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DIRECT AND INVERSE PROBLEMS FOR DIFFUSION OPERATOR WITH DISCONTINUITY POINTS

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Abstract. In this study, the diffusion operator with discontinuity points has been considered. Under certain initial and jump conditions, integral equations have been derived for solutions and integral representation have been presented. Some important spectral properties of eigenvalue and eigenfunctions have been obtained. Reconstruction of the diffusion operator with discontinuity points problem have been proved by Weyl function, spectral datas and two sectra.

Keywords: Integral equation, Sturm-Liouville, Diffusion operator, inverse problems.

AMS Subject Classification: 34K08, 34L05, 34L10, 34E05

1. Introduction

Let's define the following boundary value problem;

$$l\left(y\right) := -y'' + \left[2\lambda p\left(x\right) + q\left(x\right)\right] y = \lambda^2 y,\ x \in (0, a_1) \cup (a_1, a_2) \cup (a_2, \pi) \tag{1}$$

$$U\left(y\right) = y'\left(0\right) = 0, V\left(y\right) = y\left(\pi\right) = 0 \tag{2}$$

$$y\left(a_1 + 0\right) = \alpha_1 y\left(a_1 - 0\right) \tag{3}$$

$$y'\left(a_1 + 0\right) = \beta_1 y'\left(a_1 - 0\right) + i\lambda\gamma_1 y\left(a_1 - 0\right) \tag{4}$$

$$y\left(a_2 + 0\right) = \alpha_2 y\left(a_2 - 0\right) \tag{5}$$

$$y'\left(a_2 + 0\right) = \beta_2 y'\left(a_2 - 0\right) + i\lambda\gamma_2 y\left(a_2 - 0\right) \tag{6}$$

where λ is a spectral parameter, $q(x) \in L_2\left[0, \pi\right]$, $p(x) \in W_2^1\left[0, \pi\right]$, $a_1, a_2 \in (0, \pi)$,$a_1 < a_2$, $\left|\alpha_1 - 1\right|^2 + \gamma_1^2 \neq 0$,$\left|\alpha_2 - 1\right|^2 + \gamma_2^2 \neq 0$,$\left(\beta_i = \frac{1}{\alpha_i}\left(i = 1, 2\right)\right)$.

The fundamental studies on the spectral theory of the Sturm-Liouville equations were performed by Bernoulli, Euler, Sturm and Liouville. The first study, which is considered to be the beginning of the inverse problem theory for the differential equations, was put forth by Ambartsumyan. A lot of study were done about the inverse problem in $[1-15]$.

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The inverse problem is called reconstruction of the operator, whose spectral characteristics are given in sequences. Spectral theory has a wide usage and plays an important role in engineering and geophysics and natural sciences.
Jaulent and Jean have investigated spectral and inverse problems of diffusion operators in[3] . In [6] , Gasymov and Guseinov have focused on spectral theory of diffusion operators.
The inverse quasiperiodic problem for the diffusion operator was studied by Nabiev [8]. Inverse spectral problem for pencils of differential operators on a finite interval from the Weyl function was studied Buterin and Yurko in [19] . In [20] , another considerable paper have performed by Panakhov and Koyunbakan, the half inverse problem for diffusion operators have been considered. Yang investigated the inverse nodal problem for diffusion operators in [21] .

2. Preliminaries.

In this section, we will recall some basic definitions and concepts in order to fresh memories. Let $\varphi\left(x,\lambda\right)$, $\psi\left(x,\lambda\right)$ be solution of (1) respectively under the initial conditions

$$\phi\left(0,\lambda\right)=1,\phi'\left(0,\lambda\right)=0 \tag{7}$$

$$\psi\left(\pi,\lambda\right)=0,\psi'\left(\pi,\lambda\right)=1 \tag{8}$$

and discontinuity conditions $(3),(6)$.
The function $\varphi\left(x,\lambda\right)$ provides $(9),(10),(11)$.
In this case, if $0\leq x<a_1$, then we have

$$\phi\left(x,\lambda\right)=e^{i\lambda x}+\frac{1}{\lambda}\int_0^x\sin\lambda\left(x-t\right)\chi\left(t\right)y\left(t,\lambda\right)dt, \tag{9}$$

if $a_1<x<a_2$, we have

$$\begin{aligned}&\phi\left(x,\lambda\right)=\tfrac{1}{2}\left(\alpha_1+\beta_1\right)e^{i\lambda x}+\tfrac{1}{2}\left(\alpha_1-\beta_1\right)e^{i\lambda(2a_1-x)}+\tfrac{\gamma_1}{2}e^{i\lambda x}-\tfrac{\gamma_1}{2}e^{i\lambda(2a_1-x)}\\&+\tfrac{1}{2}\left(\alpha_1+\beta_1\right)\int_0^{a_1}\tfrac{\sin\lambda(x-t)}{\lambda}\chi\left(t\right)y\left(t,\lambda\right)dt\\&-\tfrac{1}{2}\left(\alpha_1-\beta_1\right)\int_0^{a_1}\tfrac{\sin\lambda(x+t-2a_1)}{\lambda}\chi\left(t\right)y\left(t,\lambda\right)dt-i\tfrac{\gamma_1}{2}\int_0^{a_1}\tfrac{\cos\lambda(x-t)}{\lambda}\chi\left(t\right)y\left(t,\lambda\right)dt\\&+i\tfrac{\gamma_1}{2}\int_0^{a_1}\tfrac{\cos\lambda(x+t-2a_1)}{\lambda}\chi\left(t\right)y\left(t,\lambda\right)dt+\int_{a_1}^{x}\tfrac{\sin\lambda(x-t)}{\lambda}\chi\left(t\right)y\left(t,\lambda\right)dt.\end{aligned} \tag{10}$$

if $a_2<x\leq\pi$, we can write

$$\begin{aligned}&\phi\left(x,\lambda\right)=\alpha_1^+\alpha_2^+e^{i\lambda x}+\alpha_1^-\alpha_2^-e^{i\lambda(2a_1-2a_2+x)}+\alpha_1^+\alpha_2^-e^{i\lambda(2a_2-x)}+\alpha_1^-\alpha_2^+e^{i\lambda(2a_1-x)}+\tfrac{\gamma_1\alpha_2^+}{2}e^{i\lambda x}\\&-\tfrac{\gamma_1\alpha_2^-}{2}e^{i\lambda(2a_1-2a_2+x)}+\tfrac{\gamma_1\alpha_2^-}{2}e^{i\lambda(2a_2-x)}-\tfrac{\gamma_1\alpha_2^+}{2}e^{i\lambda(2a_1-x)}-\tfrac{\gamma_1\gamma_2}{4}e^{i\lambda x}-\tfrac{\gamma_1\gamma_2}{4}e^{i\lambda(2a_1-2a_2+x)}\\&-\tfrac{\gamma_2\alpha_1^+}{2}e^{i\lambda(2a_2-x)}-\tfrac{\gamma_2\alpha_1^-}{2}e^{i\lambda(2a_1-x)}+\tfrac{\gamma_2\alpha_1^+}{2}e^{i\lambda x}+\tfrac{\gamma_2\alpha_1^-}{2}e^{i\lambda(2a_1-2a_2+x)}-\tfrac{\gamma_1\gamma_2}{4}e^{i\lambda(2a_2-x)}\\&+\tfrac{\gamma_1\gamma_2}{4}e^{i\lambda(2a_1-x)}+\left(\alpha_1^+\alpha_2^++\tfrac{\gamma_1\gamma_2}{4}\right)\int_0^{a_1}\tfrac{\sin\lambda(x-t)}{\lambda}\chi\left(t\right)y\left(t,\lambda\right)dt\\&+\left(-\alpha_1^+\alpha_2^-+\tfrac{\gamma_1\gamma_2}{4}\right)\int_0^{a_1}\tfrac{\sin\lambda(x+t-2a_2)}{\lambda}\chi\left(t\right)y\left(t,\lambda\right)dt\\&+\left(-\alpha_1^-\alpha_2^+-\tfrac{\gamma_1\gamma_2}{4}\right)\int_0^{a_1}\tfrac{\sin\lambda(x+t-2a_1)}{\lambda}\chi\left(t\right)y\left(t,\lambda\right)dt-\alpha_2^-\int_{a_1}^{a_2}\tfrac{\sin\lambda(x+t-2a_2)}{\lambda}\chi\left(t\right)y\left(t,\lambda\right)dt\\&+\left(\alpha_1^-\alpha_2^--\tfrac{\gamma_1\gamma_2}{4}\right)\int_0^{a_1}\tfrac{\sin\lambda(2a_1-2a_2+x-t)}{\lambda}\chi\left(t\right)y\left(t,\lambda\right)dt+\alpha_2^+\int_{a_1}^{a_2}\tfrac{\sin\lambda(x-t)}{\lambda}\chi\left(t\right)y\left(t,\lambda\right)dt\\&-\tfrac{i}{2}\left(\gamma_1\alpha_2^+-\gamma_2\alpha_1^+\right)\int_0^{a_1}\tfrac{\cos\lambda(x-t)}{\lambda}\chi\left(t\right)y\left(t,\lambda\right)dt\\&+\tfrac{i}{2}\left(-\gamma_1\alpha_2^--\gamma_2\alpha_1^-\right)\int_0^{a_1}\tfrac{\cos\lambda(x+t-2a_2)}{\lambda}\chi\left(t\right)y\left(t,\lambda\right)dt\\&+\tfrac{i}{2}\left(\gamma_1\alpha_2^++\gamma_2\alpha_2^-\right)\int_0^{a_1}\tfrac{\cos\lambda(x+t-2a_1)}{\lambda}\chi\left(t\right)y\left(t,\lambda\right)dt+\tfrac{i\gamma_2}{2}\int_{a_1}^{a_2}\tfrac{\cos\lambda(x+t-2a_2)}{\lambda}\chi\left(t\right)y\left(t,\lambda\right)dt\\&+\tfrac{i}{2}\left(\gamma_1\alpha_2^-+\gamma_2\alpha_2^-\right)\int_0^{a_1}\tfrac{\cos\lambda(2a_1-2a_2+x-t)}{\lambda}\chi\left(t\right)y\left(t,\lambda\right)dt-\tfrac{i\gamma_2}{2}\int_{a_1}^{a_2}\tfrac{\cos\lambda(x-t)}{\lambda}\chi\left(t\right)y\left(t,\lambda\right)dt\\&+\int_{a_2}^{x}\tfrac{\sin\lambda(x-t)}{\lambda}\chi\left(t\right)y\left(t,\lambda\right)dt.\end{aligned} \tag{11}$$

The function $\psi\left(x,\lambda\right)$ provides $(12),(13),(14)$.

if $a_2 < x \le \pi$, we have

$$\psi(x,\lambda) = \frac{\sin\lambda(x-\pi)}{\lambda} + \int_x^\pi \frac{\sin\lambda(x-t)}{\lambda}\chi(t)\psi(t,\lambda)\,dt. \tag{12}$$

if $a_1 < x < a_2$, we have

$$\begin{gathered}\psi(x,\lambda) = \tfrac{\alpha_2^+}{\lambda}\sin\lambda(x-\pi) - \tfrac{\alpha_2^-}{\lambda}\sin\lambda(2a_2-x-\pi) + \alpha_2^+\int_{a_2}^\pi \tfrac{\sin\lambda(x-t)}{\lambda}\chi(t)\psi(t,\lambda)\,dt \\ -\alpha_2^-\int_{a_2}^\pi \tfrac{\sin\lambda(x+t-2a_2)}{\lambda}\chi(t)\psi(t,\lambda)\,dt + \tfrac{i\gamma_2}{2\lambda\alpha_2\beta_2}\cos\lambda(x-\pi) - \tfrac{i\gamma_2}{2\lambda\alpha_2\beta_2}\cos\lambda(2a_2-x-\pi) \\ +\tfrac{i\gamma_2}{2\alpha_2\beta_2}\int_{a_2}^\pi \tfrac{\cos\lambda(x-t)}{\lambda}\chi(t)\psi(t,\lambda)\,dt - \tfrac{i\gamma_2}{2\alpha_2\beta_2}\int_{a_2}^\pi \tfrac{\cos\lambda(x+t-2a_2)}{\lambda}\chi(t)\psi(t,\lambda)\,dt \\ +\int_x^{a_2}\tfrac{\sin\lambda(x-t)}{\lambda}\chi(t)\psi(t,\lambda)\,dt\end{gathered} \tag{13}$$

if $0 \le x < a_1$, then it has the form

$$\begin{gathered}\psi(x,\lambda) = \alpha_1^+\alpha_2^+\tfrac{\sin\lambda(x-\pi)}{\lambda} - \alpha_1^-\alpha_2^+\tfrac{\sin\lambda(2a_1-x-\pi)}{\lambda} + \alpha_1^-\alpha_2^-\tfrac{\sin\lambda(2a_2-2a_1+x-\pi)}{\lambda} \\ -\alpha_1^+\alpha_2^-\tfrac{\cos\lambda(2a_2-x-\pi)}{\lambda} + \tfrac{i\alpha_1^+\gamma_2}{2\alpha_2\beta_2}\tfrac{\cos\lambda(x-\pi)}{\lambda} - \tfrac{i\alpha_1^-\gamma_2}{2\alpha_2\beta_2}\tfrac{\cos\lambda(2a_1-x-\pi)}{\lambda} \\ -\tfrac{i\alpha_1^+\gamma_2}{2\alpha_2\beta_2}\tfrac{\cos\lambda(2a_2-x-\pi)}{\lambda} + \tfrac{i\alpha_1^-\gamma_2}{2\alpha_2\beta_2}\tfrac{\cos\lambda(2a_2-2a_1+x-\pi)}{\lambda} - \tfrac{i\alpha_2^+\gamma_2}{2\alpha_1\beta_1}\tfrac{\cos\lambda(2a_1-x-\pi)}{\lambda} \\ +\tfrac{i\alpha_2^+\gamma_1}{2\alpha_1\beta_1}\tfrac{\cos\lambda(x-\pi)}{\lambda} + \tfrac{i\alpha_2^-\gamma_2}{2\alpha_1\beta_1}\tfrac{\cos\lambda(2a_2-x-\pi)}{\lambda} - \tfrac{i\alpha_2^-\gamma_1}{2\alpha_1\beta_1}\tfrac{\cos\lambda(2a_2-2a_1+x-\pi)}{\lambda} \\ +\tfrac{\gamma_1\gamma_2}{4\alpha_1\beta_1\alpha_2\beta_2}\tfrac{\sin\lambda(x-\pi)}{\lambda} - \tfrac{\gamma_1\gamma_2}{4\alpha_1\beta_1\alpha_2\beta_2}\tfrac{\sin\lambda(2a_1-x-\pi)}{\lambda} + \tfrac{\gamma_1\gamma_2}{4\alpha_1\beta_1\alpha_2\beta_2}\tfrac{\sin\lambda(2a_2-x-\pi)}{\lambda} \\ -\tfrac{\gamma_1\gamma_2}{4\alpha_1\beta_1\alpha_2\beta_2}\tfrac{\sin\lambda(2a_2-2a_1+x-\pi)}{\lambda} + \left(\alpha_1^+\alpha_2^+ + \tfrac{\gamma_1\gamma_2}{4\alpha_1\beta_1\alpha_2\beta_2}\right)\int_{a_2}^\pi \tfrac{\sin\lambda(x-t)}{\lambda}\chi(t)\psi(t,\lambda)\,dt \\ +\left(\alpha_1^-\alpha_2^+ + \tfrac{\gamma_1\gamma_2}{4\alpha_1\beta_1\alpha_2\beta_2}\right)\int_{a_2}^\pi \tfrac{\sin\lambda(x+t-2a_1)}{\lambda}\chi(t)\psi(t,\lambda)\,dt \\ -\left(\alpha_1^+\alpha_2^- + \tfrac{\gamma_1\gamma_2}{4\alpha_1\beta_1\alpha_2\beta_2}\right)\int_{a_2}^\pi \tfrac{\sin\lambda(x+t-2a_2)}{\lambda}\chi(t)\psi(t,\lambda)\,dt \\ -\left(\alpha_1^-\alpha_2^- + \tfrac{\gamma_1\gamma_2}{4\alpha_1\beta_1\alpha_2\beta_2}\right)\int_{a_2}^\pi \tfrac{\sin\lambda(2a_2-2a_1+x-\pi)}{\lambda}\chi(t)\psi(t,\lambda)\,dt \\ +i\left(\tfrac{\alpha_1^+\gamma_2}{2\alpha_2\beta_2} + \tfrac{\alpha_2^+\gamma_1}{2\alpha_1\beta_1}\right)\int_{a_2}^\pi \tfrac{\cos\lambda(x-t)}{\lambda}\chi(t)\psi(t,\lambda)\,dt \\ -i\left(\tfrac{\alpha_1^-\gamma_2}{2\alpha_2\beta_2} + \tfrac{\alpha_2^+\gamma_1}{2\alpha_1\beta_1}\right)\int_{a_2}^\pi \tfrac{\cos\lambda(x+t-2a_1)}{\lambda}\chi(t)\psi(t,\lambda)\,dt \\ -i\left(\tfrac{\alpha_1^+\gamma_2}{2\alpha_2\beta_2} + \tfrac{\alpha_2^-\gamma_1}{2\alpha_1\beta_1}\right)\int_{a_2}^\pi \tfrac{\cos\lambda(x+t-2a_2)}{\lambda}\chi(t)\psi(t,\lambda)\,dt \\ -i\left(\tfrac{\alpha_1^+\gamma_2}{2\alpha_2\beta_2} + \tfrac{\alpha_2^-\gamma_1}{2\alpha_1\beta_1}\right)\int_{a_2}^\pi \tfrac{\cos\lambda(2a_2-2a_1+x-\pi)}{\lambda}\chi(t)\psi(t,\lambda)\,dt \\ +\alpha_1^+\int_{a_1}^{a_2}\tfrac{\sin\lambda(x-t)}{\lambda}\chi(t)\psi(t,\lambda)\,dt + \alpha_1^-\int_{a_1}^{a_2}\tfrac{\sin\lambda(x+t-2a_1)}{\lambda}\chi(t)\psi(t,\lambda)\,dt \\ +i\tfrac{\gamma_1}{2\alpha_1\beta_1}\int_{a_1}^{a_2}\tfrac{\cos\lambda(x-t)}{\lambda}\chi(t)\psi(t,\lambda)\,dt - i\tfrac{\gamma_1}{2\alpha_1\beta_1}\int_{a_1}^{a_2}\tfrac{\cos\lambda(x+t-2a_1)}{\lambda}\chi(t)\psi(t,\lambda)\,dt \\ +\int_x^{a_1}\tfrac{\sin\lambda(x-t)}{\lambda}\chi(t)\psi(t,\lambda)\,dt.\end{gathered} \tag{14}$$

Where $\chi(t) = 2\lambda p(t) + q(t)$.

Theorem 2.1. *If $p(x) \in W_2^1(0,\pi)$ and $q(x) \in L_2(0,\pi)$; $y_\upsilon(x,\lambda)$ solution of the equations* (1) *, providing initial conditions* (2) *and discontinuity conditions* (3) − (6) *, then it has the form*

$$y_\upsilon(x,\lambda) = y_{0\upsilon}(x,\lambda) + \int_{-x}^x K_\upsilon(x,\lambda)\,e^{i\lambda t}dt\ \left(\upsilon = \overline{1,3}\right)$$

where

$$y_{0\upsilon}(x,\lambda) = \begin{cases} R_0(x)\,e^{i\lambda x} & ; 0 \le x < a_1 \\ R_1(x)\,e^{i\lambda x} + R_2(x)\,e^{i\lambda(2a_1-x)} & ; a_1 < x < a_2 \\ R_3(x)\,e^{i\lambda(2a_1-x)} + R_4(x)\,e^{i\lambda(2a_1-2a_2+x)} & ; a_2 < x \le \pi \end{cases}$$

$$\begin{aligned} R_0(x) &= e^{-i\int_0^x p(x)dx}, R_1(x) = R_0(a_1)\left(\alpha_1^+ + \frac{\gamma_1}{2}\right)e^{-i\int_{a_1}^x p(t)dt}, \\ R_2(x) &= R_0(a_1)\left(\alpha_1^- - \frac{\gamma_1}{2}\right)e^{i\int_{a_1}^x p(t)dt}, \end{aligned}$$

$$R_3(x) = \left(-\frac{\gamma_2}{2}\left(\alpha_1^+ + \frac{\gamma_1}{2}\right)R_0(a_1) - \alpha_2^- R_1(a_2)\right)e^{i\int_{a_2}^x p(t)dt},$$

$$R_4(x) = \left(\frac{\gamma_2}{2}\left(\alpha_1^- + \frac{\gamma_1}{2}\right)R_0(a_1) + \alpha_2^- R_2(a_2)\right)e^{-i\int_{a_2}^x p(t)dt}$$

and $\sigma(x) = \int_0^x (2|p(t)| + (x-t)|q(t)|)\,dt$ *, where the function* $K_\upsilon(x,t)$ *satisfies the condition*

$$\int_{-x}^{x} |K_\upsilon(x,\lambda)|\,dt \le e^{c_\upsilon \sigma(x)} - 1$$

with $c_1 = 1$ *,*$c_2 = \left(\alpha_1^+ + |\alpha_1^-| + \frac{\gamma_1}{2} + 2\right)$ $c_3 = \left[\left(|\alpha_2^-| + \gamma_2\right)(\alpha_1 + 1) + \alpha_2^+ + 1\right]$.

Theorem 2.2. *Let* $p(x) \in W_2^1(0,\pi)$ *,*$q(x) \in L_2(0,\pi)$*.* $\mathbf{M}(\mathbf{x},\mathbf{t})$ *,*$N(x,t)$ *are summable functions on* $[0,\pi]$ *such that the representation for each* $x \in [0,\pi] / \{a_1, a_2\}$

$$\phi(x,\lambda) = \phi_0(x,\lambda) + \int_0^x M(x,t)\cos\lambda t dt + \int_0^x N(x,t)\sin\lambda t dt$$

is satisfied,

$$a_1 < x < a_2$$

$$\begin{gathered} \phi(x,\lambda) = A_4\cos[\lambda x - \beta(x)] + A_5\cos[\lambda(2a_1 - x) - \beta(x)] \\ + \int_0^x M(x,t)\cos\lambda t dt + \int_0^x N(x,t)\sin\lambda t dt \end{gathered} \tag{15}$$

where $A_4 = \left(\alpha_1^+ + \frac{\gamma_1}{2}\right)$, $A_5 = \left(\alpha_1^- - \frac{\gamma_1}{2}\right)$.

$$a_2 < x \le \pi$$

$$\begin{aligned} \phi(x,\lambda) =& A_1\cos\left[\lambda(2a_1 - x) + \left(\int_{a_2}^x p(t)\,dt - \int_0^{a_1} p(t)\,dt\right)\right] \\ &+ A_2\cos\left[\lambda(2a_1 - x) + \left(\int_{a_2}^x p(t)\,dt - \int_0^{a_2} p(t)\,dt\right)\right] \\ &- A_1\cos\left[\lambda(2a_1 - 2a_2 + x) - \left(\int_{a_2}^x p(t)\,dt + \int_0^{a_1} p(t)\,dt\right)\right] \\ &+ A_3\cos\left[\lambda(2a_1 - 2a_2 + x) - \int_0^x p(t)\,dt\right] \\ &+ \int_0^x M(x,t)\cos\lambda t dt + \int_0^x N(x,t)\sin\lambda t dt \end{aligned} \tag{16}$$

where $A_1 = -\frac{\gamma_2}{2}\left(\alpha_1^+ + \frac{\gamma_1}{2}\right)$, $A_2 = -\alpha_2^-\left(\alpha_1^+ + \frac{\gamma_1}{2}\right)$, $A_3 = \alpha_2^-\left(\alpha_1^- - \frac{\gamma_1}{2}\right)$.
Thus , following the relations hold;

$$\cos\beta(x)\cdot M(x,x) + \sin\beta(x)\cdot N(x,x) = \frac{A_1}{2}\int_0^x \left(q(t) + p^2(t)\right)dt \tag{17}$$

$$p(x) = p(0) + \frac{2}{A_1}\left[M(x,x)\sin\beta(x) - N(x,x)\cos\beta(x)\right] \tag{18}$$

$$\left[M(x,t)\cos\beta(x) - N(x,t)\sin\beta(x)\right]_{t=2a_1-x-0}^{t=2a_1-x+0} = \frac{A_2}{2}\int_0^x \left(q(t) + p^2(t)\right)dt \tag{19}$$

$$\begin{gathered} \left[(A_1\cos\beta_3(x) - A_3\cos\beta_4(x))M(x,t) + (A_1\sin\beta_3(x) - A_3\sin\beta_4(x))N(x,t)\right]_{t=2a_1-2a_2+x-0}^{t=2a_1-2a_2+x+0} \\ = \frac{1}{2}\left(A_1^2 + A_3^2 - 2A_1A_3\cos(\beta_3 - \beta_4)\right)\int_0^x \left(q(t) + p^2(t)\right)dt \end{gathered} \tag{20}$$

$$N(x,0) = \left.\frac{\partial M(x,t)}{\partial t}\right|_{t=0} = 0. \tag{21}$$

If $p(x) \in W_2^2(0,\pi)$, $q(x) \in W_2^1(0,\pi)$, *so system* (22) *is provided for* $M(x,t)$, $N(x,t)$.

$$\begin{cases} \frac{\partial^2 M(x,t)}{\partial x^2} - M(x,t)\,q(x) - 2p(x)\frac{\partial N(x,t)}{\partial t} = \frac{\partial^2 M(x,t)}{\partial t^2} \\ \frac{\partial^2 N(x,t)}{\partial x^2} - N(x,t)\,q(x) + 2p(x)\frac{\partial M(x,t)}{\partial t} = \frac{\partial^2 N(x,t)}{\partial t^2} \end{cases} \tag{22}$$

On the contrary , if $M(x,t)$, $N(x,t)$*are summable on* $[0,\pi]$ *for each* $x \in [0,\pi] / \{a_1, a_2\}$*and satisfy equalities* (22) *and* $(17) - (21)$, *then the function* $\phi(x,\lambda)$ *which is defined by* $(15) - (16)$*is a solution of* $(1) - (6)$.

Definition 2.3. If $y_0(x)$ a nontrivial solution of equation (1) with conditions $(2) - (6)$, then λ_0 is called eigenvalue. Additionally, $y_0(x)$ is called the eigenfunction of the problem corresponding to the eigenvalue λ_0.
We suppose that the function $q(x)$ provides the condition

$$\int_0^\pi \left[\left|y'(x)\right|^2 + q(x)\left|y(x)\right|^2\right] dx > 0 \tag{23}$$

for all $y(x) \in W_2^2[0,a_1) \cup (a_1,a_2) \cup (a_2,\pi]$ such that $y(x) \neq 0$ and

$$y'(0) \cdot \overline{y(0)} - y'(\pi) \cdot \overline{y(\pi)} = 0.$$

Lemma 2.4. Eigenvalues $\{\lambda_n\}$ for the problem $(1) - (6)$ are real.
Proof. We set $l(y) := [-y'' + q(x)\,y]$. Then,

$$\begin{aligned}(l(y), y) &= \int_0^\pi l(y) \cdot \overline{y(x)} dx \\ &= \int_0^\pi \left\{|y'(x)|^2 + |y(x)|^2 q(x)\right\} dx\end{aligned}$$

due to condition (23), follow this $(l(y), y) > 0$.
Lemma 2.5. Eigenfunctions corresponding to different eigenvalues of problem $(1) - (6)$ are orthogonal in the sense of the equality

$$(\lambda_n + \lambda_k)\int_0^\pi y(x,\lambda_n)\,y(x,\lambda_k)\,dx - 2\int_0^\pi p(x)\,y(x,\lambda_n)\,y(x,\lambda_k)\,dx = 0. \tag{24}$$

3. Properties of the spectrum

$\psi(x,\lambda)$,$\phi(x,\lambda)$ are any two solution functions of equation (1).

$$W[\psi(x,\lambda), \phi(x,\lambda)] = \psi(x,\lambda)\,\phi'(x,\lambda) - \psi'(x,\lambda)\,\phi(x,\lambda)$$

One can see that Wronskian is independent by x,
$\frac{dW[\psi,\phi]}{dx} = 0$ and

$$\begin{aligned} W[\psi(x,\lambda), \phi(x,\lambda)]_{a_1+0} &= W[\psi(x,\lambda), \phi(x,\lambda)]_{a_1-0}, \\ W[\psi(x,\lambda), \phi(x,\lambda)]_{a_2+0} &= W[\psi(x,\lambda), \phi(x,\lambda)]_{a_2-0} \end{aligned}$$

.
Then it can be shown as $W[\psi,\varphi] = \Delta(\lambda)$.
The $\Delta(\lambda)$ function, which is defined as the characteristic function of $(1) - (6)$, is entire for λ. In that case, $\{\lambda_n\}$ is countable set of zeros of characteristic function.
Lemma 3.1. Eigenvalues of the problem $(1) - (6)$ and zeros of the characteristic function are overlapped. $\psi(x,\lambda_0)$,$\phi(x,\lambda_0)$ functions corresponding to eigenvalue λ_n; then there is such a (η_n) sequence that provides

$$\psi(x,\lambda_n) = \eta_n \phi(x,\lambda_n) \;,\; \eta_n \neq 0. \tag{25}$$

The proof of the lemma is done as in [7].

Let use denote

$$\alpha_n = \int_0^\pi \varphi^2(x,\lambda_n)\,dx - \frac{1}{\lambda_n}\int_0^\pi p(x)\,\varphi^2(x,\lambda_n)\,dx,\ n=1,2,3,... \tag{26}$$

$\{\alpha_n\}$ are the normalized numbers of problem (1) – (6).

Lemma 3.2. The equality $\dot{\Delta}(\lambda_n) = 2\lambda_n\beta_n\alpha_n$ is held. Here $\dot{\Delta} = \frac{d\Delta}{d\lambda}$.

Proof. Since $\phi(x,\lambda_0)$, $\psi(x,\lambda_0)$ are the solutions of (1), the following equations are satisfied;

$$\begin{array}{l} -\phi''(x,\lambda) + [2\lambda p(x) + q(x)]\,\phi(x,\lambda) = \lambda^2\phi(x,\lambda) \\ -\psi''(x,\lambda) + [2\lambda p(x) + q(x)]\,\psi(x,\lambda) = \lambda^2\psi(x,\lambda) \end{array} \tag{27}$$

If equations (27) are differentiated with respect to λ, we get

$$\begin{array}{l} -\dot{\phi}''(x,\lambda) + [2\lambda p(x) + q(x)]\,\dot{\phi}(x,\lambda) = \lambda^2\dot{\phi}(x,\lambda) + \phi(x,\lambda)\{2\lambda - 2p(x)\} \\ -\dot{\psi}''(x,\lambda) + [2\lambda p(x) + q(x)]\,\dot{\psi}(x,\lambda) = \lambda^2\dot{\psi}(x,\lambda) + \psi(x,\lambda)\{2\lambda - 2p(x)\} \end{array} \tag{28}$$

are held. Namely,

$$\begin{array}{l} \frac{d}{dx}\left\{\phi(x,\lambda)\cdot\dot{\psi}'(x,\lambda) - \phi'(x,\lambda)\cdot\dot{\psi}(x,\lambda)\right\} = -[2\lambda - 2p(x)]\,\phi(x,\lambda)\,\psi(x,\lambda) \\ \frac{d}{dx}\left\{\dot{\phi}(x,\lambda)\cdot\psi'(x,\lambda) - \dot{\phi}'(x,\lambda)\cdot\psi(x,\lambda)\right\} = [2\lambda - 2p(x)]\,\phi(x,\lambda)\,\psi(x,\lambda) \end{array} \tag{29}$$

are held. If equations (29) are integrated from 0 to π , respectively, we obtain

$$-\left\{\phi(\xi,\lambda)\cdot\dot{\psi}'(\xi,\lambda) - \phi'(\xi,\lambda)\cdot\dot{\psi}(\xi,\lambda)\right\}_x^\pi = \int_x^\pi [2\lambda - 2p(\xi)]\,\phi(\xi,\lambda)\,\psi(\xi,\lambda)\,d\xi \tag{30}$$

and

$$\begin{aligned} &\frac{d}{dx}\left\{\dot{\phi}(\xi,\lambda)\cdot\psi'(\xi,\lambda) - \dot{\phi}'(\xi,\lambda)\cdot\psi(\xi,\lambda)\right\} \\ = \ &[2\lambda - 2p(x)]\,\phi(\xi,\lambda)\,\psi(\xi,\lambda) \end{aligned}$$

$$\begin{aligned} &\int_0^x \frac{d}{dx}\left\{\dot{\phi}(\xi,\lambda)\cdot\psi'(\xi,\lambda) - \dot{\phi}'(\xi,\lambda)\cdot\psi(\xi,\lambda)\right\}d\xi \\ = \ &\int_0^x [2\lambda - 2p(\xi)]\,\phi(\xi,\lambda)\,\psi(\xi,\lambda)\,d\xi \end{aligned}$$

$$\begin{array}{l} \int_0^{a_1-0} \frac{d}{dx}\left\{\dot{\phi}(\xi,\lambda)\cdot\psi'(\xi,\lambda) - \dot{\phi}'(\xi,\lambda)\cdot\psi(\xi,\lambda)\right\}d\xi \\ + \int_{a_1+0}^{a_2-0} \frac{d}{dx}\left\{\dot{\phi}(\xi,\lambda)\cdot\psi'(\xi,\lambda) - \dot{\phi}'(\xi,\lambda)\cdot\psi(\xi,\lambda)\right\}d\xi \\ + \int_{a_2+0}^{x} \frac{d}{dx}\left\{\dot{\phi}(\xi,\lambda)\cdot\psi'(\xi,\lambda) - \dot{\phi}'(\xi,\lambda)\cdot\psi(\xi,\lambda)\right\}d\xi \\ = \int_0^{a_1-0} [2\lambda - 2p(\xi)]\,\phi(\xi,\lambda)\,\psi(\xi,\lambda)\,d\xi + \int_{a_1+0}^{a_2-0} [2\lambda - 2p(\xi)]\,\phi(\xi,\lambda)\,\psi(\xi,\lambda)\,d\xi \\ + \int_{a_2+0}^{x} [2\lambda - 2p(\xi)]\,\phi(\xi,\lambda)\,\psi(\xi,\lambda)\,d\xi \end{array}$$

$$\left\{\overset{\bullet}{\phi}(\xi,\lambda)\cdot\psi'(\xi,\lambda)-\overset{\bullet}{\phi}'(\xi,\lambda)\cdot\psi(\xi,\lambda)\right\}_0^{a_1-0}$$
$$+\left\{\overset{\bullet}{\phi}(\xi,\lambda)\cdot\psi'(\xi,\lambda)-\overset{\bullet}{\phi}'(\xi,\lambda)\cdot\psi(\xi,\lambda)\right\}_{a_1+0}^{a_2-0}$$
$$+\left\{\overset{\bullet}{\phi}(\xi,\lambda)\cdot\psi'(\xi,\lambda)-\overset{\bullet}{\phi}'(\xi,\lambda)\cdot\psi(\xi,\lambda)\right\}_{a_2+0}^{x}$$
$$=\int_0^x[2\lambda-2p(\xi)]\,\phi(\xi,\lambda)\,\psi(\xi,\lambda)\,d\xi$$

$$\left\{\overset{\bullet}{\phi}(\xi,\lambda)\cdot\psi'(\xi,\lambda)-\overset{\bullet}{\phi}'(\xi,\lambda)\cdot\psi(\xi,\lambda)\right\}_0^{x} \tag{31}$$
$$=\int_0^x[2\lambda-2p(\xi)]\,\phi(\xi,\lambda)\,\psi(\xi,\lambda)\,d\xi.$$

By addition side by side, we obtain the equality

$$W\left[\phi(\xi,\lambda),\dot{\psi}(\xi,\lambda)\right]+W\left[\dot{\phi}(\xi,\lambda),\psi(\xi,\lambda)\right]$$
$$=-\dot{\Delta}(\lambda)=\int_0^\pi[2\lambda-2p(\xi)]\,\phi(\xi,\lambda)\,\psi(\xi,\lambda)\,d\xi$$

for $\lambda\to\lambda_n$. This yield

$$\overset{\bullet}{\Delta}(\lambda_n)=-\int_0^\pi[2\lambda-2p(\xi)]\,\eta_n\phi^2(\xi,\lambda_n)\,d\xi$$
$$=2\lambda_n\eta_n\int_0^\pi\phi^2(\xi,\lambda_n)\,d\xi-\frac{1}{\lambda_n}\int_0^\pi p(\xi)\,\phi^2(\xi,\lambda_n)\,d\xi=2\lambda_n\eta_n\alpha_n.$$

Let denote $\Gamma_n=\left\{\lambda:|\lambda|=\left|\lambda_n^0\right|+\delta,\delta>0,n=0,1,2,...\right\}$,
$T_n=\left\{\lambda:\left|\lambda-\lambda_n^0\right|\geq\delta,\delta>0,n=0,1,2,...\right\}$,
where $\delta>0$.

Lemma 3.3. For enough large values of n ,

$$|\Delta(\lambda)-\Delta_0(\lambda)|<\frac{C_\delta}{2}e^{|\tau|(2a_1-2a_2+\pi)},\lambda\in\Gamma_n \tag{32}$$

is provided.

Proof. As it is shown in [26] , $|\Delta_0(\lambda)|\geq C_\delta e^{|Im\lambda|\pi}$ for all $\lambda\in\bar{T}_\delta$, where $C_\delta>0$ is constant. On the other hand, since

$$\lim_{|\lambda|\to\infty}e^{-|Im\lambda|\pi}\left(\Delta(\lambda)-\Delta_0(\lambda)\right) \tag{33}$$
$$=\lim_{|\lambda|\to\infty}e^{-|Im\lambda|\pi}\left(\int_0^\pi\tilde{M}(\pi,t)\cos\lambda t dt+\int_0^\pi\tilde{N}(\pi,t)\sin\lambda t dt\right)=0$$

for enough large values of n (see[1]) we get (3) . The lemma is proved.

Lemma 3.4. The boundary value problem (1) – (6) has a countable number of eigenvalues, that grow unlimitedly, when that are ordered according to their absolute value. In addition, eigenvalues can also be shown asymptotically as the following.

$$\lambda_n=\lambda_n^0+\frac{s_n}{\lambda_n^0}+\frac{t_n}{\lambda_n^0}\ ,\ n\to\infty$$
$$\lambda_n^0=\frac{n\pi}{2a_1-2a_2+\pi}+\psi_1(n)\ ;\ \sup_n|\psi_1(n)|=c<+\infty$$

where $t_n\in l_2$ and s_n is a bounded sequence.

Proof. By the lemma 3.3., if $\lambda\in\Gamma_n$, $|\Delta_0(\lambda)|\geq C_\delta e^{|Im\lambda|\pi}>\frac{C_\delta}{2}e^{|Im\lambda|\pi}>|\Delta(\lambda)-\Delta_0(\lambda)|$ is true. Applying Rouche's theorem inside the Γ_n, the functions $\Delta(\lambda)=\Delta_0(\lambda)+\{\Delta(\lambda)-\Delta_0(\lambda)\}$ and $\Delta_0(\lambda)$ have the same number of zeros. Namely , $\lambda_0,\lambda_1,\lambda_2,...,\lambda_n$ are zeros. Similarly, according to the Rouche's theorem the function $\Delta(\lambda)$ has a unique zero inside each circle $\left|\lambda-\lambda_k^0\right|<\delta$ for sufficiently large values of k. Since $\delta>0$, $\lambda_n=\lambda_n^0+\varepsilon_n$. where $\lim_{n\to\infty}\varepsilon_n=0$. If $\Delta(\lambda_n)=0$, we have

$$\Delta_0\left(\lambda_n^0+\varepsilon_n\right)+\int_0^\pi M(\pi,t)\cos\left(\lambda_n^0+\varepsilon_n\right)tdt+\int_0^\pi N(\pi,t)\sin\left(\lambda_n^0+\varepsilon_n\right)tdt=0 \quad (34)$$

$$\begin{aligned}&\Delta_0\left(\lambda_n^0+\varepsilon_n\right)=A_1\cos\left[\left(\lambda_n^0+\varepsilon_n\right)(2a_1-x)+\left(\int_{a_2}^x p(t)\,dt-\int_0^{a_1}p(t)\,dt\right)\right]\\&+A_2\cos\left[\left(\lambda_n^0+\varepsilon_n\right)(2a_1-x)+\left(\int_{a_2}^x p(t)\,dt-\int_0^{a_2}p(t)\,dt\right)\right]\\&-A_1\cos\left[\left(\lambda_n^0+\varepsilon_n\right)(2a_1-2a_2+x)-\left(\int_{a_2}^x p(t)\,dt+\int_0^{a_1}p(t)\,dt\right)\right]\\&+A_3\cos\left[\left(\lambda_n^0+\varepsilon_n\right)(2a_1-2a_2+x)-\int_0^x p(t)\,dt\right]=0\end{aligned} \quad (35)$$

$$\Delta_0\left(\lambda_n^0+\varepsilon_n\right)=\left[\dot{\underset{0}{\Delta}}\left(\lambda_n^0\right)+o(1)\right]\varepsilon_n\ ,\ n\to\infty$$

(35) takes the form of

$$\begin{aligned}&\left[\dot{\Delta}_0\left(\lambda_n^0\right)+o(1)\right]\varepsilon_n+\int_0^{2a_1-\pi-0}M(\pi,t)\cos\left(\lambda_n^0+\varepsilon_n\right)tdt\\&+\int_{2a_1-\pi+0}^{2a_1-2a_2+\pi-0}M(\pi,t)\cos\left(\lambda_n^0+\varepsilon_n\right)tdt\\&+\int_{2a_1-2a_2+\pi+0}^{\pi}M(\pi,t)\cos\left(\lambda_n^0+\varepsilon_n\right)tdt+\int_0^{2a_1-\pi-0}N(\pi,t)\sin\left(\lambda_n^0+\varepsilon_n\right)tdt\\&+\int_{2a_1-\pi+0}^{2a_1-2a_2+\pi-0}N(\pi,t)\sin\left(\lambda_n^0+\varepsilon_n\right)tdt+\int_{2a_1-2a_2+\pi+0}^{\pi}N(\pi,t)\sin\left(\lambda_n^0+\varepsilon_n\right)tdt=0\end{aligned}$$

İt is easy to see that the function $\Delta_0(\lambda)=0$ is type of [17] , so there is a $\eta_\delta>0$, such that $\left|\dot{\Delta}_0\left(\lambda_n^0\right)\right|\geq\eta_\delta>0$ is satisfied for all n. We also have

$$\lambda_n^0=\frac{n\pi}{2a_1-2a_2+\pi}+\psi_1(n) \quad (36)$$

where $\sup_n|\psi_1(n)|<M$ for some constant $M>0$ [18]. In addition to, when we replace (36) into (35) and calculate certain transformations, we reach $\varepsilon_n\in l_2$. We can obtain exactly

$$\begin{aligned}&\varepsilon_n=\\&-\frac{1}{\left(\dot{\Delta}_0(\lambda_n^0)+o(1)\right)(\lambda_n^0+\varepsilon_n)}\left[-\left(M(\pi,2a_1-\pi+0)-M(\pi,2a_1-\pi-0)\right)\sin\left(\lambda_n^0+\varepsilon_n\right)(2a_1-\pi)\right.\\&-\left(M(\pi,2a_1-2a_2+\pi+0)-M(\pi,2a_1-2a_2+\pi-0)\right)\sin\left(\lambda_n^0+\varepsilon_n\right)(2a_1-2a_2+\pi)\\&+M(\pi,\pi)\sin\left(\lambda_n^0+\varepsilon_n\right)-\int_0^\pi M_t(\pi,t)\sin\left(\lambda_n^0+\varepsilon_n\right)tdt\\&+\left(N(\pi,2a_1-\pi+0)-N(\pi,2a_1-\pi-0)\right)\cos\left(\lambda_n^0+\varepsilon_n\right)(2a_1-\pi)\\&+\left(N(\pi,2a_1-2a_2+\pi+0)-N(\pi,2a_1-2a_2+\pi-0)\right)\cos\left(\lambda_n^0+\varepsilon_n\right)(2a_1-2a_2+\pi)+N(\pi,0)\\&\left.-N(\pi,\pi)\cos\left(\lambda_n^0+\varepsilon_n\right)\pi+\int_0^\pi N_t(\pi,t)\cos\left(\lambda_n^0+\varepsilon_n\right)tdt\right]\end{aligned}$$

Since $\left(\int_0^\pi M_t(\pi,t)\sin\left(\lambda_n^0+\varepsilon_n\right)tdt\right)\in l_2$ and $\left(\int_0^\pi N_t(\pi,t)\cos\left(\lambda_n^0+\varepsilon_n\right)tdt\right)\in l_2$, we have

$$\begin{aligned}&\varepsilon_n=\tfrac{1}{2\lambda_n^0\Delta_0(\lambda_n^0)}\left\{\left[-A_5\sin\left[\beta_4(\pi)+\lambda_n^0(2a_1-\pi)\right]+A_1\sin\left[\beta_3(\pi)-\lambda_n^0(2a_1-2a_2+\pi)\right]\right.\right.\\&\left.+A_3\sin\left[\beta_4(\pi)+\lambda_n^0(2a_1-2a_2+\pi)\right]-A_4\sin\left[\beta_4(\pi)-\lambda_n^0\pi\right]\right]\int_0^\pi\left(q(t)+p^2(t)\right)dt\\&+\left[A_5\cos\left[\beta_4(\pi)+\lambda_n^0(2a_1-\pi)\right]-A_1\cos\left[\beta_3(\pi)-\lambda_n^0(2a_1-2a_2+\pi)\right]\right.\\&\left.+A_3\cos\left[\beta_4(\pi)-\lambda_n^0(2a_1-2a_2+\pi)\right]-A_4\cos\left[\beta_4(\pi)+\lambda_n^0\pi\right]\right]\left[p(\pi)-p(0)\right]+\tfrac{t_n}{\lambda_n^0}\end{aligned}$$

where $t_n\in l_2$. So we obtain

$$\lambda_n=\lambda_n^0+\frac{s_n}{\lambda_n^0}+\frac{t_n}{\lambda_n^0}$$

where

$$\begin{array}{l} s_n = \frac{1}{2\Delta_0(\lambda_n^0)} \left\{ \left[-A_5 \sin\left[\beta_4(\pi) + \lambda_n^0 (2a_1 - \pi)\right] + A_1 \sin\left[\beta_3(\pi) - \lambda_n^0 (2a_1 - 2a_2 + \pi)\right] \right.\right. \\ +A_3 \sin\left[\beta_4(\pi) + \lambda_n^0 (2a_1 - 2a_2 + \pi)\right] - A_4 \sin\left[\beta_4(\pi) - \lambda_n^0 \pi\right]\Big] \int_0^\pi \left(q(t) + p^2(t)\right) dt \\ + \left[A_5 \cos\left[\beta_4(\pi) + \lambda_n^0 (2a_1 - \pi)\right] - A_1 \cos\left[\beta_3(\pi) - \lambda_n^0 (2a_1 - 2a_2 + \pi)\right] \right. \\ +A_3 \cos\left[\beta_4(\pi) - \lambda_n^0 (2a_1 - 2a_2 + \pi)\right] - A_4 \cos\left[\beta_4(\pi) + \lambda_n^0 \pi\right]\Big] \left[p(\pi) - p(0)\right] \end{array}$$

s_n is a bounded sequence. This completes the proof. □

4. Inverse problem

Let's show problem $(1)-(6)$ as $L(\alpha, a_1, a_2)$.

$$\tilde{L}(\alpha, a_1, a_2) := \begin{cases} -y'' + [2\lambda\tilde{p}(x) + \tilde{q}(x)]\, y = \lambda^2 y, \;\; x \in (0,\pi) \\ U(y) = y'(0) = 0, V(y) = y(\pi) = 0 \\ y(\tilde{a}_1 + 0) = \tilde{\alpha}_1 y(\tilde{a}_1 - 0) \\ y'(\tilde{a}_1 + 0) = \tilde{\beta}_1 y'(\tilde{a}_1 - 0) + i\lambda\tilde{\gamma}_1 y(\tilde{a}_1 - 0) \\ y(\tilde{a}_2 + 0) = \tilde{\alpha}_2 y(\tilde{a}_2 - 0) \\ y'(\tilde{a}_2 + 0) = \tilde{\beta}_2 y'(\tilde{a}_2 - 0) + i\lambda\tilde{\gamma}_2 y(\tilde{a}_2 - 0) \end{cases}$$

we consider the boundary value problem $\tilde{L}(\alpha, a_1, a_2)$. $\tilde{L}(\alpha, a_1, a_2)$ has the same form $L(\alpha, a_1, a_2)$, but only its coefficients $(\tilde{q}, \tilde{p}, \tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{a}_1, \tilde{a}_2, \tilde{\gamma}_1, \tilde{\gamma}_2)$ different. We suppose to the coefficient $(\tilde{q}, \tilde{p}, \tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{a}_1, \tilde{a}_2, \tilde{\gamma}_1, \tilde{\gamma}_2)$ provide conditions of problem $(1)-(6)$. Let $\Phi(x,\lambda)$ is a solution of equation (1) with the conditions $U(\Phi) = 1, V(\Phi) = 0$ and jump contitions $(3)-(6)$. We define $M(\lambda) := \Phi(0,\lambda)$.
The functions $\Phi(x,\lambda)$and $M(\lambda)$ are called the Weyl solution and Weyl function for the boundary value problem $(1)-(6)$.
Using the solution $\phi(x,\lambda)$ defined in the previous sections one has

$$\Phi(x,\lambda) := -\frac{\psi(x,\lambda)}{\Delta(\lambda)} = S(x,\lambda) + M(\lambda)\cdot\phi(x,\lambda)\,, \;\; M(\lambda) = -\frac{\psi(0,\lambda)}{\Delta(\lambda)} \tag{37}$$

where $\psi(x,\lambda)$ is a solution of (1) with the conditon $\psi(\pi,\lambda) = 0, \psi'(\pi,\lambda) = -1$ and $(3)-(6)$. $S(x,\lambda)$ is defined from the equality

$$\psi(x,\lambda) = \psi(0,\lambda)\cdot\phi(x,\lambda) - \Delta(\lambda)\cdot S(x,\lambda) \tag{38}$$

then,

$$\langle \Phi(x,\lambda), \phi(x,\lambda) \rangle \equiv 1$$

$\langle \Phi(x,\lambda), \phi(x,\lambda) \rangle \equiv -\Delta(\lambda)$ for $x \neq a_1, a_2$.
The following theorem states the significance of the Weyl function.

Theorem 4.1. *If $M(\lambda) = \widetilde{M}(\lambda)$, then $L(\alpha, a_1, a_2) = \tilde{L}(\alpha, a_1, a_2)$. Thus, the boundary value problem $L(\alpha, a_1, a_2)$is uniquely defined by the Weyl function.*

Proof. Let us define the matrix $P(x,\lambda) = [P_{j,k}(x,\lambda)], (j,k = 1,2)$

$$P(x,\lambda)\cdot\begin{pmatrix} \widetilde{\phi}(x,\lambda) & \widetilde{\Phi}(x,\lambda) \\ \widetilde{\phi}'(x,\lambda) & \widetilde{\Phi}'(x,\lambda) \end{pmatrix} = \begin{pmatrix} \phi(x,\lambda) & \Phi(x,\lambda) \\ \phi'(x,\lambda) & \Phi'(x,\lambda) \end{pmatrix}$$

So, we can write

$$\left.\begin{array}{l} P_{11}(x,\lambda) = -\phi(x,\lambda)\frac{\widetilde{\psi}'(x,\lambda)}{\widetilde{\Delta}(\lambda)} + \widetilde{\phi}'(x,\lambda)\frac{\psi(x,\lambda)}{\Delta(\lambda)} \\ P_{12}(x,\lambda) = -\widetilde{\phi}(x,\lambda)\frac{\psi(x,\lambda)}{\Delta(\lambda)} + \phi(x,\lambda)\frac{\widetilde{\psi}(x,\lambda)}{\widetilde{\Delta}(\lambda)} \\ P_{21}(x,\lambda) = -\phi'(x,\lambda)\frac{\widetilde{\psi}'(x,\lambda)}{\widetilde{\Delta}(\lambda)} - \widetilde{\phi}'(x,\lambda)\frac{\psi'(x,\lambda)}{\Delta(\lambda)} \\ P_{22}(x,\lambda) = -\widetilde{\phi}(x,\lambda)\frac{\psi'(x,\lambda)}{\Delta(\lambda)} + \phi'(x,\lambda)\frac{\widetilde{\psi}(x,\lambda)}{\widetilde{\Delta}(\lambda)} \end{array}\right\} \quad (39)$$

then we have

$$\begin{array}{l} P_{11}(x,\lambda) = \phi(x,\lambda)\left[\widetilde{S}'(x,\lambda) + \widetilde{M}(\lambda)\cdot\widetilde{\phi}'(x,\lambda)\right] \\ -\widetilde{\phi}'(x,\lambda)\left[S(x,\lambda) + M(\lambda)\cdot\phi(x,\lambda)\right] \\ = \phi(x,\lambda)\widetilde{S}'(x,\lambda) - \widetilde{\phi}'(x,\lambda)S(x,\lambda) + \left[\widetilde{M}(\lambda) - M(\lambda)\right]\phi(x,\lambda)\widetilde{\phi}'(x,\lambda) \end{array}$$

$$\begin{array}{l} P_{12}(x,\lambda) = \widetilde{\phi}(x,\lambda)\left[S(x,\lambda) + M(\lambda)\cdot\phi(x,\lambda)\right] - \phi(x,\lambda)\left[\widetilde{S}(x,\lambda) + \widetilde{M}(\lambda)\cdot\widetilde{\phi}(x,\lambda)\right] \\ = \widetilde{\phi}(x,\lambda)S(x,\lambda) - \phi(x,\lambda)\widetilde{S}(x,\lambda) + \left[M(\lambda) - \widetilde{M}(\lambda)\right]\phi(x,\lambda)\widetilde{\phi}(x,\lambda) \end{array}$$

$$\begin{array}{l} P_{21}(x,\lambda) = \phi'(x,\lambda)\left[\widetilde{S}'(x,\lambda) + \widetilde{M}(\lambda)\cdot\widetilde{\phi}'(x,\lambda)\right] \\ -\widetilde{\phi}'(x,\lambda)\left[S'(x,\lambda) + M(\lambda)\cdot\phi'(x,\lambda)\right] \\ = \phi'(x,\lambda)\widetilde{S}'(x,\lambda) - \widetilde{\phi}'(x,\lambda)S'(x,\lambda) + \left[\widetilde{M}(\lambda) - M(\lambda)\right]\phi'(x,\lambda)\widetilde{\phi}'(x,\lambda) \end{array}$$

$$\begin{array}{l} P_{22}(x,\lambda) = \widetilde{\phi}(x,\lambda)\left[S'(x,\lambda) + M(\lambda)\cdot\phi'(x,\lambda)\right] + \phi'(x,\lambda)\left[\widetilde{S}(x,\lambda) + \widetilde{M}(\lambda)\cdot\widetilde{\phi}(x,\lambda)\right] \\ = \phi'(x,\lambda)S'(x,\lambda) - \phi'(x,\lambda)\widetilde{S}(x,\lambda) + \left[M(\lambda) - \widetilde{M}(\lambda)\right]\phi'(x,\lambda)\widetilde{\phi}(x,\lambda). \end{array}$$

If $M(\lambda) \equiv \widetilde{M}(\lambda)$,then

$$P_{11}(x,\lambda) = \phi(x,\lambda)\widetilde{S}'(x,\lambda) - \widetilde{\phi}'(x,\lambda)S(x,\lambda)$$

$$P_{12}(x,\lambda) = \widetilde{\phi}(x,\lambda)S(x,\lambda) - \phi(x,\lambda)\widetilde{S}(x,\lambda)$$

$$P_{21}(x,\lambda) = \phi'(x,\lambda)\widetilde{S}'(x,\lambda) - \widetilde{\phi}'(x,\lambda)S'(x,\lambda)$$

$$P_{22}(x,\lambda) = \phi'(x,\lambda)S'(x,\lambda) - \phi'(x,\lambda)\widetilde{S}(x,\lambda).$$

If $M(\lambda) \equiv \widetilde{M}(\lambda)$, the functions $P_{j,k}(x,\lambda), (j,k = 1,2)$are entire for λ . On the other hand, the function $\phi(x,\lambda)$ has asymtotic expression due to $(9),(10),(11)$ and the function $\psi(x,\lambda)$ has asymtotic expression due to $(12),(13),(14)$

$$\begin{array}{rcl} \phi'(x,\lambda) & = & \left\{\begin{array}{ll} O\left(|\lambda|e^{|\sigma|x}\right) & ;\ 0<x<a_1 \\ O\left(|\lambda|e^{|\sigma|(2a_1-x)}\right) & ;\ a_1<x<a_2 \\ O\left(|\lambda|e^{|\sigma|(2a_1-2a_2+x)}\right) & ;\ a_2<x<\pi \end{array}\right. , \\ \psi'(x,\lambda) & = & \left\{\begin{array}{ll} O\left(|\lambda|e^{|\lambda|(\pi-x)}\right) & ;\ 0<x<a_1 \\ O\left(|\lambda|e^{|\lambda|\alpha_1(\pi-x)}\right) & ;\ a_1<x<a_2 \\ O\left(|\lambda|e^{|\lambda|\alpha_2(\pi-x)}\right) & ;\ a_2<x<\pi \end{array}\right. . \end{array}$$

We replace $(9),(10),(11)$ into (39) for $\forall x \in [0,\pi] / \{a_1, a_2, \widetilde{a}_1, \widetilde{a}_2\}$
$|P_{11}(x,\lambda)| \le c_\delta$ and $|P_{12}(x,\lambda)| \le C_\delta$

is held. c_δ , C_δ can be shown presence of constants. We calculate certain transformation of Liouville theorem, then
$P_{11}(x,\lambda) \equiv A(x)$ and $P_{12}(x,\lambda) \equiv 0$ are held. Now by using (39), we obtain

$$\begin{array}{l} \phi(x,\lambda) \cdot \Phi'(x,\lambda) - \widetilde{\phi}'(x,\lambda) \cdot \Phi(x,\lambda) = A(x) \\ \widetilde{\phi}(x,\lambda) \cdot \Phi(x,\lambda) - \phi(x,\lambda) \cdot \widetilde{\Phi}(x,\lambda) = 0 \end{array}$$

and

$$\left.\begin{array}{l} \phi(x,\lambda) = \widetilde{\phi}(x,\lambda) \cdot A(x) \\ \Phi(x,\lambda) = \widetilde{\Phi}(x,\lambda) \cdot A(x) \end{array}\right\} \tag{40}$$

therefore with (37) and (38)

$$\begin{array}{l} W[\phi,\Phi] = W\left[\phi(x,\lambda), -\frac{\psi(x,\lambda)}{\Delta(\lambda)}\right] \\ = \frac{1}{\Delta(\lambda)} W[\phi(x,\lambda), -\psi(0,\lambda)\phi(x,\lambda) + \Delta(\lambda) S(x,\lambda)] \\ = -\frac{\psi(0,\lambda)}{\Delta(\lambda)} W[\phi(x,\lambda), \phi(x,\lambda)] + W[\phi(x,\lambda), S(x,\lambda)] = 1 \end{array}$$

and similarly $W\left[\widetilde{\phi},\widetilde{\Phi}\right] = 1$. According to (40)$1 = W[\phi(x,\lambda), \Phi(x,\lambda)] = W\left[A(x)\widetilde{\phi}(x,\lambda), A(x)\widetilde{\Phi}(x,\lambda)\right]$

$$= A^2(x) W\left[\widetilde{\phi}(x,\lambda), \widetilde{\Phi}(x,\lambda)\right] = A^2(x)$$

Function$\phi(x,\lambda)$ has the following the representation in $D = \{\lambda : \arg\lambda \in [\varepsilon, \pi-\varepsilon]\}$
for $|\lambda| \to \infty$,
if $a_1 < x < a_2$;

$$\varphi(x,\lambda) = \frac{k}{2}\exp(-i(\lambda x - \beta(x)))\left(1 + O\left(\frac{1}{\lambda}\right)\right) \tag{41}$$

$k = 1$ for $x < a_1$, $k = A_4$ for $x > a_1$.
If $a_2 < x \le \pi$;

$$\varphi(x,\lambda) = \frac{k}{2}\exp(-i(\lambda(2a_1 - 2a_2 + \pi) - (\beta_1(x) + \beta_2(x))))\left(1 + O\left(\frac{1}{\lambda}\right)\right) \tag{42}$$

$k = A_1$. Because of $(40), (41), (42)$;$a_1 = \tilde{a}_1$,$a_2 = \tilde{a}_2$ are true. In addition to $A(x) = 1$,$A_1 = \tilde{A}_1$ ve $A_4 = \tilde{A}_4$ are true. Because of (42), $\phi(x,\lambda) \equiv \tilde{\phi}(x,\lambda)$, $\Phi(x,\lambda) \equiv \tilde{\Phi}(x,\lambda)$ is held.
Furthermore , when we replace $\varphi(x,\lambda) \equiv \tilde{\varphi}(x,\lambda)$ and operation are calculated, we reach$p(x) = \tilde{p}(x)$, $q(x) = \tilde{q}(x)$. Thus , for each λ,

$$\begin{array}{l} (\alpha_1 - \tilde{\alpha}_1)\varphi(a_1 - 0,\lambda) = 0 \\ \left(\beta_1 - \tilde{\beta}_1\right)\varphi'(a_1 - 0,\lambda) + (\gamma_1 - \tilde{\gamma}_1)\varphi(a_1 - 0,\lambda) = 0 \end{array}$$

$$\begin{array}{l} (\alpha_2 - \tilde{\alpha}_2)\varphi(a_2 - 0,\lambda) = 0 \\ \left(\beta_2 - \tilde{\beta}_2\right)\varphi'(a_2 - 0,\lambda) + (\gamma_2 - \tilde{\gamma}_2)\varphi(a_2 - 0,\lambda) = 0 \end{array}$$

$\alpha_1 = \tilde{\alpha}_1$, $\alpha_2 = \tilde{\alpha}_2$, $\gamma_1 = \tilde{\gamma}_1$ ve $\gamma_2 = \tilde{\gamma}_2$ are held. Consequently $L(\alpha, a_1, a_2) = \tilde{L}(\alpha, a_1, a_2)$.
The theorem is proved. □

Theorem 4.2. *If* $\mu_n = \widetilde{\mu}_n$, $\lambda_n = \widetilde{\lambda}_n$, ; $n = 0, 1, 2, ...,$ *then* $L(\alpha, a_1, a_2) = \tilde{L}(\alpha, a_1, a_2)$.

Proof. Obviously, the characteristic function $\Delta(\lambda)$ and $\psi(0,\lambda)$ are uniquely identified by the sequences $\{\lambda_n^2\}$ and $\{\mu_n^2\}$. If $\lambda_n = \widetilde{\lambda}_n$ and $\mu_n = \widetilde{\mu}_n$,$(n = 0, 1, 2, ...)$, then $\Delta(\lambda) \equiv \widetilde{\Delta}(\lambda)$ and $\psi(0,\lambda) \equiv \widetilde{\psi}(0,\lambda)$. İt follows from (38) that $M(\lambda) \equiv \widetilde{M}(\lambda)$. Therefore, applying Theorem 10, we conclude that $L(\alpha, a_1, a_2) = \tilde{L}(\alpha, a_1, a_2)$. The theorem is proved. □

Theorem 4.3. *If* $\alpha_n = \widetilde{\alpha}_n\,, \lambda_n = \widetilde{\lambda}_n\,, ; n = 0, 1, 2, ...,$ *then* $L(\alpha, a_1, a_2) = \tilde{L}(\alpha, a_1, a_2)$. *So, spectral data* $\{\lambda_n, \alpha_n\}$ *uniquely determines the problem* $L(\alpha, a_1, a_2)$.

Proof. The Weyl function $M(\lambda)$ is meromorphic with simple poles at points λ_n^2. The expression

$$\Delta(\lambda) = \Delta_0(\lambda) + \int_0^\pi A(\pi, \lambda) \cos \lambda t dt + \int_0^\pi B(\pi, \lambda) \sin \lambda t dt$$

and equalities $2\lambda_n \beta_n \alpha_n = -\dot{\Delta}(\lambda_n)$, $\psi(x, \lambda_n) = \beta_n \varphi(x, \lambda_n)$, we have

$$\underset{\lambda=\lambda_n}{Res} M(\lambda) = -\frac{\psi(0, \lambda_n)}{\dot{\Delta}(\lambda_n)} = -\frac{\beta_n}{\dot{\Delta}(\lambda_n)} = \frac{1}{2\lambda_n \alpha_n}.$$

because the $M(\lambda)$is regular for $\lambda \in \Gamma_n$ by Rouche's theorem. So,

$$M(\lambda) = \frac{1}{2\pi i} \int_{\Gamma_{n1}} \frac{M(\mu)}{\mu - \lambda} d\mu\ ,\ \lambda \in \Gamma_n.$$

It follows from (38) that $|M(\lambda)| \leq C_\delta |\lambda|^{-1}$, $\lambda \in C_\delta$. Where $\Gamma_{n1} = \left\{\lambda : |\lambda| = \left|\lambda_n^0\right|, n = 1, 2, ...\right\}$. Because of residue theorem,
$M(\lambda) = \sum_{-\infty}^{\infty} \frac{1}{2\lambda_n \alpha_n (\lambda - \lambda_n)}$ is held. At last, from equality$M(\lambda) \equiv \widetilde{M}(\lambda)$ by Theorem 10, we conclude that $L(\alpha, a_1, a_2) = \tilde{L}(\alpha, a_1, a_2)$. The theorem is proved. □

5. Conclusions

In this study diffusion operator with discontinuity points is considered. Firstly , important definitions and theorems which are used frequently in spectral theory of differential operators are given. Integral equations for solition which satisfy certain initial and jump condition of given equation has been obtained and useful integral represantation some important properties of eigenvalues and eigenfunction have been investigated. Finally, consist of the inverse problem. It has been proven that the coefficients of the given problem are uniquely determined by the Weyl function and spectral data.

References

[1] Marchenko, V. A., (1986), Sturm-Liouville Operators and Applications, AMS Chelsea Publishing, Basel.

[2] Levitan, B. M. and Gasymov, M. G., (1964), Determination of a differential equation by two spectra. Uspekhi Mathematicheskikh Nauk, 19, 2, 3-63.

[3] Jaulent, M . and Jean, C., (1976), The Inverse Problem for the one-dimensional Schrödinger equation with an energy-dependent potential. Annales de L'Institut Henri Poincare A, 25, 2, 105-137.

[4] Yurko, V. A., (2000), Introduction to the Theory of Inverse Spectral Problems, Fizmatlit, MOscow, Russia, 2007, English translation, Inverse Spectral Problems dor Differantial Operators and Their Applications, Gordon and Breach, Amsterdam, The Netherlands.

[5] Levitan, B. M., (1978), Inverse Sturm-Liouville Problems, Nauka, Moscow, Russia.

[6] Gasymov M. G. and Guseinov, G. S., (1981), Determination of a diffusion operator from spectral data, Akademiya Nauk Azerbaidzhanskoi SSR. Doklady, 37, 2, 19-23.

[7] Yurko, V. A., (2000), An inverse problem for differential operator pencils, Methematicheskikh Sbornik, 191, 10, 137-160.

[8] Nabiev, M. I., (2004), The inverse spectral problem for he diffusion operator on an interval. Mathematicheskaya Fizika, Analiz, Geometriya, 11, 3, 302-313,

[9] Guseinov, G. S., (1986), Inverse spectral problems for a quadratic pencil of Sturm-Liouville operators on a finite interval. Spectral Theory of Operators and Its Applications, 51-101, Elm, Baku, Azerbaijan.

[10] Freiling G. and Yurko V., (2001), Inverse Sturm-Liouville problems and their applications, Nova Science, Huntington, NY, USA.

[11] Yurko, V. A., (2000), Inverse spectral problems for differential operators and their applications, vol. 2 of Analytical Methods and Special Functions, Gordon and Breach, Amsterdam, The Netherlands.

[12] Hald, O. H., (1984), Discontinuous inverse eigenvalue problems, Communications on Pure and Applied Mathematics, 37, 5, 539–577.
[13] Amirov, R. Kh., (2006), On Sturm-Liouville operators with discontinuity conditions inside an interval.Journal of Mathematical Analysis and Applications, 317, 1, 163-176.
[14] AmirovR. Kh. and Yurko, V. A., (2001), On differential operators with a singularity and discontinuity conditions inside an interval.Ukrainian Mathematical Journal, 53,11,1443–1457.
[15] FreilingG.and Yurko, V., (2002), Inverse spectral problems for singular non-selfadjoint differential operators with discontinuities in an interior point. Inverse Problems, 18, 3, 757-773.
[16] Levin, B. Ya., (1971), Entire Functions, MGV, Moscow, Russia.
[17] Jdanovich, B. F. (1960), Formulae for the zeros of Drichlet polynomials and quasi-polynomials. Doklady Akademii Nauk SSSR, 135, 8, 1046-1049.
[18] Kreyn M. and Levin, B. Ya., (1949), On entire almost periodic functions of exponential type, Doklady Akademii Nauk SSSR, 64, 285-287.
[19] Buterin S. A., Yurko V.A., (2006), Inverse spectral problem for pencils of differential operators on a finite interval. Vesn. Bashkir. Univ, 4, 8-12.
[20] Koyunbakan H., Panakhov E. S., (2007), Half-inverse problem for diffusion operators on the finite interval.Journal of Mathematical Analysis and Applications, 326, 1024-1030.
[21] Yang C. F.,(2010), Reconstruction of the diffusion operator from nodal data Zeitschrift fur Naturforschung, 65, 100-106.
[22] Amirov R. Kh., Ergun A., (2015), İki noktada süreksizlik kosullarına sahip difuzyon denkleminin cozumleri icin integral gosterilim, Cumhuriyet Universty Science Journal, doi:10.17776/csj.06773.

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