve{T}}(\breve{S}'(x,y), \breve{T}'(x,y)).$$

Proposition 2.7 Let $\breve{T}$ be a left continuous intuitionistic fuzzy t-norm and $\breve{I}_{\breve{T}}$ be the corresponding intuitionistic fuzzy R-implicator then it holds:

$$x \leq_{L^*} y \Longrightarrow \breve{T}_M(\breve{I}_{\breve{T}}(x,y), \breve{I}_{\breve{T}}(y,x)) = \breve{I}_{\breve{T}}(\breve{S}_M(x,y), \breve{T}_M(x,y)).$$

Proof Suppose $x \leq_{L^*} y$. Then we obtain
$\breve{I}_{\breve{T}}(x,y) = 1_{L^*}, \breve{T}_M(x,y) = x$ and $\breve{S}_M(x,y) = y$.
Hence, $\breve{T}_M(\breve{I}_{\breve{T}}(x,y), \breve{I}_{\breve{T}}(y,x)) = \breve{T}_M(1_{L^*}, \breve{I}_{\breve{T}}(y,x))$
$= \breve{I}_{\breve{T}}(y,x) = \breve{I}_{\breve{T}}(\breve{S}_M(x,y), \breve{T}_M(x,y))$.

Proposition 2.8 Let $\breve{T}'$ be an intuitionistic fuzzy t-norm, $\breve{T}$ a left continuous intuitionistic fuzzy t-norm and $\breve{I}_{\breve{T}}$ be the corresponding intuitionistic fuzzy R-implicator, then the intuitionistic fuzzy $\beta-$bi-implicator IBI_β satisfies the following properties for all $x, y \in L^*$:
$(b'1)$. $IBI_\beta(x,y) = 1_{L^*}$ if $x = y$ (reflexivity);
$(b'2)$. $IBI_\beta(x,y) = IBI_\beta(y,x)$ (symmetry);
$(b'3)$. $IBI_\beta(x,y) = \breve{T}'(\breve{I}_{\breve{T}}(x,y), \breve{I}_{\breve{T}}(y,x)) = \breve{T}_M(\breve{I}_{\breve{T}}(x,y), \breve{I}_{\breve{T}}(y,x))$ provided either $x \leq_{L^*} y$ or $y \leq_{L^*} x$;
$(b'4)$. $IBI_\beta(x,y) = \breve{I}_{\breve{T}}(\breve{S}_M(x,y), \breve{T}_M(x,y))$ provided $x \leq_{L^*} y$.

Proof
$(b'1)$. $IBI_\beta(x,x) = \breve{T}'(\breve{I}_{\breve{T}}(x,x), \breve{I}_{\breve{T}}(x,x)) = \breve{T}'(1_{L^*}, 1_{L^*})) = 1_{L^*}$.
$(b'2)$. $IBI_\beta(x,y) = \breve{T}'(\breve{I}_{\breve{T}}(x,y), \breve{I}_{\breve{T}}(y,x)) = \breve{T}'(\breve{I}_{\breve{T}}(y,x), \breve{I}_{\breve{T}}(x,y)) = IBI_\beta(y,x)$.
$(b'3)$. From $x \leq_{L^*} y$ we get $\breve{I}_{\breve{T}}(x,y) = 1_{L^*}$
and hence $IBI_\beta(x,y) = \breve{T}'(\breve{I}_{\breve{T}}(x,y), \breve{I}_{\breve{T}}(y,x)) = \breve{T}'(1_{L^*}, \breve{I}_{\breve{T}}(y,x))$
$= \breve{I}_{\breve{T}}(y,x) = \breve{T}_M(\breve{I}_{\breve{T}}(y,x), 1_{L^*}) = \breve{T}_M(1_{L^*}, \breve{I}_{\breve{T}}(y,x)) = \breve{T}_M(\breve{I}_{\breve{T}}(x,y), \breve{I}_{\breve{T}}(y,x))$.
$(b'4)$. Let $x \leq_{L^*} y$.
Then $IBI_\beta(x,y) = \breve{T}_M(\breve{I}_{\breve{T}}(x,y), \breve{I}_{\breve{T}}(y,x)) = \breve{I}_{\breve{T}}(\breve{S}_M(x,y), \breve{T}_M(x,y))$ (By Proposition 2.7).

Proposition 2.9 An intuitionistic fuzzy bi-implicator IBI is an intuitionistic fuzzy $\kappa-$bi-implicator if and only if it is an intuitionistic fuzzy $\beta-$bi-implicator.

Proof By taking $\breve{T}' = \breve{T}$ in the definition of intuitionistic fuzzy $\beta-$bi-implicator we can get an intuitionistic fuzzy $\kappa-$bi-implicator. Conversely, we need to show that a $\kappa-$bi-implicator is an intuitionistic fuzzy $\beta-$bi-implicator. Let IBI_κ be an intuitionistic fuzzy $\kappa-$bi-implicator. Then we show that it satisfies all the axioms of Proposition 2.7 to become a $\beta-$bi-implicator.
$(b'1)$. $IBI_\kappa(x,x) = \breve{T}(\breve{I}_{\breve{T}}(x,x), \breve{I}_{\breve{T}}(x,x)) = \breve{T}(1_{L^*}, 1_{L^*})) = 1_{L^*}$by (LOP) and Definition 1.4.
$(b'2)$. $IBI_\kappa(x,y) = \breve{T}(\breve{I}_{\breve{T}}(x,y), \breve{I}_{\breve{T}}(y,x))$
$= \breve{T}(\breve{I}_{\breve{T}}(y,x), \breve{I}_{\breve{T}}(x,y)) = IBI_\kappa(y,x)$.
$(b'3)$. Suppose $x \leq_{L^*} y$ we get $\breve{I}_{\breve{T}}(x,y) = 1_{L^*}$

and hence $IBI_\kappa(x,y) = \check{T}(\breve{I}_{\check{T}}(x,y), \breve{I}_{\check{T}}(y,x)) = \check{T}(1_{L^*}, \breve{I}_{\check{T}}(y,x))$
$= \breve{I}_{\check{T}}(y,x) = \check{T}_M(\breve{I}_{\check{T}}(y,x), 1_{L^*}) = \check{T}_M(1_{L^*}, \breve{I}_{\check{T}}(y,x))$
$= \check{T}_M(\breve{I}_{\check{T}}(x,y), \breve{I}_{\check{T}}(y,x))$.
$(b'4)$. Suppose $x \leq_{L^*} y$.
Then it follows: $IBI_\kappa(x,y) = \check{T}_M(\breve{I}_{\check{T}}(x,y), \breve{I}_{\check{T}}(y,x))$
$= \breve{I}_{\check{T}}(\check{S}_M(x,y), \check{T}_M(x,y))$ (by Proposition 2.7).

Proposition 2.10 An intuitionistic fuzzy $\kappa-$bi-implicator satisfies the axioms of Definition 2.1.

Proof Let $\check{T}$ be a left continuous intuitionistic fuzzy t-norm and $\breve{I}_{\check{T}}$ be its intuitionistic fuzzy R-implicator and IBI_κ be the intuitionistic fuzzy $\kappa-$ bi-implicator based on $\check{T}$ and $\breve{I}_{\check{T}}$. Then we only have to prove that an intuitionistic fuzzy $\kappa-$ bi-implicator satisfies the axioms $(b2)$ and $(b4)$, as $(b1) = (b'2)$ and $(b3) = (b'1)$ have already been proved in Proposition 2.9.
$(b2)$. $IBI_\kappa(0_{L^*}, 1_{L^*}) = \check{T}(\breve{I}_{\check{T}}(0_{L^*}, 1_{L^*}), \breve{I}_{\check{T}}(1_{L^*}, 0_{L^*})) = \check{T}(1_{L^*}, 0_{L^*}) = 0_{L^*}$.
$(b4)$. Suppose $w \leq_{L^*} x \leq_{L^*} y \leq_{L^*} z$. Then we obtain:
$IBI_\kappa(w,z) = \check{T}_M(\breve{I}_{\check{T}}(w,z), \breve{I}_{\check{T}}(z,w))$ by $(b'3)$
implies $IBI_\kappa(w,z) = \check{T}_M(1_{L^*}, \breve{I}_{\check{T}}(z,w)) = \breve{I}_{\check{T}}(z,w)$
implies $IBI_\kappa(w,z) = \breve{I}_{\check{T}}(z,w) \leq_{L^*} \breve{I}_{\check{T}}(y,w) \leq_{L^*} \breve{I}_{\check{T}}(y,x)$ as $y \leq_{L^*} z$ and $w \leq_{L^*} x$
implies $IBI_\kappa(w,z) \leq_{L^*} \breve{I}_{\check{T}}(y,x) = \check{T}_M(1_{L^*}, \breve{I}_{\check{T}}(y,x)) = \check{T}_M(\breve{I}_{\check{T}}(x,y), \breve{I}_{\check{T}}(y,x))$
implies $IBI_\kappa(w,z) \leq_{L^*} IBI_\kappa(x,y)$.

Proposition 2.11 An intuitionistic fuzzy $\check{T}'\check{S}'-$bi-implicator satisfies the properties $(b1)$ and $(b2)$ but may fail to satisfy the properties $(b3)$ and $(b4)$.

Proof Let $\check{T}'$ be an intuitionistic fuzzy t-norm, $\check{T}$ a left continuous intuitionistic fuzzy t-norm, $\breve{I}_{\check{T}}$ be its intuitionistic fuzzy R-implicator and $\check{S}'$ be an intuitionistic fuzzy conorm. Let $IBI_{\check{T}'\check{S}'}$ be the intuitionistic fuzzy $\check{T}'\check{S}'-$bi-implicator based on $\check{T}$, $\breve{I}_{\check{T}}$ and $\check{S}'$. Then we show that $IBI_{\check{T}'\check{S}'}$satisfies the properties $(b1)$ and $(b2)$ but may fail to satisfy the properties $(b3)$ and $(b4)$. For all $x, y \in L^*$
$(b1)$. $IBI_{\check{T}'\check{S}'}(x,y) = \breve{I}_{\check{T}}(\check{S}'(x,y), \check{T}'(x,y)) = \breve{I}_{\check{T}}(\check{S}'(y,x), \check{T}'(y,x))$
$= IBI_{\check{T}'\check{S}'}(y,x)$.
$(b2)$. $IBI_{\check{T}'\check{S}'}(0_{L^*}, 1_{L^*}) = \breve{I}_{\check{T}}(\check{S}'(0_{L^*}, 1_{L^*}), \check{T}'(0_{L^*}, 1_{L^*})) = \breve{I}_{\check{T}}(1_{L^*}, 0_{L^*}) = 0_{L^*}$.
In order to show that $IBI_{\check{T}'\check{S}'}$ fails to satisfy $(b3)$ and $(b4)$ we shall present the following counter examples:
Let $x = (0.7, 0.1) \in L^*$. Then by Definitions (1.4),(1.5) and Remark 1.12 we have $\check{S}_P(x,x) = (0.91, 0.01)$ and $\check{T}_L(x,x) = (0.4, 0.2)$, which implies that
$IBI_{\check{T}_L\check{S}_P}(x,x) = \breve{I}_{\check{T}_M}(\check{S}_P(x,x), \check{T}_L(x,x)) = (0.4, 0.2) \neq 1_{L^*}$.
Hence $IBI_{\check{T}'\check{S}'}$ fails to satisfy $(b3)$.
Let $w = (0.2, 0.7), x = (0.3, 0.4)$, $y = (0.5, 0.3)$, $z = (0.8, 0.1) \in L^*$. Then
$\check{S}_L(w,z) = (1,0), \check{T}_P(w,z) = (0.16, 0.73)$ and $\check{S}_L(x,y) = (0.8, 0), \check{T}_P(x,y) = (0.15, 0.58)$, which implies that $IBI_{\check{T}_P\check{S}_L}(w,z) = \breve{I}_{\check{T}_M}(\check{S}_L(w,z), \check{T}_P(w,z)) = (0.16, 0.73)$ and
$IBI_{\check{T}_P\check{S}_L}(x,y) = \breve{I}_{\check{T}_M}(\check{S}_L(x,y), \check{T}_P(x,y)) = (0.15, 0.58)$.
Clearly, we see that $IBI_{\check{T}_P\check{S}_L}(w,z) \leq_{L^*} IBI_{\check{T}_P\check{S}_L}(x,y)$ as $0.16 > 0.15$ and $0.58 < 0.73$.
Hence, $IBI_{\check{T}_P\check{S}_L}$ fails to satisfy the property $(b4)$.

Remark 2.12 It must be noted that if we restrict ourself to the choice of all those $x, y \in L^*$ such that either $x \leq_{L^*} y$ or $y \leq_{L^*} x$ and $\check{T}$ to be a left continuous intuitionistic fuzzy t-norm and $\breve{I}_{\check{T}}$ be the corresponding intuitionistic fuzzy R-implicator then, the class of all intuitionistic fuzzy $\kappa-$bi-implicators and the class of all intuitionistic fuzzy $\beta-$bi-implicators satisfies $(b^{'}4)$. Thus they become the subclasses of the class of all intuitionistic fuzzy $\check{T}'\check{S}'-$bi-implicators.

3. Lukasiewicz Intuitionistic Fuzzy Bi-implicator

Next, we shall study in detail the properties of intuitionistic fuzzy $\beta-$ bi-implicator by specifying intuitionistic fuzzy t-norms $\check{T}^{'}=\check{T}_M$ and $\check{T}=\check{T}_L$ with an R-implicator $\check{I}=\check{I}_{\check{T}_L}$ respectively in its definition. For simplicity in results we drop the index β in notation IBI_β and from here onward we will use IBI for such an intuitionistic fuzzy $\beta-$ bi-implicator. Thus, for all $A,B\in IFS(X)$ and $x\in X$ we have:

$$\begin{aligned}IBI(A,B)(x) &= \check{T}_M(\check{I}_{\check{T}_L}(A(x),B(x)),\check{I}_{\check{T}_L}(B(x),A(x)))\\ &= (\min(1,b_1-a_1+1,a_2-b_2+1,a_1-b_1+1,b_2-a_2+1),\\ &\max(0,a_1+b_2-1,b_1+a_2-1))\end{aligned}$$

where $A(x)=(a_1,a_2)=(\mu_A(x),\nu_A(x))$, $B(x)=(b_1,b_2)=(\mu_B(x),\nu_B(x))\in L^*$.

Proposition 3.1 For all $A,B\in IFS(X)$,

(a). $IBI_\beta(A,B)(x)=1_{L^*}$ if and only if $A(x)=B(x)$;

(b). $IBI_\beta(A,B)=\widetilde{1_X}$ if and only if $A=B$;

(c). $IBI_\beta(A,B)(x)=0_{L^*}$ if and only if $x\in Ker(A)\cap(Supp(B))^c$or $x\in Ker(B)\cap(Supp(A))^c$;

(d). $IBI_\beta(A,B)=\widetilde{0_X}$ implies $Ker(A)\cap(Supp(B))^c\neq\phi$ or $Ker(B)\cap(Supp(A))^c\neq\phi$.

Proof Let $A,B\in IFS(X)$,

(a). $IBI_\beta(A,B)(x)=1_{L^*}$ for any $x\in X$,
if and only if $\check{T}_M(\check{I}_{\check{T}_L}(A(x),B(x)),\check{I}_{\check{T}_L}(B(x),A(x)))=1_{L^*}$
if and only if $(\min(1,b_1-a_1+1,a_2-b_2+1,a_1-b_1+1,b_2-a_2+1),\max(0,a_1+b_2-1,b_1+a_2-1))=1_{L^*}$
if and only if $\min(1,b_1-a_1+1,a_2-b_2+1,a_1-b_1+1,b_2-a_2+1)=1$ and $\max(0,a_1+b_2-1,b_1+a_2-1)=0$
if and only if $b_1-a_1+1\geq 1,a_2-b_2+1\geq 1,\ a_1-b_1+1\geq 1,b_2-a_2+1\geq 1$ and $a_1+b_2-1\leq 0,b_1+a_2-1\leq 0$
if and only if $b_1\geq a_1,a_2\geq b_2$ and $a_1\geq b_1,b_2\geq a_2$
if and only if $a_1=b_1$ and $a_2=b_2$
if and only if $A(x)=B(x)$.

(b). $IBI_\beta(A,B)=\widetilde{1_X}$
if and only if $IBI_\beta(A,B)(x)=1_{L^*}$ for all $x\in X$
if and only if $A(x)=B(x)$ for all $x\in X$
if and only if $A=B$.

(c). $IBI_\beta(A,B)(x)=0_{L^*}$ for any $x\in X$,
if and only if $\check{T}_M(\check{I}_{\check{T}_L}(A(x),B(x)),\check{I}_{\check{T}_L}(B(x),A(x)))=0_{L^*}$
if and only if $(\min(1,b_1-a_1+1,a_2-b_2+1,a_1-b_1+1,b_2-a_2+1),\max(0,a_1+b_2-1,b_1+a_2-1))=0_{L^*}$
if and only if $\min(1,b_1-a_1+1,a_2-b_2+1,a_1-b_1+1,b_2-a_2+1)=0$ and $\max(0,a_1+b_2-1,b_1+a_2-1)=1$
if and only if either $\{b_1-a_1+1=0$ and $\max(0,a_1+b_2-1,b_1+a_2-1)=1\}$
or $\{a_2-b_2+1=0$ and $\max(0,a_1+b_2-1,b_1+a_2-1)=1\}$
or $\{a_1-b_1+1=0$ and $\max(0,a_1+b_2-1,b_1+a_2-1)=1\}$
or $\{b_2-a_2+1=0$ and $\max(0,a_1+b_2-1,b_1+a_2-1)=1\}$.

Next, we discuss all these cases one by one such that they have a mutual relation of "or" between them.
Case 1: If $b_1 - a_1 + 1 = 0$ then we have $b_1 = 0$, $a_1 = 1$, $a_2 = 0$ and $b_2 \leq 1$. However, the condition $\max(0, a_1 + b_2 - 1, b_1 + a_2 - 1) = 1$ will enforce $b_2 = 1$.
Thus, we have $(a_1, a_2) = (1, 0)$ and $(b_1, b_2) = (0, 1)$ and hence $x \in Ker(A) \cap (Supp(B))^c$.
Case 2: If $a_2 - b_2 + 1 = 0$ then we get $a_2 = 0$, $b_2 = 1$, $b_1 = 0$ and $a_1 \leq 1$. However, the condition $\max(0, a_1 + b_2 - 1, b_1 + a_2 - 1) = 1$ will enforce $a_1 = 1$.
Thus, we have $(a_1, a_2) = (1, 0)$ and $(b_1, b_2) = (0, 1)$ and hence $x \in Ker(A) \cap (Supp(B))^c$.
Case 3: If $a_1 - b_1 + 1 = 0$ then we have $a_1 = 0$, $b_1 = 1$, $b_2 = 0$ and $a_2 \leq 1$. Likewise, to above two cases the condition $\max(0, a_1 + b_2 - 1, b_1 + a_2 - 1) = 1$ will enforce $a_2 = 1$.
Thus, we have $(a_1, a_2) = (0, 1)$ and $(b_1, b_2) = (1, 0)$ and hence $x \in Ker(B) \cap (Supp(A))^c$.
Case 4: If we choose $b_2 - a_2 + 1 = 0$ then we get $b_2 = 0$, $a_2 = 1$, $a_1 = 0$ and $b_1 \leq 1$. The condition $\max(0, a_1 + b_2 - 1, b_1 + a_2 - 1) = 1$ will enforces $b_1 = 1$.
Thus, we have $(a_1, a_2) = (0, 1)$ and $(b_1, b_2) = (1, 0)$ and hence $x \in Ker(B) \cap (Supp(A))^c$.
Thus, all of these situations lead to the result:
$IBI_\beta(A, B)(x) = 0_{L^*}$ implies $x \in Ker(A) \cap (Supp(B))^c$or $x \in Ker(B) \cap (Supp(A))^c$.
Conversely,
let $x \in Ker(A) \cap (Supp(B))^c$
implies that $x \in Ker(A)$ and $x \in (Supp(B))^c$
implies that $(a_1, a_2) = (1, 0)$ and $x \in (Supp(B))^c$.
Now, $x \in Supp(B) \Leftrightarrow (b_1 \neq 0$ or $b_2 \neq 1)$ and hence $x \in (Supp(B))^c \Leftrightarrow (b_1 = 0$ and $b_2 = 1) \Leftrightarrow (b_1, b_2) = (0, 1)$
implies that $[(a_1, a_2) = (1, 0)$ and $(b_1, b_2) = (0, 1)]$
implies that $\min(1, b_1 - a_1 + 1, a_2 - b_2 + 1, a_1 - b_1 + 1, b_2 - a_2 + 1) = 0$ and $\max(0, a_1 + b_2 - 1, b_1 + a_2 - 1) = 1$
implies that$(\min(1, b_1 - a_1 + 1, a_2 - b_2 + 1, a_1 - b_1 + 1, b_2 - a_2 + 1), \max(0, a_1 + b_2 - 1, b_1 + a_2 - 1)) = (0, 1)$
implies that $IBI_\beta(A, B)(x) = 0_{L^*}$.
Similarly, for $x \in Ker(B) \cap (Supp(A))^c$ we get $IBI_\beta(A, B)(x) = 0_{L^*}$.
Thus, $x \in Ker(A) \cap (Supp(B))^c$or $x \in Ker(B) \cap (Supp(A))^c$ implies $IBI_\beta(A, B)(x) = 0_{L^*}$.
(d). $IBI_\beta(A, B) = \widetilde{0_X}$
implies that $IBI_\beta(A, B)(x) = 0_{L^*}$ for all $x \in X$
implies that either $[x \in Ker(A) \cap (Supp(B))^c]$ or $[x \in Ker(B) \cap (Supp(A))^c]$ for all $x \in X$
implies that $Ker(A) \cap (Supp(B))^c \neq \phi$ or $Ker(B) \cap (Supp(A))^c \neq \phi$.
Proposition 3.2 For all $A, B \in IFS(X)$,
$IBI_\beta(A, B) = IBI_\beta(B, A)$.
Proof The result holds due to commutativity of $\check{T}_M$.
Corollary 3.3 For $A \in IFS(X)$,
(a). $IBI_\beta(A, A^c)(x) = 1_{L^*}$if and only if $A(x) = (a_1, a_2)$ such that $a_1 = a_2$;
(b). $IBI_\beta(A, A^c) = \widetilde{1_X}$ if and only if $A(x) = (a_1, a_2)$ such that $a = a_2$ for all $x \in X$;
(c). $IBI_\beta(A, A^c)(x) = 0_{L^*}$ if and only if either $A(x) = 1_{L^*}$ or $A(x) = 0_{L^*}$;
(d). $IBI_\beta(A, A^c) = \widetilde{0_X}$ if and only if $A = \widetilde{1_X}$ or $A = \widetilde{0_X}$.
Proof Follows directly from Proposition 3.1 by taking $B = A^c$.
Proposition 3.4 For $A, B \in IFS(X)$
$IBI(A, B) = IBI(B^c, A^c)$.
Proof The result holds due to contrapositivity of the Lukasiewicz intuitionistic fuzzy implicator $\check{I}_{\check{T}_L}$ used in Definition 2.5.
Proposition 3.5 For any $A, B, C \in IFS(X)$ such that $A \subseteq B \subseteq C$ we have:

(a). $IBI_\beta(A,C) \subseteq \left\{ \begin{array}{c} IBI_\beta(A,B) \\ IBI_\beta(B,C) \end{array} \right\}$ i.e., the first partial mapping $IBI_\beta(\cdot,B)$ of IBI_β is increasing and the second partial mapping $IBI_\beta(A,\cdot)$ is decreasing;
(b). $IBI_\beta(A,C) \subseteq \check{T}_M(IBI_\beta(A,B), IBI_\beta(B,C))$.
Proof Let $A, B, C \in IFS(X)$ such that $A \subseteq B \subseteq C$.
(a). $A(x) \leq_{L^*} B(x) \leq_{L^*} C(x)$ for all $x \in X$
implies that $a_1 \leq b_1 \leq c_1$ and $a_2 \geq b_2 \geq c_2$.
Now as, $[a_1 \leq b_1$ and $a_2 \geq b_2$ and $a_1 + a_2 \leq 1]$
implies that$1 \leq b_1 - a_1 + 1$ and $a_2 - b_2 + 1 \geq 1$
implies that $\min(1, b_1 - a_1 + 1, a_2 - b_2 + 1) = 1$ and $\max(0, a_1 + b_2 - 1) = 0$
implies that $\check{I}_{\check{T}_L}(A(x), B(x)) = (\min(1, b_1 - a_1 + 1, a_2 - b_2 + 1), \max(0, a_1 + b_2 - 1))$ $= (1, 0) = 1_{L^*}$
implies that $IBI_\beta(A,B)(x) = \check{T}_M(\check{I}_{\check{T}_L}(A(x), B(x)), \check{I}_{\check{T}_L}(B(x), A(x)))$
$= \check{I}_{\check{T}_L}(B(x), A(x)) = (\min(1, a_1 - b_1 + 1, b_2 - a_2 + 1), \max(0, b_1 + a_2 - 1))$.
Similarly, we have $IBI_\beta(B,C)(x) = (\min(1, b_1 - c_1 + 1, c_2 - b_2 + 1), \max(0, c_1 + b_2 - 1))$ and $IBI_\beta(A,C)(x) = (\min(1, a_1 - c_1 + 1, c_2 - a_2 + 1), \max(0, c_1 + a_2 - 1))$.
Now as, $[a_1 - b_1 + 1 \geq a_1 - c_1 + 1$ and $b_2 - a_2 + 1 \geq c_2 - a_2 + 1$ also $c_1 + a_2 - 1 \geq b_1 + a_2 - 1]$
implies that $[\min(1, a_1 - c_1 + 1, c_2 - a_2 + 1) \leq \min(1, a_1 - b_1 + 1, b_2 - a_2 + 1)$
and $\max(0, c_1 + a_2 - 1) \geq \max(0, b_1 + a_2 - 1)]$
implies that $(\min(1, a_1 - c_1 + 1, c_2 - a_2 + 1), \max(0, c_1 + a_2 - 1))$
$\leq_{L^*} (\min(1, a_1 - b_1 + 1, b_2 - a_2 + 1), \max(0, b_1 + a_2 - 1))$
implies that $IBI_\beta(A,C)(x) \leq_{L^*} IBI_\beta(A,B)(x)$ for all $x \in X$
implies that $IBI_\beta(A,C) \subseteq IBI_\beta(A,B)$.
Moreover, $[a_1 - c_1 + 1 \leq b_1 - c_1 + 1,\ c_2 - a_2 + 1 \leq c_2 - b_2 + 1$ and $c_1 + a_2 - 1 \geq c_1 + b_2 - 1]$
implies that $\min(1, a_1 - c_1 + 1, c_2 - a_2 + 1) \leq \min(1, b_1 - c_1 + 1, c_2 - b_2 + 1)$
and $\max(0, c_1 + a_2 - 1) \geq \max(0, c_1 + b_2 - 1)$
implies that $(\min(1, a_1 - c_1 + 1, c_2 - a_2 + 1), \max(0, c_1 + a_2 - 1))$
$\leq_{L^*} (\min(1, b_1 - c_1 + 1, c_2 - b_2 + 1), \max(0, c_1 + b_2 - 1))$
implies that $IBI_\beta(A,C)(x) \leq_{L^*} IBI_\beta(B,C)(x)$ for all $x \in X$
implies that $IBI_\beta(A,C) \subseteq IBI_\beta(B,C)$.
(b). $A(x) \leq_{L^*} B(x) \leq_{L^*} C(x)$ for all $x \in X$
implies that $a_1 \leq b_1 \leq c_1$ and $a_2 \geq b_2 \geq c_2$
implies that $\check{I}_{\check{T}_L}(A(x), B(x)) = 1_{L^*}$, $\check{I}_{\check{T}_L}(A(x), C(x)) = 1_{L^*}$and $\check{I}_{\check{T}_L}(B(x), C(x)) = 1_{L^*}$
implies that $IBI_\beta(A,B)(x) = (\min(1, a_1 - b_1 + 1, b_2 - a_2 + 1), \max(0, b_1 + a_2 - 1))$, $IBI_\beta(B,C)(x) = (\min(1, b_1 - c_1 + 1, c_2 - b_2 + 1), \max(0, c_1 + b_2 - 1))$ and $IBI_\beta(A,C)(x) = (\min(1, a_1 - c_1 + 1, c_2 - a_2 + 1), \max(0, c_1 + a_2 - 1))$.
Now $[a_1 - b_1 + 1 \geq a_1 - c_1 + 1,\ b_2 - a_2 + 1 \geq c_2 - a_2 + 1, b_1 - c_1 + 1 \geq a_1 - c_1 + 1,$ $c_2 - b_2 + 1 \geq c_2 - a_2 + 1$ and $c_1 + b_2 - 1 \leq c_1 + a_2 - 1,\ b_1 + a_2 - 1 \leq c_1 + a_2 - 1]$
implies that $\min(1, a_1 - c_1 + 1, c_2 - a_2 + 1) \leq \min(1, b_2 - a_2 + 1, a_1 - b_1 + 1, b_1 - c_1 + 1, c_2 - b_2 + 1)$
and $\max(0, c_1 + a_2 - 1) \geq \max(0, c_1 + b_2 - 1, b_1 + a_2 - 1)$
implies that $\min(1, a_1 - c_1 + 1, c_2 - a_2 + 1) \leq \min(\min(1, b_2 - a_2 + 1, a_1 - b_1 + 1), \min(1, b_1 - c_1 + 1, c_2 - b_2 + 1))$ and $\max(0, c_1 + a_2 - 1) \geq \max(\max(0, c_1 + b_2 - 1), \max(0, b_1 + a_2 - 1))$
implies that $(\min(1, a_1 - c_1 + 1, c_2 - a_2 + 1), \max(0, c_1 + a_2 - 1))$
$\leq_{L^*} (\min(\min(1, b_2 - a_2 + 1, a_1 - b_1 + 1), \min(1, b_1 - c_1 + 1, c_2 - b_2 + 1)), \max(\max(0, c_1 + b_2 - 1), \max(0, b_1 + a_2 - 1)))$
implies that $IBI_\beta(A,C)(x) \leq_{L^*} \check{T}_M(IBI_\beta(A,B)(x), IBI_\beta(B,C)(x))$ for all $x \in X$
implies that $IBI_\beta(A,C) \subseteq \check{T}_M(IBI_\beta(A,B), IBI_\beta(B,C))$.

Definition 3.6 For any $A, B \in IFS(X)$, A is said to be point wise comparable with B if for all $x \in X$ either $A(x) \leq_{L^*} B(x)$ or $B(x) \leq_{L^*} A(x)$. Moreover, it may be noted that:
1. A is point wise comparable to A for all $A \in IFS(X)$;
2. If A is pointwise comparable with B then B is pointwise comparable to A;
3. If A is pointwise comparable to B and B is point wise comparable to C then A is comparable to C.

Proposition 3.7 For any $A, B \in IFS(X)$, such that A and B are pointwise comparable:
(a). $IBI_\beta(A, \check{T}_M(A, B)) = IBI_\beta(B, \check{S}_M(A, B))$;
(b). $IBI_\beta(A, \check{S}_M(A, B)) = IBI_\beta(B, \check{T}_M(A, B))$.

Proof
(a). Let $A, B \in IFS(X)$, such that A and B are pointwise comparable. Then for all $x \in X$, either $A(x) \leq_{L^*} B(x)$ or $B(x) \leq_{L^*} A(x)$
implies that either $\check{T}_M(A(x), B(x)) = A(x)$ and $\check{S}_M(A(x), B(x)) = B(x)$
or $\check{T}_M(A(x), B(x)) = B(x)$ and $\check{S}_M(A(x), B(x)) = A(x)$
implies that $IBI_\beta(A(x), \check{T}_M(A(x), B(x))) = IBI_\beta(A(x), A(x)) = 1_{L^*}$
and $IBI_\beta(B(x), \check{S}_M(A(x), B(x))) = IBI_\beta(B(x), B(x)) = 1_{L*}$
or $IBI_\beta(A(x), \check{T}_M(A(x), B(x))) = IBI_\beta(A(x), B(x))$
and $IBI_\beta(B(x), \check{S}_M(A(x), B(x))) = IBI_\beta(B(x), A(x)) = IBI_\beta(A(x), B(x))$
implies that $IBI_\beta(A, \check{T}_M(A, B))(x) = IBI_\beta(B, \check{S}_M(A, B))(x)$
implies that $IBI_\beta(A, \check{T}_M(A, B)) = IBI_\beta(B, \check{S}_M(A, B))$.
(b). The proof can be constructed in a similar way as part (a).

Proposition 3.8 For any $A, B \in IFS(X)$, such that A and B are pointwise comparable, the following intuitionistic fuzzy sets are equal:
(a). $IBI_\beta(A, B)$;
(b). $\check{T}_M(IBI_\beta(\check{T}_M(A, B), A), IBI_\beta(A, \check{S}_M(A, B)))$;
(c). $\check{T}_M(IBI_\beta(\check{T}_M(A, B), B), IBI_\beta(B, \check{S}_M(A, B)))$;
(d). $IBI_\beta(\check{T}_M(A, B), \check{S}_M(A, B))$;
(e). $\check{T}_M(IBI_\beta(A, \check{S}_M(A, B)), IBI_\beta(B, \check{S}_M(A, B)))$;
(f). $\check{T}_M(IBI_\beta(A, \check{T}_M(A, B)), IBI_\beta(B, \check{T}_M(A, B)))$.

Proof Let $A, B \in IFS(X)$, such that A and B are pointwise comparable. Then, for all $x \in X$, either $A(x) \leq_{L^*} B(x)$ or $B(x) \leq_{L^*} A(x)$
implies that for any $x \in X$, either $\check{T}_M(A, B)(x) = A(x)$ and $\check{S}_M(A, B) = B(x)$
or $\check{T}_M(A, B)(x) = B(x)$ and $\check{S}_M(A, B) = A(x)$.
For simplicity of proofs we consider the cases of all those $x \in X$ for which $A(x) \leq_{L^*} B(x)$ i.e., $\check{T}_M(A, B)(x) = A(x)$ and $\check{S}_M(A, B) = B(x)$ then,
(b)=(a). $\check{T}_M(IBI_\beta(\check{T}_M(A(x), B(x)), A(x)), IBI_\beta(A(x), \check{S}_M(A(x), B(x))))$
$= \check{T}_M(IBI_\beta(A(x), A(x)), IBI_\beta(A(x), B(x))))$
$= \check{T}_M(1_{L^*}, IBI_\beta(A, B)(x)) = IBI_\beta(A, B)(x)$.
(c)=(a). $\check{T}_M(IBI_\beta(\check{T}_M(A(x), B(x)), B(x)), IBI_\beta(B(x), \check{S}_M(A(x), B(x))))$
$= \check{T}_M(IBI_\beta(A(x), B(x)), IBI_\beta(B(x), B(x)))$
$= \check{T}_M(IBI_\beta(A, B)(x), 1_{L^*}) = IBI_\beta(A, B)(x)$.
(d)=(a). $IBI_\beta(\check{T}_M(A, B), \check{S}_M(A, B))(x)$
$= IBI_\beta(A(x), B(x)) = IBI_\beta(A, B)(x)$.
(e)=(a). $\check{T}_M(IBI_\beta(A, \check{S}_M(A, B)), IBI_\beta(B, \check{S}_M(A, B)))(x)$
$= \check{T}_M(IBI_\beta(A(x), B(x)), 1_{L*}) = IBI_\beta(A, B)(x)$.
(f)=(a). $\check{T}_M(IBI_\beta(A, \check{T}_M(A, B)), IBI_\beta(B, \check{T}_M(A, B)))(x)$
$= \check{T}_M(IBI_\beta(A(x), A(x)), IBI_\beta(B(x), A(x)))$

$= \check{T}_M(1_{L^*}, IBI_\beta(B(x), A(x)))$
$= IBI_\beta(B(x), A(x)) = IBI_\beta(A(x), B(x)) = IBI_\beta(A, B)(x)$ (because of the symmetry of IBI_β).
The above results also hold for all those $x \in X$, for which $B(x) \leq_{L^*} A(x)$.

Conclusion

In this research a detailed study of intuitionistic fuzzy bi-implicators was presented. Several new classes of intuitionistic fuzzy bi-implicators were introduced. The inter-relationship of these classes was also studied. Moreover, the properties of one of the introduced classes called $\beta-$bi-implicators were developed by employing the intuitionistic fuzzy Lukasiewicz implicator along with intuitionistic fuzzy Min t-norm in its definition. Such a knowledge not only provides a better understanding about the structural details of the particular class but also signifies the role of a bi-implicator in defining any similarity relation between two intuitionistic fuzzy sets.

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