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VULNERABILITY IN NETWORKS

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Abstract. Recently defined exponential domination number is reported as a new measure to graph vulnerability. It is a methodology, emerged in graph theory, for vulnerability analysis of networks. Also, it gives more sensitive results than other available measures. Exponential domination number has great significance both theoretically and practically for designing and optimizing networks. In this paper, it is studied how some of the graph types perform when they suffer a vertex failure. When its vertices are corrupted, the vulnerability of a graph can be calculated by the exponential domination number which gives more information about the characterization of the network.

Keywords: Graph vulnerability, network design and communication, domination, exponential domination number, robustness, thorn graphs.

AMS Subject Classification: 05C40, 05C69, 68M10, 68R10

1. Introduction

Networks surround us. In the real world, networks with non-trivial topology have a broad variety of utilizations. These real-world utilizations can be exemplified as: The Internet, world trade Web, metabolic networks, electricity networks, supply chain networks, road networks, etc. With the introduction of small-world networks and complex networks' scale-free properties in the literature, trying to understand the principles of organization of complex networks has attracted considerable interest within half a decade. Complex networks have a multidisciplinary research and application domain. Within this domain, there are also branches of different sciences such as social sciences and information as well as basic sciences. The staple topic that is used to get the measure of stability and robustness of complex networks is that of vulnerability. The total resistance of a network's underlying graph can be defined as the vulnerability of that network. While obtaining an underlying graph of a network, main components are the nodes and the links. The links connect two nodes that mutually send information. Both the node and link vulnerability of complex networks can be examined; that is, it is possible to discuss how the network is affected by removing any combination of nodes and links from the network.
Various methods have been introduced to characterize vulnerability of networks. Assume that $G = (V(G), E(G))$ is a graph which has these characteristics : non-directional, simple and connected. Generally, a network is modeled as a non-directional and simple graph.

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In this model, while processors are taken as vertices, connections between the processors are taken as edges. As the most powerful mathematical tool, graph theory plays an important role to analyze and understand the architecture of a network. When the network requirements are stated within theoretical graph parameters, the analyzing and designing problem of networks turns into obtaining a graph G that meets some particularized requirements.

In the literature, several metrics have been given to determine the robustness value of nets. Moreover, to calculate the reliability of nets some parameters which belong to graph theory have been used. For instance, connectivity, toughness, integrity, domination, scattering [3, 13, 16]

In this study, the non-directional, finite and simple graph is within our care. Also, it does not have loops and multiple edges. Let's take a graph $G = (V(G), E(G))$ by assuming V is a set of vertices and E is a set of edges. Wherein $V(G) \neq \emptyset$ and $E(G)$ is a subset of $V(G) \times V(G)$. Let's remember some common definitions that we will encounter and use in this study. The complement $\overline{G}$ of a graph G has $V(G)$ as its vertex sets, but two vertex are adjacent in $\overline{G}$ if only if they are not adjacent in G. That is, to generate the complement of a graph, one fills in all the missing edges required to form a complete graph, and removes all the edges that were previously there. The open neighborhood set of any vertex v in $V(G)$ is formed by the vertices $u \in V(G)$ which are neighbors of the vertex v and this set is represented as $N_G(v)$. Besides, the closed neighborhood set of any vertex $v \in V(G)$ can be obtained from by adding v to its open neighborhood set. The closed neighborhood set represented by $N_G[v]$. The cardinality of an open neighborhood set of a vertex v gives us the degree of v and it is represented by $deg(v)$. Consider the path with the minimum length connecting any two u_1 and u_2 vertices of G, the length of this path is denoted by $d(u_1, u_2)$. This is indicated by $d(u_1, u_2) = \infty$ if there is no path connecting u_1 and u_2 vertices. It is also indicated by $d(u_1, u_2) = 0$ if u_1 and u_2 are the same. The maximum value of the minimum paths between each pair of (u_1, u_2) vertices in G, is defined as diameter. It is denoted by $diam(G)$[8-9].

Let S is a subset of the vertices set of G. If the elements of S are subtracted from the vertex set $V(G)$, we get difference set. If all vertices of the set we have is connected to at least one vertex in the set S, the set we have obtained by subtracting is a dominating set. Within G's all dominating sets, the size of the dominating set which has the minimum cardinality is named as the domination number of graph G and it is indicated by $\gamma(G)$.

The definition given immediately above is a rapidly growing and significant research topic that has attracted interest in the graph theory in recent years. This rapid growth can be explained by the variety of its applicability to real-world problems as well as to theory. For instance, facility location problems are modeled naturally as dominating sets in graph theory. The number of domination is foremost significant vulnerability parameters for nets. There are several domination parameters in the literature [2, 4-5, 12].

In real life applications, we can encounter that a vertex can affect both its neighborhood vertices and all vertices within a given distance. Distance domination is a kind of this situation. There has been no framework yet in which the effect of a vertex broadens over its neighbors while decreasing by distance. Exponential domination can be a model for the reliability of a spreading information or a hearsay [6]. In this model, distance exponentially reduces the dominating strategy of any vertex of a graph G, by the factor $1/2$. Therefore, it is possible that a vertex v is suppressed by one of its neighbors or by some vertices that are closer to v. It is assumed that hearsay gathered straight from a source is completely trustworthy, whereas passed one from person to person misses its reliability by the factor $1/2$ in each person. The exponential domination number can be found by

calculating the smallest quantity of origins required so that each person gets fully reliable information.
This study deals with the exponential domination number of graphs which is a new characteristic for graph vulnerability introduced by Dankelmann [6]. This new parameter is closely in relation with a distance of each pair of vertices. Let G as a graph and $S \subseteq V(G)$. When G is induced by the S vertices subset, we get a subgraph of the original graph G. This subgraph is indicated by $\langle S\rangle$. $\forall u_1 \in S$ and $\forall u_2 \in V(G)-S$, if there is a path between u_1 and u_2 vertices, $\overline{d}(u_1, u_2) = \overline{d}(u_2, u_1)$ defined as the shortest path length in $\langle V(G)-(S-\{u_1\})\rangle$, otherwise $\overline{d}(u_1, u_2) = \infty$. Let $v \in V(G)$. The definition is

$$w_s(v) = \begin{cases} \sum_{u\in S} 1/2^{\overline{d}(u,v)-1}, & \text{if } v \notin S \\ 2, & \text{otherwise} \end{cases}$$

We refer to $w_s(v)$ as the weight of S at v. If $w_s(v) \geq 1$, $\forall v \in V(G)$ then S can be an exponential dominating set. Within all exponential dominating set of G, the size of the exponential dominating set with the smallest cardinality is named as the exponential domination number of graph G and it is indicated by $\gamma_e(G)$. Also, minimum exponential dominating set of G denoted by γ_e *−set*. Let $u_1 \in S$ and $u_2 \in V(G)-S$, u_1 exponentially dominates u_2, if $\dfrac{1}{2^{\overline{d}(u_1,u_2)-1}} > 0$[1, 6-7, 11].
The study continues as follow. Section 2 gives the theoretical background and an overview of results on the exponential domination number. Main results for the exponential domination number of particular types of graphs and regular caterpillars are provided in Section 3 while giving an insight of how to evaluate the parameter and derive formula on path type networks.

2. Basic results

In this part of the study, some known basic results related to exponential domination number in the literature are given.

Theorem 2.1. *[6] The exponential domination number of*

(a) the path graph P_n of order $n \geq 2$ is $\gamma_e(P_n) = \lceil \frac{n+1}{4} \rceil$.

(b) the cycle graph C_n of order $n \geq 4$ is $\gamma_e(C_n) = \begin{cases} 2, & \text{if } n = 4 \\ \lceil \frac{n}{4} \rceil, & \text{if } n \neq 4 \end{cases}$

Theorem 2.2. *[6] For every graph $G, \gamma_e(G) \leq \gamma(G)$, and also $\gamma_e(G) = 1$ if and only if $\gamma(G) = 1$.*

Theorem 2.3. *[1] If G is connected graph, $|V(G)| = n$ and there is a vertex of G with a degree of $n-1$. Then, $\gamma_e(G) = 1$.*

Theorem 2.4. *[1] If G is a connected graph, $|V(G)| = n$, diam $(G) = 2$ and there isn't any vertex such that $deg(v) = n-1$. Then, $\gamma_e(G) = 2$.*

Theorem 2.5. *[6] If G is a connected graph of diameter d, then $\gamma_e(G) = \lceil \frac{d+2}{4} \rceil$.*

Theorem 2.6. *[6] If G is a connected graph and $|V(G)| = n$ then $\gamma_e(G) \leq \frac{2}{5}(n+2)$.*

Theorem 2.7. *[6] If G is a connected graph, $|V(G)| = n$ and T is G's spanning tree. Then $\gamma_e(G) \leq \gamma_e(T)$.*

Theorem 2.8. *If the diameter of G is at least 3, then $\gamma_e(\overline{G}) = 2$.*

Proof. Assume that u and v are two vertices such that $d_G(u, v) = 3$. Since $d_G(u, v) = 3$, $N(u) \cap N(v) = \emptyset$. Assume that $\overline{S}$ is a minimum exponential dominating set in $\overline{G}$. Firstly, we assume that $\overline{S}$ consists of the vertex u, that is $\overline{S} = \{u\}$. For $x \in V(G) - \overline{S}$, two cases arise.
Case 1. Let $x \notin N_G(u)$.
In this case, u is connected to x in $\overline{G}$. Therefore, $\overline{S}$ in $\overline{G}$ is exponentially dominates the vertex x.
Case 2. Let $x \in N_G(u)$.
In this case, $d_{\overline{G}}(u, x) = 2$. The vertex u contributes $\frac{1}{2}$ to $w_s(x)$. Thus we must add new one vertex to $\overline{S}$. When this new vertex is added to $\overline{S}$, at least one vertex of $\overline{S}$ should not be connected to the vertex x in G. This requires that v, because $d_G(u, v) = 3$. Hence, we get $\overline{S} = \{u, v\}$ and $|\overline{S}| = 2$.
By Case 1 and Case 2, the exponential domination number of $\overline{G}$ is $\gamma_e(\overline{G}) = 2$. □

In graph theory, complement graphs discussed in the previous theorem have a great importance. Because, there are lots of theoretical graph concepts which contains complement graph concept. Therefore, the information we obtain about the complement of a graph will allow us to make comments for other related graph concepts as well. For example, a graph without edges is the complement of a complement graph. Also, regular graphs play an important role in determining isomorphic structures due to their specific properties.

Theorem 2.9. *Let G be a r–regular graph with order n. Then, $\gamma_e(G) = \lceil \frac{2(n+1)}{3r+2} \rceil$.*

Proof. If u and v are vertices in G then $[u, v]$ will denote the G's all vertices set that lie on at least one $u - v$ geodesic. Let S be a minimum exponentially dominating set of G. It is obvious that each vertex of $G - S$ is exponentially dominated by at most two vertices of S. Let $u \in S$ and $v \in S$, on G such that $[u, v] \cap S = \{u, v\}$ and $u = u_0, u_1, \ldots, u_{\frac{r}{2}}, u_{\frac{r}{2}+1}, u_r, u_{r+1}, \ldots, u_{\frac{3r}{2}+1} = v$ the $u - v$ path in G. The vertex u contributes $\frac{1}{2}$ to $w_s(u_{\frac{r}{2}+1})$, so v must contribute at least $\frac{1}{2}$ to $w_s(u_{\frac{r}{2}+1})$. This implies that $d(u, v) \leq 4$. Since $\forall v \in V(G)$, $deg(v) = r$, the number of the vertices in $[u, v]$ is at most $\frac{3r+2}{2}$. This leads to $n \leq (\frac{3r+2}{2})\gamma_e(G) - 1$.

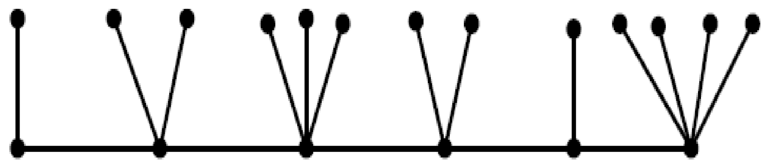
To see that $\gamma_e(G) = \lceil \frac{2(n+1)}{3r+2} \rceil$, we note that easy to construct exponential dominating set with $|S| = \lceil \frac{2(n+1)}{3r+2} \rceil$ □

3. The exponential domination number of some thorn networks

In this section, we give a definition of thorn network. Then, we calculate the exponential domination number of some thorn networks.

Definition 3.1. *[10] Let p_i be in the set $\mathbb{Z}^* = \{0\} \cup \mathbb{Z}^+, V(G)$ be the vertices of graph G and $u_i \in V(G)$ for i with $1 \leq i \leq n$. If we attach p_i new vertices with the degree one to the each vertex $u_i \in V(G)$ respectively then we get the thorn graph of the graph G with parameters p_i. We denote u_{ij} the thorn which come from the vertex u_i, for all i and j*

with $1 \leq i \leq n$ and $1 \leq j \leq p_i$. The representation of the thorn graph of the graph G is in the form G^* or $G^*(p_1, p_2, \ldots, p_n)$.



$P_6^*(1, 2, 3, 2, 1, 4)$

FIGURE 1. A thorn graph of P_6

Theorem 3.1. *Let P_n^* be thorn graph of P_n with $p_j \geq 1, i \in \{1, 2, \ldots, n\}$. Then $\gamma_e(P_n^*) = \lfloor \frac{n+2}{2} \rfloor$.*

Proof. Label the vertices of P_n by $u_1, u_2, \ldots, u_n$ and let the new vertex attached to the vertex u_i of the graph be $u_{ij}, j \in \{1, 2, \ldots, p_i\}$. It is obvious that $deg(u_{ij}) = 1$ and $deg(u_i) \geq 2$. Let $S = \{u_{2i+1} | i \in \{0, 1, \ldots, \lfloor \frac{diam}{2} \rfloor\}\}$. Any vertex u in S dominates all adjacent vertices. Consider the vertices $x \in (V(P_n^*) - \sum_{u \in S} N[u])$. The vertices x are the thorns of vertices in $V(P_n) - S$. These vertices are at distance 2 to exactly two vertices in S. This implies $w_{S(v)} \geq 1$ for $\forall v \in V(P_n^*)$. So, the elements of S set dominate either all vertices of $V(P_n^*)$ or some vertices of $V(P_n^*)$ remain undominated and then we have

$$\gamma_e(P_n^*) \geq |S| = 1 + \lfloor \frac{diam}{2} \rfloor.$$

Let S^* be a minimum exponential dominating set of P_n^* and S^* contains all vertices of S. Depending on the value of n, two cases arise.

Case 1. Let $n \equiv 0 \ (mod\ 2)$.
In this case, $S^* = \{u_1, u_2, \ldots, u_{n-1}\}$. Then, the set S^* contributes $1/2$ to $w_{S^*}(u_{nj})$. For the vertex $u_{nj}, w_{s^*}(u_{nj}) \geq 1$ does not satisfied. So, one vertex which is either any vertex u_{nj} or the vertex u_n must be added to S^*. Then we have

$$\gamma_e(P_n^*) \geq |S^*| = 1 + 1 + \frac{diam - 1}{2} = \frac{diam + 3}{2}.$$

Case 2. Lets $n \equiv 1 \ (mod\ 2)$.
If S^* contains all vertices of S and only them, all vertices of $V(P_n^*)$ are exponentially dominated by S^*. Thus, $|S^*| = |S|$ for this case. Hence, we get

$$\gamma_e(P_n^*) \geq |S^*| = |S| = 1 + \frac{diam}{2} = \frac{diam + 2}{2}$$

It is easy to say that from Case1, the set $\{u_1, u_2\}$ is $\gamma_e - set$ of P_2^* and $\gamma_e(P_2^*) = 2$. But, $S = \{u_1\}$ for P_2^* according to the definition of S given above. $S^* - \{u_1\}$ is $\gamma_e - set$ of P_{n-2}^*. That is $\{u_1, u_2\} - \{u_1\} = \{u_2\}$ must be removed from S^*. By an inductive argument, we obtain

$$\gamma_e(P_n^*) \leq \gamma_e(P_2^*) + \gamma_e(P_{n-2}^*) - 1$$

We examine all cases using $diam = n-1$ and inductive argument. Thus, we obtain from Case1,

$$\frac{n-1+3}{2} = \frac{n+2}{2} \leq \gamma_e(P_n^*) \leq 2 + \frac{(n-3)+3}{2} - 1 = \frac{n+2}{2}.$$

Hence, we get $\gamma_e(P_n^*) = \dfrac{n+2}{2}$.

Similarly, we have from Case2,

$$\frac{n-1+2}{2} = \frac{n+1}{2} \leq \gamma_e(P_n^*) \leq 2 + \frac{(n-3)+2}{2} - 1 = \frac{n+1}{2}$$

Thus, we have $\gamma_e(P_n^*) = \dfrac{n+1}{2}$.

It is obvious that $\dfrac{n+1}{2} = \lfloor \dfrac{n+2}{2} \rfloor$ and $\dfrac{n+2}{2} = \lfloor \dfrac{n+2}{2} \rfloor$. As a result, $\gamma_e(P_n^*) = \lfloor \dfrac{n+2}{2} \rfloor$ is obtained.

□

Theorem 3.2. *Let C_n^* be thorn graph of C_n with $p_i \geq 1, i \in \{1,\ 2, \ldots, n\}$. Then $\gamma_e(C_n^*) = \lfloor \dfrac{n+1}{2} \rfloor$.*

Proof. The proof can be made like the proof of Theorem 3.1. □

Theorem 3.3. *Let $K_{m,n}^*$ be thorn graph of $K_{m,n}$ with $m \geq n$ and $p_i \geq 1, i \in \{1,\ 2, \ldots, n\}$. Then $\gamma_e(K_{m,n}^*)$ is 2.*

Proof. Let $(K_{m,n}^*) = V_1 \cup V_2 \cup V_1' \cup V_2'$, where
$V_1 = \{u_1, u_2, \ldots, u_m\}$,
$V_2 = \{u_1, u_2, \ldots, u_{m+n}\}$,
$V_1' = \{u_{ij} | 1 \leq i \leq m \ and \ 1 \leq j \leq p_i\}$. Let u_{ij} be the thorn the vertices of V_1.
$V_2' = \{u_{rs} | m+1 \leq r \leq m+n \ and \ 1 \leq s \leq p_r\}$. Let u_{rs} be the thorn the vertices of V_2.
The distance between the vertices of these vertices sets is as follows:
Let u_{ij} and u_{xy} be distinct vertices in V_1'.

$$d(u_{ij}, u_{xy}) = \begin{cases} 2, & \text{if } i = x \\ 4, & \text{if } i \neq x \end{cases}$$

This value is also some for any two distinct vertices of V_2'.
The distance between any two vertices, one in V_1' and the other V_2', is 3.
The distance between any two vertices, one in V_1' and the other V_2 or one in V_2' and the other V_1 or one in V_1 and the other V_1 or one in V_2 and the other V_2, is 2.

Let S^* be a minimum exponential domination set $K_{m,n}^*$. According to the distance examined above, S^* must contain exactly two vertices of V_1 or V_2. Hence, $|S^*| = 2$ and all vertices of $K_{m,n}^*$ are exponentially dominated. Consequently, the exponential domination number of $K_{m,n}^*$ is 2. □

Theorem 3.4. *Let $K_{1,n-1}^*$ be thorn graph of $K_{1,n-1}$ with $p_i \geq 1, i \in \{1, 2, \ldots, n\}$ and $n \geq 5$. Then $\gamma_e(K_{1,n-1}^*) = 4$.*

Proof. Let S^* be a minimum exponential domination set $K_{1,n-1}^*$. As might be seen that S^* must not contain the central vertex c in $K_{1,n-1}$. If S^*consists of all vertices in $(V(K_{1,n-1}) - \{c\})$, all vertices in $K_{1,n-1}^*$are exponentially dominated. But, in this case S^* must not be minimum exponential domination set. Therefore, S^* should be consist of some vertices in $(V(K_{1,n-1}) - \{c\})$. The distance between the thorn vertex u and the vertex $v \in (V(K_{1,n-1}) - \{c\})$ is

$$d(u,\ v) = \begin{cases} 1, & \text{if } v \in N(u) \\ 3, & \text{otherwise.} \end{cases}$$

We assume that S^* contains only one vertex $v \in (V(K_{1,n-1}) - \{c\})$. We must find minimum number of vertices of S^* that provides

$$w_{S^*}(v) = \sum_{u \in S^*} (\frac{1}{2})^{\overline{d}(u,v)-1} \geq 1$$

where v is thorn vertex of $K^*_{1,n-1}$. Since $\overline{d}(u,\ v) = 3$,

$$\begin{aligned} |S^* \backslash \{u\}| &= 2^{\overline{d}(u,v)-1} - 1 \\ &= 2^2 - 1 = 3. \end{aligned}$$

Hence, $|S^*| = 4$. Then, all vertices in $K^*_{1,n-1}$ are exponentially dominated by S^*. Consequently, the exponential domination number of $K^*_{1,n-1}$with $n \geq 5$ is

$$\gamma_e(K^*_{1,n-1}) = 4.$$

□

Theorem 3.5. *Let $W^*_{1,n-1}$ be thorn graph of $W_{1,n-1}$ with $p_i \geq 1, i \in \{1,\ 2, \ldots, n\}$ and $n \geq 5$. Then $\gamma_e(W^*_{1,n-1}) = 4$.*

Proof. The proof can be made like the proof of Theorem 3.4. □

The concept of thorn graphs proposed recently by Ivan Gutman to study of chemical graphs [10]. Danail Bonchev and Douglas J Klein extended this idea to a more general concept of thorny graph. Since it represents the structural formula of aliphatic hydrocarbons and aromatic hydrocarbons, this graphs' class has great significance in spectral theory.

Calculation of exponential domination number for some thorn graph types is important. Because when more complex networks are fragmented into smaller networks, for an optimization problem, the solutions on small networks can be unified as a solution on a large network, in some circumstances.

4. Conclusion

Exponential domination can be a model for the reliability of a spreading information or a hearsay. In this model, distance exponentially reduces the dominating strategy of any vertex of a graph G, by the factor 1/2. By using exponential domination number, the vertices within a network can be identified which are more important than others and responsible for the fast communication flow. The vertices that give the number of exponential dominating of a graph are fast in distributing information over the network.

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