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A NEW FACTOR THEOREM ON ABSOLUTE MATRIX SUMMABILITY METHODS

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Abstract. The aim of this paper is to obtain a new theorem dealing with absolute matrix summability factors.

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1. Introduction

Let $A=(a_{nv})$ be a normal matrix and (s_n) be the sequence of the nth partial sums of the series $\sum a_n$, then we define

$$A_n(s)=\sum_{v=0}^{n} a_{nv}s_v. \tag{1}$$

Let (θ_n) be any sequence of positive constants. The series $\sum a_n$ is said to be summable $|A,\theta_n|_k$, $k\geq 1$, if (see [2])

$$\sum_{n=1}^{\infty} \theta_n^{k-1}|\bar{\Delta}A_n(s)|^k<\infty. \tag{2}$$

where

$$\bar{\Delta}A_n(s)=A_n(s)-A_{n-1}(s). \tag{3}$$

One can also see [1] for this method. If we take $\theta_n=n$, then the $|A,\theta_n|_k$ summability reduces to $|A|_k$ summability (see [3]).
Given a normal matrix $A=(a_{nv})$, we associate two lower semimatrices $\bar{A}=(\bar{a}_{nv})$ and $\hat{A}=(\hat{a}_{nv})$ as follows:

$$\bar{a}_{nv}=\sum_{i=v}^{n} a_{ni}, \quad n,v=0,1,... \quad \bar{\Delta}a_{nv}=a_{nv}-a_{n-1},v \quad a_{-1,0}=0 \tag{4}$$

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and

$$\hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{\Delta}\bar{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad n = 1, 2, ... \tag{5}$$

It may be noted that $\bar{A}$ and $\hat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we have

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v = \sum_{v=0}^{n} \bar{a}_{nv} a_v \tag{6}$$

and

$$\bar{\Delta} A_n(s) = \sum_{v=0}^{n} \hat{a}_{nv} a_v. \tag{7}$$

We say that A is a normal matrix if A is lower triangular and $a_{nn} \neq 0$ for all n.

2. The Known Result

Sulaiman [4] has proved the following theorem for matrix summability methods.

Theorem 2.1 Let (λ_n), (X_n) be two sequences such that $\sum_{n=1}^{\infty} n^{-1}\lambda_n X_n$ is convergent, and the conditions

$$n\Delta\lambda_n = O(\lambda_n), \quad n \to \infty, \tag{8}$$

$$\sum_{v=1}^{n} \lambda_v = O(n\lambda_n), \quad n \to \infty, \tag{9}$$

are satisfied. Let A be a lower triangular with non-negative entries satisfying

$$\bar{a}_{n0} = 1, \; n = 0, 1, ..., \tag{10}$$

$$a_{n-1,v} \geq a_{nv}, \; \text{for } n \geq v + 1, \tag{11}$$

$$na_{nn} = O(1), \quad 1 = O(na_{nn}) \tag{12}$$

$$\sum_{v=1}^{n-1} a_{vv} \hat{a}_{n,v} = O(a_{nn}). \tag{13}$$

If $t_v^k = O(1)(C, 1)$, where $t_v = \frac{1}{v+1} \sum_{r=1}^{v} r a_r$, then the series $\sum a_n \lambda_n X_n$ is summable $|A|_k$, $k \geq 1$.

3. The Main Result

The aim of this paper is to generalize Theorem 2.1 for $|A, \theta_n|_k$ summability method in the following form.

Theorem 3.1 Let A be a positive normal matrix satisfying the conditions (10)-(13) of Theorem 2.1. Let $(\theta_n a_{nn})$ be a non-increasing sequence. If (θ_n) is any sequence of positive constants such that

$$\sum_{n=v+1}^{\infty} (\theta_n a_{nn})^{k-1} \hat{a}_{n,v} = O\left\{(\theta_v a_{vv})^{k-1}\right\}, \tag{14}$$

$$\sum_{n=v+1}^{\infty} (\theta_n a_{nn})^{k-1} |\bar{\Delta} a_{nv}| = O\left\{(\theta_v a_{vv})^{k-1} a_{vv}\right\}, \tag{15}$$

and all the conditions of Theorem 2.1 are satisfied, then the series $\sum a_n \lambda_n X_n$ is summable $|A, \theta_n|_k$, $k \geq 1$, where (λ_n) and (X_n) are as in Theorem 2.1.

We need the following lemmas for the proof of Theorem 3.1.

Lemma 3.1[4] If $\sum n^{-1}\lambda_n$ is convergent, then (λ_n) is non-negative, non-decreasing, $\lambda_n logn = O(1)$, and $n\Delta\lambda_n = O(1/(logn)^2)$.

Lemma 3.2[4] If $\sum n^{-1}\lambda_n X_n$ is convergent, and the conditions (8) and (9) of Theorem 2.1 are satisfied, then

$$n\lambda_n\Delta X_n = O(1), \tag{16}$$

$$\sum_{n=1}^{\infty}\lambda_n\Delta X_n = O(1), \quad n\to\infty, \tag{17}$$

$$\sum_{n=1}^{m} n\lambda_n\Delta^2 X_n = O(1), \quad m\to\infty. \tag{18}$$

Lemma 3.3[4] Under the conditions (10) and (11) of Theorem 2.1, we have

$$\sum_{v=0}^{n-1}|\bar{\Delta}a_{nv}| \leq a_{n,n}, \tag{19}$$

$$\hat{a}_{n,v+1} \geq 0, \tag{20}$$

$$\sum_{n=v+1}^{m+1}\hat{a}_{n,v+1} = O(1). \tag{21}$$

Proof of Theorem 3.1

Let (V_n) denotes the A-transform of the series $\sum_{n=1}^{\infty} a_n\lambda_n X_n$. We write $\varphi_n = \lambda_n X_n$, so we have

$$\bar{\Delta}V_n = \sum_{v=1}^{n}\hat{a}_{n,v}a_v\varphi_v = \sum_{v=1}^{n} v^{-1}\hat{a}_{n,v}va_v\varphi_v$$

Applying Abel's transformation to this sum, we have that

$$\begin{aligned}
\bar{\Delta}V_n &= \sum_{v=1}^{n-1}\Delta_v(\hat{a}_{n,v}\varphi_v v^{-1})\sum_{r=1}^{v} ra_r + a_{nn}\varphi_n n^{-1}\sum_{v=1}^{n} va_v \\
&= \sum_{v=1}^{n-1}(v+1)t_v(v^{-1}(v+1)^{-1}\hat{a}_{n,v}\varphi_v + (v+1)^{-1}\bar{\Delta}a_{nv}\varphi_v + (v+1)^{-1}\hat{a}_{n,v+1}\Delta\varphi_v) + \frac{n+1}{n}a_{nn}\varphi_n t_n \\
&= \sum_{v=1}^{n-1} v^{-1}t_v\hat{a}_{n,v}\varphi_v + \sum_{v=1}^{n-1} t_v\bar{\Delta}a_{nv}\varphi_v + \sum_{v=1}^{n-1} t_v\hat{a}_{n,v+1}\Delta\varphi_v + \frac{n+1}{n}a_{nn}\varphi_n t_n \\
&= V_{n,1} + V_{n,2} + V_{n,3} + V_{n,4}.
\end{aligned}$$

To complete the proof of Theorem 3.1, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty}\theta_n^{k-1} \mid V_{n,r}\mid^k < \infty, \quad \text{for} \quad r = 1,2,3,4. \tag{22}$$

First, by applying Hölder's inequality with indices k and k', where $k > 1$ and $\frac{1}{k} + \frac{1}{k'} = 1$, we have that

$$\sum_{n=2}^{m+1} \theta_n^{k-1} \mid V_{n,1} \mid^k = \sum_{n=2}^{m+1} \theta_n^{k-1} \left| \sum_{v=1}^{n-1} v^{-1} \hat{a}_{n,v} t_v \varphi_v \right|^k$$
$$\leq \sum_{n=2}^{m+1} \theta_n^{k-1} \sum_{v=1}^{n-1} v^{-k} t_v^k a_{vv}^{1-k} \hat{a}_{n,v} \varphi_v^k \left(\sum_{v=1}^{n-1} a_{vv} \hat{a}_{n,v} \right)^{k-1}$$
$$= O(1) \sum_{n=2}^{m+1} (\theta_n a_{nn})^{k-1} \sum_{v=1}^{n-1} t_v^k a_{vv} \varphi_v^k \hat{a}_{n,v} = O(1) \sum_{v=1}^{m} a_{vv} t_v^k \varphi_v^k \sum_{n=v+1}^{m+1} (\theta_n a_{nn})^{k-1} \hat{a}_{n,v}$$
$$= O(1) \sum_{v=1}^{m} (\theta_v a_{vv})^{k-1} a_{vv} t_v^k \varphi_v^k = O(1) \sum_{v=1}^{m} (\theta_v a_{vv})^{k-1} \varphi_v^{k-1} \varphi_v t_v^k v^{-1},$$

using $nX_n \Delta\lambda_n = O(\lambda_n X_n) = O(1)$ from Lemma 3.2 and writing $\varphi_n = \lambda_n X_n$ we have that

$$\sum_{n=2}^{m+1} \theta_n^{k-1} \mid V_{n,1} \mid^k = O(1) \sum_{v=1}^{m} (\theta_v a_{vv})^{k-1} \varphi_v t_v^k v^{-1} = O(1)(\theta_1 a_{11})^{k-1} \sum_{v=1}^{m} \varphi_v t_v^k v^{-1}$$
$$= O(1) \sum_{v=1}^{m-1} \left(\sum_{r=1}^{v} t_r^k \right) \Delta(v^{-1} \varphi_v) + \left(\sum_{v=1}^{m} t_v^k \right) m^{-1} \varphi_m$$
$$= O(1) \sum_{v=1}^{m-1} v(v^{-2} \varphi_v + (v+1)^{-1} \Delta\varphi_v) + O(1)\varphi_m$$
$$= O(1) \sum_{v=1}^{m-1} v^{-1} \varphi_v + O(1) \sum_{v=1}^{m-1} \Delta\varphi_v + O(1)\varphi_m$$
$$= O(1) \sum_{v=1}^{m-1} \frac{\lambda_v X_v}{v} + O(1)\lambda_m X_m = O(1), \quad \text{as} \quad m \to \infty,$$

by virtue of the hypotheses of Theorem 3.1 and Lemma 3.3. Now, using Hölder's inequality, and by the hypotheses of Theorem 3.1 and Lemma 3.3. we have that

$$\sum_{n=2}^{m+1} \theta_n^{k-1} \mid V_{n,2} \mid^k = \sum_{n=2}^{m+1} \theta_n^{k-1} \left| \sum_{v=1}^{n-1} \bar{\Delta} a_{nv} t_v \varphi_v \right|^k$$
$$\leq \sum_{n=2}^{m+1} \theta_n^{k-1} \sum_{v=1}^{n-1} t_v^k |\bar{\Delta} a_{nv}| \varphi_v^k \left(\sum_{v=1}^{n-1} |\bar{\Delta} a_{nv}| \right)^{k-1}$$
$$= O(1) \sum_{n=2}^{m+1} (\theta_n a_{nn})^{k-1} \sum_{v=1}^{n-1} t_v^k \varphi_v^k |\bar{\Delta} a_{nv}| = O(1) \sum_{v=1}^{m} t_v^k \varphi_v^k \sum_{n=v+1}^{m+1} (\theta_n a_{nn})^{k-1} |\bar{\Delta} a_{nv}|$$
$$= O(1) \sum_{v=1}^{m} (\theta_v a_{vv})^{k-1} a_{vv} t_v^k \varphi_v^k = \sum_{v=1}^{m} (\theta_v a_{vv})^{k-1} v^{-1} t_v^k \varphi_v = O(1), \quad \text{as} \quad m \to \infty,$$

as in the case of $V_{n,1}$. Furthermore, we have that

$$\sum_{n=2}^{m+1} \theta_n^{k-1} \mid V_{n,3} \mid^k = \sum_{n=2}^{m+1} \theta_n^{k-1} \left| \sum_{v=1}^{n-1} \hat{a}_{n,v+1} t_v \Delta\varphi_v \right|^k$$

$$\leq \sum_{n=2}^{m+1} \theta_n^{k-1} \sum_{v=1}^{n-1} t_v^k a_{vv}^{1-k} \hat{a}_{n,v+1} (\Delta\varphi_v)^k \left(\sum_{v=1}^{n-1} a_{vv} \hat{a}_{n,v+1} \right)^{k-1}$$

$$= O(1) \sum_{n=2}^{m+1} (\theta_n a_{nn})^{k-1} \sum_{v=1}^{n-1} t_v^k a_{vv}^{1-k} \hat{a}_{n,v+1} (\Delta\varphi_v)^k$$

$$= O(1) \sum_{v=1}^{m} t_v^k a_{vv}^{1-k} (\Delta\varphi_v)^k \sum_{n=v+1}^{m+1} (\theta_n a_{nn})^{k-1} \hat{a}_{n,v+1}$$

$$= O(1) \sum_{v=1}^{m} (\theta_v a_{vv})^{k-1} a_{vv}^{1-k} t_v^k (\Delta\varphi_v)^k = O(1) \sum_{v=1}^{m} (\theta_v a_{vv})^{k-1} v^{k-1} t_v^k (\Delta\varphi_v)^{k-1} \Delta\varphi_v$$

$$= O(1) \sum_{v=1}^{m} (\theta_v a_{vv})^{k-1} t_v^k \Delta\varphi_v \, (v\Delta\varphi_v)^{k-1},$$

by using $n\Delta(\lambda_n X_n) = O(1)$ from Lemma 3.1 we have that

$$\sum_{n=2}^{m+1} \theta_n^{k-1} \mid V_{n,3} \mid^k = O(1)(\theta_1 a_{11})^{k-1} \sum_{v=1}^{m} t_v^k \Delta\varphi_v$$

$$= O(1) \sum_{v=1}^{m} t_v^k (\Delta\lambda_v X_v + \lambda_{v+1} \Delta X_v) = O(1) \quad \text{as} \quad m \to \infty, \quad \text{(see [4] for detail).}$$

Finally, as in the case of $V_{n,1}$, we have that

$$\sum_{n=1}^{m} \theta_n^{k-1} \mid V_{n,4} \mid^k = \sum_{n=1}^{m} \theta_n^{k-1} \left| \frac{n+1}{n} a_{nn} t_n \varphi_n \right|^k$$

$$= O(1) \sum_{n=1}^{m} (\theta_n a_{nn})^{k-1} a_{nn} t_n^k \varphi_n^k = O(1) \sum_{n=1}^{m} (\theta_n a_{nn})^{k-1} n^{-1} t_n^k \varphi_n = O(1) \quad \text{as} \quad m \to \infty,$$

by the hypotheses of Theorem 3.1 and Lemma 3.3. This completes the proof of Theorem 3.1.

In the special case, if we take $\theta_n = n$ and A as a lower triangular matrix in Theorem 3.1, then we obtain Theorem 2.1.

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