TWMS J. App. and Eng. Math. V.9, No.1, Special Issue, 2019, pp. 38-48

ANALYSIS OF TWO-DIMENSIONAL NON-LINEAR BURGERS' EQUATIONS

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Abstract. In this paper,we prove the existence, uniqueness and continuously of solution of two-dimensional Burgers' equations by iteration method.

Keywords: Two Dimensional Parabolic Equation, Periodic Boundary Condition, Finite Difference.

AMS Subject Classification: 65Bxx, 65Gxx, 65Mxx

1. Introduction

The nonlinear coupled Burgers' equations are a special form of incompressible Navier-Stokes equation without having pressure term and continuity equation. Burgers' equations are an important partial differential equation from fluid dynamics, and are widely used to describe various natural phenomena such as mathematically shown that the turbulence and modelling of gas dynamics, shock waves, etc. Due to its wide range of applicability some researchers have been interested in studying its solution using various numerical methods. For a survey of these methods one refers to [2] and references cited there in [5-16]. Construction of new solutions by superposition of known ones is a familiar tool in nonlinear partial differential equations. The idea of superpositions for nonlinear differential operators originated in 1893 by Vessiot [6]. Key references can be found in [6, 7, 8, 9, 10]. In this study, we use the superposition principle for nonlinear Burgers' equations.

Consider two-dimensional coupled nonlinear Burgers' equations taken from [2],

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{R}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{1}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = \frac{1}{R}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{2}$$

two-dimensional non linear Burgers' equations is given by the initial conditions:

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§ Selected papers of International Conference on Life and Engineering Sciences (ICOLES 2018), Kyrenia, Cyprus, 2-6 September, 2018.

$$\begin{aligned} u(x,y,0) &= f(x,y), x,y \in D \\ v(x,y,0) &= g(x,y), x,y \in D \end{aligned} \tag{3}$$

and boundary conditions

$$\begin{aligned} u(x,y,t) &= f_1(x,y,t), x,y \in \partial D, t>0 \\ v(x,y,t) &= g_1(x,y,t), x,y \in \partial D, t>0 \end{aligned} \tag{4}$$

Here

$$D = \{(x,y) : a \le x \le b, a \le y \le b\}$$

∂D denotes the boundary of D, $u(x,y,t)$ and $v(x,y,t)$ are the velocity components to be determinant f, g and $f_{1,}g_1$ are known functions and R is Reynolds number.

In this paper, an iterative method is presented to find numerical solutions of two dimensional non linear Burgers' equations.To solve two dimensional non linear Burgers' equations, we use the superposition principle for nonlinear partial equations [6, 7, 8, 9, 10]. Computed results are compared with analytical and other numerical results using various values of the Reynolds number R.

D denotes the domain

$D := \{(x,y,t) : 0 < x < \pi,\ 0 < y < \pi, 0 < t < T\}$

Partial differential equation (1) can be solved by splitting it into two one dimensional equation from [6] rather than discretising the complete two-dimensional Burgers' equation to give an aproximating equation based on two-dimensional computational molecule, as seen [11]. Let consider

$$\begin{aligned} u_x &\approx f(x,t,u) \\ u_y &\approx f(y,t,u). \end{aligned}$$

We can write equation (1) as equations (5) and (6)

$$\frac{1}{2}u_t - \frac{1}{R}u_{xx} = uf(x,t,u), (x,t)\epsilon D \tag{5}$$

$$\frac{1}{2}u_t - \frac{1}{R}u_{yy} = vf(y,t,u), (y,t)\epsilon D \tag{6}$$

Applying the same estimations for equation (2) we can write as equations (7) and (8)

$$\frac{1}{2}v_t - \frac{1}{R}v_{xx} = ug(x,t,v), (x,t)\epsilon D \tag{7}$$

$$\frac{1}{2}v_t - \frac{1}{R}v_{yy} = vg(y,t,v), (y,t)\epsilon D \tag{8}$$

with the initial condition and the periodic boundary condition

$$\begin{aligned} u(x,0) &= \varphi(x), u(y,0) = \varphi(y)\ x,y \in [0,\pi] \\ u(0,t) &= u(\pi,t), u_x(0,t) = u_x(\pi,t), u_y(0,t) = u_y(\pi,t) \\ v(x,0) &= \varphi(x), v(y,0) = \varphi(y)\ x,y \in [0,\pi], \\ v(0,t) &= v(\pi,t), v_x(0,t) = v_x(\pi,t), v_y(0,t) = v_y(\pi,t) \end{aligned} \tag{9}$$

The functions $\varphi(x), \varphi(y)$ and $f(x,t,u)$,$f(y,t,u), g(x,t,u)$ and $g(y,t,u)$ are given functions on $[0,\pi]$ and $\partial D \times (-\infty,\infty)$, respectively.

The problem of finding the pair $\{u(x,t), v(x,t)\}$are solutions. In technical applications, the following boundary conditions $u(0,t) = u(\pi,t), u_x(0,t) = u_x(\pi,t)$ are encountered very often [1].

Definition 1.1. *The pair $\{u(x,t), v(x,t)\}$ from the class $(C^{2,1}(D) \cap C^{1,0}(\partial D))$ for which conditions (5)-(9) are satisfied is called the classical solution of system (5)-(9).*

2. Existence and Uniqueness of the Solution of the Problem

The main result on the existence and the uniqueness of the solution of the problem (5)-(9) is presented as follows:

We have the following assumptions on the data of the problem (5)-(9).

(A1) $\varphi(x), \varphi(y) \in C[0,\pi]$,

$\varphi(0) = \varphi(\pi), \varphi^{'}(0) = \varphi^{'}(\pi)$,

(A2) Let the function $f(x,t,u)$ and $f(y,t,u)$ is continuous with respect to all arguments in $\partial D \times (-\infty,\infty)$ and satisfies the following condition

(1)

$$\begin{aligned} |f(x,t,u) - f(x,t,\tilde{u})| &\leq b(x,t)\,|u-\tilde{u}| \\ |g(x,t,u) - g(x,t,\tilde{u})| &\leq b(x,t)\,|u-\tilde{u}|\,, \end{aligned} \tag{10}$$

where $b(x,t) \in L_2(D),\ b(x,t) \geq 0$,

$$\begin{aligned} |f(y,t,u) - f(y,t,\tilde{u})| &\leq b(y,t)\,|u-\tilde{u}| \\ |g(y,t,u) - g(y,t,\tilde{u})| &\leq b(y,t)\,|u-\tilde{u}| \end{aligned} \tag{11}$$

where $b(y,t) \in L_2(D),\ b(y,t) \geq 0$,

(2) $f(x,t,u), g(x,t,u) \in C[0,\pi]$, $t\epsilon[0,T]$,

(3) $f(x,t,u)|_{x=0} = f(x,t,u)|_{x=\pi}\,,\ f_x\,(0,t,u)|_{x=0} = f_x(\pi,t,u)|_{x=\pi}\,.$

$g(x,t,u)|_{x=0} = g(x,t,u)|_{x=\pi}\,,\ g_x\,(0,t,u)|_{x=0} = g_x(\pi,t,u)|_{x=\pi}\,.$

(4) $f(y,t,u), g(y,t,u) \in C[0,\pi]$, $t\epsilon[0,T]$,

(5) $f(y,t,u)|_{x=0} = f(y,t,u)|_{x=\pi}\,,\ f_x\,(0,t,u)|_{x=0} = f_x(\pi,t,u)|_{x=\pi}\,.$

$g(y,t,u)|_{x=0} = g(y,t,u)|_{x=\pi}\,,\ g_x\,(0,t,u)|_{x=0} = g_x(\pi,t,u)|_{x=\pi}\,.$

(Same conditions for v is considered)

By applying the standard procedure of the Fourier method, we obtain the following representation for the solution of (5)-(6) and (7)-(8)

$$\begin{aligned} u(x,t) &= \frac{u_0(t)}{2} + \sum_{k=1}^{\infty}\left[u_{ck}(t)\cos 2kx + u_{sk}(t)\sin 2kx\right], \\ v(x,t) &= \frac{v_0(t)}{2} + \sum_{k=1}^{\infty}\left[v_{ck}(t)\cos 2kx + v_{sk}(t)\sin 2kx\right], \end{aligned} \tag{12}$$

$$\begin{aligned} u(y,t) &= \frac{u_0(t)}{2} + \sum_{k=1}^{\infty}\left[u_{ck}(t)\cos 2ky + u_{sk}(t)\sin 2ky\right], \\ v(y,t) &= \frac{v_0(t)}{2} + \sum_{k=1}^{\infty}\left[v_{ck}(t)\cos 2ky + v_{sk}(t)\sin 2ky\right], \end{aligned} \tag{13}$$

For equation (5) Fourier coefficient is:

$$
\begin{aligned}
u_0(t) &= \varphi_0 + \frac{2}{\pi}\int_0^t\int_0^\pi u(\xi,\tau)\, f(\xi,\tau,u(\xi,\tau))\, d\xi d\tau, \qquad (14)\\
u_{ck}(t) &= \varphi_{ck}\, e^{\frac{-(2k)^2}{R}t} + \frac{2}{\pi}\int_0^t\int_0^\pi u(\xi,\tau)\, f(\xi,\tau,u(\xi,\tau)) \cos 2k\xi\; e^{\frac{-(2k)^2}{R}(t-\tau)} d\xi d\tau,\\
u_{sk}(t) &= \varphi_{sk}\, e^{\frac{-(2k)^2}{R}t} + \frac{2}{\pi}\int_0^t\int_0^\pi u(\xi,\tau)\, f(\xi,\tau,u(\xi,\tau)) \sin 2k\xi\; e^{\frac{-(2k)^2}{R}(t-\tau)} d\xi d\tau.
\end{aligned}
$$

where $\varphi_0 = \frac{2}{\pi}\int_0^\pi \varphi(x)dx, \varphi_{ck} = \frac{2}{\pi}\int_0^\pi \varphi(x)\cos 2kxdx, \varphi_{sk} = \frac{2}{\pi}\int_0^\pi \varphi(x)\sin 2kxdx,\ u(\xi,\tau) = \frac{u_0(\tau)}{2} + \sum_{k=1}^{\infty}\left[u_{ck}(\tau)\cos 2k\xi + u_{sk}(\tau)\sin 2k\xi\right].$

$f_0(t,u) = \frac{2}{\pi}\int_0^\pi f(x,t,u)dx,\ f_{ck}(t,u) = \frac{2}{\pi}\int_0^\pi f(x,t,u)\cos 2kxdx,\ f_{sk}(t,u) = \frac{2}{\pi}\int_0^\pi f(x,t,u)\sin 2kxdx,$ $k = 1,2,3,\ldots$.

For (6) equation Fourier coefficient is:

$$
\begin{aligned}
v_0(t) &= \varphi_0 + \frac{2}{\pi}\int_0^t\int_0^\pi v(\eta,\tau)\, f(\eta,\tau,u(\eta,\tau))\, d\eta d\tau, \qquad (15)\\
v_{ck}(t) &= \varphi_{ck}\, e^{\frac{-(2k)^2}{R}t} + \frac{2}{\pi}\int_0^t\int_0^\pi v(\eta,\tau)\, f(\eta,\tau,u(\eta,\tau)) \cos 2k\eta\; e^{\frac{-(2k)^2}{R}(t-\tau)} d\eta d\tau,\\
v_{sk}(t) &= \varphi_{sk}\, e^{\frac{-(2k)^2}{R}t} + \frac{2}{\pi}\int_0^t\int_0^\pi v(\eta,\tau)\, f(\eta,\tau,u(\eta,\tau)) \sin 2k\eta\; e^{\frac{-(2k)^2}{R}(t-\tau)} d\eta d\tau.
\end{aligned}
$$

where $\varphi_0 = \frac{2}{\pi}\int_0^\pi \varphi(y)dy, \varphi_{ck} = \frac{2}{\pi}\int_0^\pi \varphi(y)\cos 2kydy, \varphi_{sk} = \frac{2}{\pi}\int_0^\pi \varphi(y)\sin 2kydy, u(\eta,\tau) = \frac{u_0(\tau)}{2} + \sum_{k=1}^{\infty}\left[u_{ck}(\tau)\cos 2k\eta + u_{sk}(\tau)\sin 2k\eta\right]$

$f_0(t,u) = \frac{2}{\pi}\int_0^\pi vf(y,t,u)dy,\ f_{ck}(t,u) = \frac{2}{\pi}\int_0^\pi vf(y,t,u)\cos 2kydy,\ f_{sk}(t,u) = \frac{2}{\pi}\int_0^\pi vf(y,t,u)\sin 2kydy,$ $k = 1,2,3,\ldots$.

Same estimation for Fourier coefficient to (7)-(8) equation.

Definition 2.1. *Denote the set*

$$\{u(t)\} = \{u_0(t), u_{ck}(t), u_{sk}(t), k = 1,\ldots,n\},$$

of continuous on $[0,T]$ *functions satisfying the condition*
$\max\limits_{0\le t\le T}\frac{|u_0(t)|}{2} + \sum\limits_{k=1}^{\infty}\left(\max\limits_{0\le t\le T}|u_{ck}(t)| + \max\limits_{0\le t\le T}|u_{sk}(t)|\right) < \infty,$ *by* $\mathbf{B}_1$. *Let*

$$\|u(t)\| = \max_{0\le t\le T}\frac{|u_0(t)|}{2} + \sum_{k=1}^{\infty}\left(\max_{0\le t\le T}|u_{ck}(t)| + \max_{0\le t\le T}|u_{sk}(t)|\right),$$

be the norm in $\mathbf{B}_1$.

It can be shown that $\mathbf{B}_1$ *are the Banach spaces.*

$$\{v(t)\} = \{v_0(t), v_{ck}(t), v_{sk}(t), k = 1, ..., n\},$$

of continuous on $[0,T]$ *functions satisfying the condition*
$\max_{0\le t\le T}\frac{|v_0(t)|}{2} + \sum_{k=1}^{\infty}\left(\max_{0\le t\le T}|v_{ck}(t)| + \max_{0\le t\le T}|v_{sk}(t)|\right) < \infty$, *by* $\mathbf{B}_2$. *Let*

$$\|v(t)\| = \max_{0\le t\le T}\frac{|v_0(t)|}{2} + \sum_{k=1}^{\infty}\left(\max_{0\le t\le T}|v_{ck}(t)| + \max_{0\le t\le T}|v_{sk}(t)|\right),$$

be the norm in $\mathbf{B}_2$.

It can be shown that $\mathbf{B}_2$ *are the Banach spaces.*

Theorem 2.1 (optional text)**.** *Let the assumptions (A1)-(A2) be satisfied.Then the problem (5)-(9) has a unique solution for small* T.

Proof.

$$u_0^{(N+1)}(t) = u_0^{(0)}(t) + \frac{2}{\pi}\int_0^t\int_0^\pi u^{(N)}(\xi,\tau)f\left(\xi,\tau,u^{(N)}(\xi,\tau)\right)d\xi d\tau, \quad (16)$$

$$u_{ck}^{(N+1)}(t) = u_{ck}^{(0)}(t) + \frac{2}{\pi}\int_0^t\int_0^\pi u^{(N)}(\xi,\tau)f(\xi,\tau,u^{(N)}(\xi,\tau))\cos 2k\xi\; e^{\frac{-(2k)^2}{R}(t-\tau)}d\xi d\tau,$$

$$u_{sk}^{(N+1)}(t) = u_{sk}^{(0)}(t) + \frac{2}{\pi}\int_0^t\int_0^\pi u^{(N)}(\xi,\tau)f(\xi,\tau,u^{(N)}(\xi,\tau))\sin 2k\xi\; e^{\frac{-(2k)^2}{R}(t-\tau)}d\xi d\tau,$$

$$u_0^{(0)}(t) = \varphi_0, u_{ck}^{(0)}(t) = \varphi_{ck}e^{\frac{-(2k)^2}{R}}t, u_{sk}^{(0)}(t) = \varphi_{sk}e^{\frac{-(2k)^2}{R}}t.$$

From the conditions of the theorem we have $u^{(0)}(t)\epsilon\mathbf{B}_1$, $t\epsilon[0,T]$.

Let us write $N = 0$ in (16).

$$u_0^{(1)}(t) = u_0^{(0)}(t) + \frac{2}{\pi}\int_0^t\int_0^\pi u^{(0)}(\xi,\tau)f\left(\xi,\tau,u^{(0)}(\xi,\tau)\right)d\xi d\tau.$$

Adding and subtracting $\int_0^t\int_0^\pi f(\xi,\tau,0)d\xi d\tau$ to the last equation, we obtain

$$u_0^{(1)}(t) = \varphi_0(t) + \frac{2}{\pi}\int_0^t\int_0^\pi [f(\xi,\tau,u^{(0)}(\xi,\tau)) - f(\xi,\tau,0)]u^{(0)}(\xi,\tau)d\xi d\tau + \frac{2}{\pi}\int_0^t\int_0^\pi u^{(0)}(\xi,\tau)f(\xi,\tau,0)d\xi d\tau.$$

Applying Cauchy inequality,

$$\begin{aligned}\left|u_0^{(1)}(t)\right| &\leq |\varphi_0| + \left(\int_0^t d\tau\right)^{\frac{1}{2}} \left(\int_0^t \left\{\frac{2}{\pi}\int_0^\pi [f(\xi,\tau,u^{(0)}(\xi,\tau)) - f(\xi,\tau,0)] \left|u^{(0)}(\xi,\tau)\right| d\xi\right\}^2 d\tau\right)^{\frac{1}{2}} \\ &+ \left(\int_0^t d\tau\right)^{\frac{1}{2}} \left(\int_0^t \left\{\frac{2}{\pi}\int_0^\pi |f(\xi,\tau,0)| \left|u^{(0)}(\xi,\tau)\right| d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}.\end{aligned}$$

Using Lipschitzs condition in the last equation

$$\begin{aligned}\left|u_0^{(1)}(t)\right| &\leq |\varphi_0| + \sqrt{t}\left(\int_0^t \left\{\frac{2}{\pi}\int_0^\pi b(\tau,\xi) \left|u^{(0)}(\xi,\tau)\right|^2 d\xi\right\}^2 d\tau\right)^{\frac{1}{2}} \\ &+\sqrt{t}\left(\int_0^t \left\{\frac{2}{\pi}\int_0^\pi |f(\xi,\tau,0)| \left|u^{(0)}(\xi,\tau)\right| d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}.\end{aligned}$$

Applying Cauchy inequality to the last equation on $[0,\pi]$,

$$\begin{aligned}\left|u_0^{(1)}(t)\right| &\leq |\varphi_0| + 2\frac{\sqrt{t}}{\sqrt{\pi}}\left(\int_0^t \left\{\int_0^\pi b(\tau,\xi) \left|u^{(0)}(\xi,\tau)\right|^2 d\xi\right\}^2 d\tau\right)^{\frac{1}{2}} \\ &+2\frac{\sqrt{t}}{\sqrt{\pi}}\left(\int_0^t \left\{\int_0^\pi |f(\xi,\tau,0)| \left|u^{(0)}(\xi,\tau)\right| d\xi\right\}^2 d\tau\right)^{\frac{1}{2}},\end{aligned}$$

Taking the maximum of both side of the last inequality we have :

$$\begin{aligned}\max_{0\leq t\leq T}\left|u_0^{(1)}(t)\right| &\leq |\varphi_0| + 2\frac{\sqrt{T}}{\sqrt{\pi}}\|b(x,t)\|_{L_2(D)}\left\|u^{(0)}(t)\right\|_{B_1}^2 \\ &+2\frac{\sqrt{T}}{\sqrt{\pi}}\|f(x,t,0)\|_{L_2(D)}.\end{aligned}$$

$$u_{ck}^{(1)}(t) = u_{ck}^{(0)}(t) + \frac{2}{\pi}\int_0^t\int_0^\pi f(\xi,\tau,u^{(0)}(\xi,\tau))\cos 2k\xi e^{\frac{-(2k)^2}{R}(t-\tau)}d\xi d\tau$$

Adding and subtracting $\int_0^t\int_0^\pi f(\xi,\tau,0)d\xi d\tau$ to the last equation,

$$\begin{aligned}u_{ck}^{(1)}(t) &= \varphi_{ck}e^{\frac{-(2k)^2}{R}t} + \frac{2}{\pi}\int_0^t\int_0^\pi [f(\xi,\tau,u^{(0)}(\xi,\tau)) - f(\xi,\tau,0)]\left|u^{(0)}(\xi,\tau)\right|\cos 2k\xi e^{\frac{-(2k)^2}{R}(t-\tau)}d\xi d\tau \\ &+\frac{2}{\pi}\int_0^t\int_0^\pi f(\xi,\tau,0)\cos 2k\xi e^{\frac{-(2k)^2}{R}(t-\tau)}d\xi d\tau.\end{aligned}$$

Applying Cauchy,Bessel,Hölder, Lipshitzs condition and taking maximum:

$$\sum_{k=1}^{\infty}\max_{0\le t\le T}\left|u_{ck}^{(1)}(t)\right| \le \sum_{k=1}^{\infty}|\varphi_{ck}|+\frac{\sqrt{2\pi R}}{4\sqrt{3}}\left\|b(x,t)\right\|_{L_2(D)}\left\|u^{(0)}(t)\right\|_{B_1}^{2}+\frac{\sqrt{2\pi R}}{4\sqrt{3}}\left\|u^{(0)}(t)\right\|_{B_1}\left\|f(x,t,0)\right\|_{L_2(D)}.$$

Applying the same estimations we obtain,

$$\sum_{k=1}^{\infty}\max_{0\le t\le T}\left|u_{sk}^{(1)}(t)\right| \le \sum_{k=1}^{\infty}|\varphi_{sk}|+\frac{\sqrt{2\pi R}}{4\sqrt{3}}\left\|b(x,t)\right\|_{L_2(D)}\left\|u^{(0)}(t)\right\|_{B_1}^{2}+\frac{\sqrt{2\pi R}}{4\sqrt{3}}\left\|u^{(0)}(t)\right\|_{B_1}\left\|f(x,t,0)\right\|_{L_2(D)}.$$

Finally we have the following inequality:

$$\begin{aligned}
\left\|u^{(1)}(t)\right\| &= \max_{0\le t\le T}\frac{\left|u_0^{(1)}(t)\right|}{2}+\sum_{k=1}^{\infty}\left(\max_{0\le t\le T}\left|u_{ck}^{(1)}(t)\right|+\max_{0\le t\le T}\left|u_{sk}^{(1)}(t)\right|\right)\\
&\le \frac{|\varphi_0|}{2}+\sum_{k=1}^{\infty}\left(|\varphi_{ck}|+|\varphi_{sk}|\right)\\
&\quad+\left(2\sqrt{\frac{T}{\pi}}+\frac{\sqrt{2\pi R}}{2\sqrt{3}}\right)\left\|b(x,t)\right\|_{L_2(D)}\left\|u^{(0)}(t)\right\|_{B_1}^{2}\\
&\quad+\left(2\sqrt{\frac{T}{\pi}}+\frac{\sqrt{2\pi R}}{2\sqrt{3}}\right)\left\|u^{(0)}(t)\right\|_{B_1}\left\|f(x,t,0)\right\|_{L_2(D)}.
\end{aligned}$$

Same estimation for equation (6)

$$\begin{aligned}
\left\|u^{(1)}(t)\right\| &= \max_{0\le t\le T}\frac{\left|u_0^{(1)}(t)\right|}{2}+\sum_{k=1}^{\infty}\left(\max_{0\le t\le T}\left|u_{ck}^{(1)}(t)\right|+\max_{0\le t\le T}\left|u_{sk}^{(1)}(t)\right|\right)\\
&\le \frac{|\varphi_0|}{2}+\sum_{k=1}^{\infty}\left(|\varphi_{ck}|+|\varphi_{sk}|\right)\\
&\quad+\left(2\sqrt{\frac{T}{\pi}}+\frac{\sqrt{2\pi R}}{2\sqrt{3}}\right)\left\|b(y,t)\right\|_{L_2(D)}\left\|u^{(0)}(t)\right\|_{B_1}\left\|v^{(0)}(t)\right\|_{B_2}\\
&\quad+\left(2\sqrt{\frac{T}{\pi}}+\frac{\sqrt{2\pi R}}{2\sqrt{3}}\right)\left\|f(y,t,0)\right\|_{L_2(D)}\left\|v^{(0)}(t)\right\|_{B_2}.
\end{aligned}$$

$$\begin{aligned}
\left\|u^{(1)}(t)\right\|_{\mathbf{B}_1} &= \left\|u^{(1)}(t)\right\|_1 + \left\|u^{(1)}(t)\right\|_2 \\
&\leq |\varphi_0| + 2\sum_{k=1}^{\infty}\left(|\varphi_{ck}| + |\varphi_{sk}|\right) \\
&+ \left(2\sqrt{\frac{T}{\pi}} + \frac{\sqrt{2\pi R}}{2\sqrt{3}}\right)\|b(x,t)\|_{L_2(D)}\left\|u^{(0)}(t)\right\|_{B_1}^2 \\
&+ \left(2\sqrt{\frac{T}{\pi}} + \frac{\sqrt{2\pi R}}{2\sqrt{3}}\right)\left\|u^{(0)}(t)\right\|_{B_1}\|f(x,t,0)\|_{L_2(D)} \\
&+ \left(2\sqrt{\frac{T}{\pi}} + \frac{\sqrt{2\pi R}}{2\sqrt{3}}\right)\|b(y,t)\|_{L_2(D)}\left\|u^{(0)}(t)\right\|_{B_1}\left\|v^{(0)}(t)\right\|_{B_2} \\
&+ \left(2\sqrt{\frac{T}{\pi}} + \frac{\sqrt{2\pi R}}{2\sqrt{3}}\right)\|f(y,t,0)\|_{L_2(D)}\left\|v^{(0)}(t)\right\|_{B_2}.
\end{aligned}$$

From the conditions of the theorem $u^{(1)}(t) \in \mathbf{B}_1$.
Same estimation for N,

$$\begin{aligned}
\left\|u^{(N+1)}(t)\right\|_{B_1} &= \max_{0\leq t\leq T}\left|u_0^{(N)}(t)\right| + \sum_{k=1}^{\infty}\left(\max_{0\leq t\leq T}\left|u_{ck}^{(N)}(t)\right| + \max_{0\leq t\leq T}\left|u_{sk}^{(N)}(t)\right|\right) \\
&\leq |\varphi_0| + 2\sum_{k=1}^{\infty}\left(|\varphi_{ck}| + |\varphi_{sk}|\right) \\
&+ \left(2\sqrt{\frac{T}{\pi}} + \frac{\sqrt{2\pi R}}{2\sqrt{3}}\right)\|b(x,t)\|_{L_2(D)}\left\|u^{(N)}(t)\right\|_{B_1}^2 \\
&+ \left(2\sqrt{\frac{T}{\pi}} + \frac{\sqrt{2\pi R}}{2\sqrt{3}}\right)\left\|u^{(N)}(t)\right\|_{B_1}^2\|f(x,t,0)\|_{L_2(D)} \\
&+ \left(2\sqrt{\frac{T}{\pi}} + \frac{\sqrt{2\pi R}}{2\sqrt{3}}\right)\|b(y,t)\|_{L_2(D)}\left\|u^{(N)}(t)\right\|_{B_1}\left\|v^{(N)}(t)\right\|_{B_2} \\
&+ \left(2\sqrt{\frac{T}{\pi}} + \frac{\sqrt{2\pi R}}{2\sqrt{3}}\right)\|f(y,t,0)\|_{L_2(D)}\left\|v^{(N)}(t)\right\|_{B_2}.
\end{aligned} \tag{17}$$

Since $u^{(N)}(t) \in \mathbf{B}_1$ and from the conditions of the theorem, we have $u^{(N+1)}(t) \in \mathbf{B}_1$,

$$\{u(t)\} = \{u_0(t), u_{ck}(t), u_{sk}(t),\ k = 1, 2, ...\} \in \mathbf{B}_1.$$

Same estimation for (7)-(8) equations

$$\begin{aligned}\left\|v^{(N+1)}(t)\right\|_{B_2} &= \max_{0\le t\le T}\left|v_0^{(N)}(t)\right| + \sum_{k=1}^{\infty}\left(\max_{0\le t\le T}\left|v_{ck}^{(N)}(t)\right| + \max_{0\le t\le T}\left|v_{sk}^{(N)}(t)\right|\right)\\ &\le |\varphi_0| + 2\sum_{k=1}^{\infty}\left(|\varphi_{ck}|+|\varphi_{sk}|\right)\\ &+\left(2\sqrt{\frac{T}{\pi}}+\frac{\sqrt{2\pi R}}{2\sqrt{3}}\right)\|b(y,t)\|_{L_2(D)}\left\|v^{(N)}(t)\right\|_{B_2}^2 \qquad (18)\\ &+\left(2\sqrt{\frac{T}{\pi}}+\frac{\sqrt{2\pi R}}{2\sqrt{3}}\right)\left\|v^{(N)}(t)\right\|_{B_2}\|g(y,t,0)\|_{L_2(D)}\\ &+\left(2\sqrt{\frac{T}{\pi}}+\frac{\sqrt{2\pi R}}{2\sqrt{3}}\right)\|b(x,t)\|_{L_2(D)}\left\|v^{(N)}(t)\right\|_{B_2}\left\|u^{(N)}(t)\right\|_{B_1}\\ &+\left(2\sqrt{\frac{T}{\pi}}+\frac{\sqrt{2\pi R}}{2\sqrt{3}}\right)\|g(x,t,0)\|_{L_2(D)}\left\|u^{(N)}(t)\right\|_{B_1}.\end{aligned}$$

$$\{v(t)\} = \{v_0(t), v_{ck}(t), v_{sk}(t),\ k=1,2,...\}\in \mathbf{B}_2.$$

Now we prove that the iterations $u^{(N+1)}(t\ )$ and $v^{(N+1)}(t\ )$ converge in $\mathbf{B}_1$ and $\mathbf{B}_2$, as $N\to\infty$.

Applying Cauchy inequality, Bessel inequality , Hölder inequality,the Lipshitzs condition ,taking maximum of both side of the $u^{(1)}(t)-u^{(0)}(t)$ for equation (5):

$$\begin{aligned}\left\|u^{(1)}(t)-u^{(0)}(t)\right\| &\le \left(2\sqrt{\frac{T}{\pi}}+\frac{\sqrt{2\pi R}}{2\sqrt{3}}\right)\|b(x,t)\|_{L_2(D)}\left\|u^{(0)}(t)\right\|_{\mathbf{B}_1}^2\\ &+\left(\frac{2\sqrt{3T}+\pi}{2\sqrt{3\pi}}\right)\left\|u^{(0)}(t)\right\|_{\mathbf{B}_1}\|f(x,t,0)\|_{L_2(D)}.\end{aligned}$$

$$A_T=\left(2\sqrt{\frac{T}{\pi}}+\frac{\sqrt{2\pi R}}{2\sqrt{3}}\right)\left(\|b(x,t)\|_{L_2(D)}\left\|u^{(0)}(t)\right\|_{\mathbf{B}_1}^2+\left\|u^{(0)}(t)\right\|_{\mathbf{B}_1}M\right).$$

$$\left\|u^{(1)}(t)-u^{(0)}(t)\right\|_1\le A_T$$

Applying Cauchy inequality, Bessel inequality , Hölder inequality,the Lipshitzs condition ,taking maximum of both side of the $u^{(1)}(t)-u^{(0)}(t)$ for equation (6):

$$\begin{aligned}\left\|u^{(1)}(t)-u^{(0)}(t)\right\| &\le \left(2\sqrt{\frac{T}{\pi}}+\frac{\sqrt{2\pi R}}{2\sqrt{3}}\right)\|b(y,t)\|_{L_2(D)}\left\|u^{(0)}(t)\right\|_{\mathbf{B}_1}\left\|v^{(0)}(t)\right\|_{\mathbf{B}_2}\\ &+\left(\frac{2\sqrt{3T}+\pi}{2\sqrt{3\pi}}\right)\left\|v^{(0)}(t)\right\|_{\mathbf{B}_2}\|f(y,t,0)\|_{L_2(D)}.\end{aligned}$$

$$B_T=\left(2\sqrt{\frac{T}{\pi}}+\frac{\sqrt{2\pi R}}{2\sqrt{3}}\right)\left(\|b(y,t)\|_{L_2(D)}\left\|u^{(0)}(t)\right\|_{\mathbf{B}_1}+M\right)\left\|v^{(0)}(t)\right\|_{\mathbf{B}_2}.$$

$$\left\|u^{(1)}(t)-u^{(0)}(t)\right\|_2\le B_T$$

For N :

$$\left\|u^{(N+1)}(t)-u^{(N)}(t)\right\| \leq \frac{A_T}{\sqrt{N!}}\left(2\sqrt{\frac{T}{\pi}}+\frac{\sqrt{2\pi R}}{2\sqrt{3}}\right)^N \|b(x,t)\|^N_{L_2(D)}$$

$$\begin{aligned}\left\|u^{(N+1)}(t)-u^{(N)}(t)\right\| &\leq \left(2\sqrt{\frac{T}{\pi}}+\frac{\sqrt{2\pi R}}{2\sqrt{3}}\right)^N \|b(y,t)\|^N_{L_2(D)}\frac{A_T}{\sqrt{N!}}\\ &+\left(2\sqrt{\frac{T}{\pi}}+\frac{\sqrt{2\pi R}}{2\sqrt{3}}\right)^N \frac{B_T}{\sqrt{N!}}\end{aligned}$$

$$\begin{aligned}\left\|u^{(N+1)}(t)-u^{(N)}(t)\right\|_{B_1} &\leq \left\|u^{(N+1)}(t)-u^{(N)}(t)\right\|_1+\left\|u^{(N+1)}(t)-u^{(N)}(t)\right\|_2 \qquad (19)\\ &\leq \left(2\sqrt{\frac{T}{\pi}}+\frac{\sqrt{2\pi R}}{2\sqrt{3}}\right)^N \left[\|b(x,t)\|^N_{L_2(D)}+\|b(y,t)\|^N_{L_2(D)}\right]\frac{A_T}{\sqrt{N!}}\\ &+\left(2\sqrt{\frac{T}{\pi}}+\frac{\sqrt{2\pi R}}{2\sqrt{3}}\right)^N \frac{B_T}{\sqrt{N!}}\end{aligned}$$

It is easy to see that $u^{(N+1)} \to u^{(N)}$, $N \to \infty$, for (5)-(6)
Same estimations for (7)-(8)

$$\begin{aligned}\left\|v^{(N+1)}(t)-v^{(N)}(t)\right\|_{B_2} &\leq \left\|v^{(N+1)}(t)-v^{(N)}(t)\right\|_1+\left\|v^{(N+1)}(t)-v^{(N)}(t)\right\|_2 \qquad (20)\\ &\leq \left(2\sqrt{\frac{T}{\pi}}+\frac{\sqrt{2\pi R}}{2\sqrt{3}}\right)^N \left[\|b(x,t)\|^N_{L_2(D)}+\|b(y,t)\|^N_{L_2(D)}\right]\frac{A_T}{\sqrt{N!}}\\ &+\left(2\sqrt{\frac{T}{\pi}}+\frac{\sqrt{2\pi R}}{2\sqrt{3}}\right)^N \frac{B_T}{\sqrt{N!}}\end{aligned}$$

$$\lim_{N\to\infty} u^{(N+1)}(t)=u(t), \ \lim_{N\to\infty} v^{(N+1)}(t)=v(t).$$

For the uniqueness, we assume that the problems (5)-(9) have two solution pairs (u,v) and $(\overline{u},\overline{v})$ .Applying Cauchy inequality, Bessel inequality , Hölder inequality,the Lipshitzs condition ,taking maximum of both side of the $|u(t)-\overline{u}(t)|$ and $|v(t)-\overline{v}(t)|$, we obtain

$$|u(t)-\overline{u}(t)| \leq \left(2\sqrt{\frac{T}{\pi}}+\frac{\sqrt{2\pi R}}{2\sqrt{3}}\right)\left(\|u\|_{B_1}\|b(x,t)\|_{L_2(D)}+\|v\|_{B_2}\|b(y,t)\|_{L_2(D)}\right)|u(t)-\overline{u}(t)|$$

Applying Gronwall's inequality to the last equations we have $u(t)=\overline{u}(t)$
Same estimation for $|v(t)-\overline{v}(t)|$,we have $v(t)=\overline{v}(t)$ □

The theorem is proved.

Theorem 2.2. *Under assumption (A1)-(A2) the solution* (u,v) *of the problem (5)-(9) depends continuously upon the data* φ.

Proof. Let $\Phi=\{\varphi,\ f\}$ and $\overline{\Phi}=\{\overline{\varphi},f\}$ be two sets of the data, which satisfy the assumptions (A1)-(A2).

By using same estimations, we obtain:

$$|u-\overline{u}|^2 \leq \left\|\Phi-\overline{\Phi}\right\|^2 \times \exp 2M^2\left[\|b(x,t)\|_{L_2(D)}+\|b(y,t)\|_{L_2(D)}\right]^2.$$

For $\Phi \to \overline{\Phi}$ then $u \to \overline{u}$.

$\Phi = \{\varphi, g\}$ and $\overline{\Phi} = \{\overline{\varphi}, g\}$ be two sets of the data, which satisfy the assumptions (A1)-(A2),

$$|v-\overline{v}|^2 \leq \left\|\Phi-\overline{\Phi}\right\|^2 \times \exp 2M^2\left[\|b(x,t)\|_{L_2(D)}+\|b(y,t)\|_{L_2(D)}\right]^2.$$

For $\Phi \to \overline{\Phi}$ then $v \to \overline{v}$. □

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