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INEQUALITIES FOR CONVEX FUNCTIONS ON TIME SCALES

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Abstract. In this paper, we presented ostrowski type $\Delta-$integral inequalities for convex functions. Also we give some results for continuous and discrete choises of time scales.

Keywords: $\Delta-$integral, Ostrowski's inequality, Time scales, Hölder's inequality

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1. INTRODUCTION

In [1], Dinu gave definition of convex functions on time scales where $\mathbb{T}$ denotes a time scale and, for any I interval of $\mathbb{R}$ (open or closed), $I_{\mathbb{T}} = I \cap \mathbb{T}$ a time scale interval.

Definition 1.1. *A function $f : \mathbb{T} \to \mathbb{R}$ is called convex on $I_{\mathbb{T}}$, if*

$$f(\lambda t + (1-\lambda)s) \leq \lambda f(t) + (1-\lambda) f(s), \tag{1}$$

for all $t, s \in I_{\mathbb{T}}$ and all $\lambda \in [0,1]$ such that $\lambda t + (1-\lambda)s \in I_{\mathbb{T}}$.

The function f is strictly convex on $I_{\mathbb{T}}$ if the inequality (1) is strict for distinct $t, s \in I_{\mathbb{T}}$ and $\lambda \in (0,1)$.

In [5], Alomari and Darus proved the following Ostrovski type Inequality:

Theorem 1.1. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds:*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(u)\,du \right| \tag{2}$$

$$\leq \left[\frac{1}{6} + \frac{1}{3}\left(\frac{x-a}{b-a} \right)^3 \right] |f'(b)| + \left[\frac{1}{6} + \frac{1}{3}\left(\frac{b-x}{b-a} \right)^3 \right] |f'(a)|,$$

for each $x \in [a,b]$.

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1.1. **Preliminaries.** A time scale (or measure chain) $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$ (together with the topology of subspace of $\mathbb{R}$). The most popular examples of time scales are the real numbers $\mathbb{R}$, the integers $\mathbb{Z}$ or $h\mathbb{Z}$.

Throughout this paper $\mathbb{T}$ will denote a time scale and, for any I interval of $\mathbb{R}$ (open or closed), $I_{\mathbb{T}} = I \cap \mathbb{T}$ a time scale interval. Next we define the concepts of forward and backward jump operators:

For all $t \in \mathbb{T}$, we define the *forward jump operator* σ and the *backward jump operator* ρ by the formulas:

$$\sigma(t) = \inf\{\tau \in \mathbb{T}\colon \tau > t\} \in \mathbb{T}, \qquad \rho(t) = \sup\{\tau \in \mathbb{T}\colon \tau < t\} \in \mathbb{T}.$$

In this definition, the convention is $\inf \oslash = \sup \mathbb{T}$ and $\sup \oslash = \inf \mathbb{T}$.

If $\sigma(t) > t$, then we say that t is $right-scattered$, and if $\rho(t) < t$, then we say that t is $left-scattered$. Points that are right-scattered and left-scattered at the same time are called *isolated*. Also, if $\sigma(t) = t$, then t is said to be $right-dense$, and if $\rho(t) = t$, then t is said to be $left-dense$. Points that are simultaneously right-dense and left-dense are called *dense*. If $\mathbb{T}$ has a left-scattered maximum M, then we define $\mathbb{T}^{\kappa} = \mathbb{T}\setminus\{M\}$; otherwise $\mathbb{T}^{\kappa} = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum m, then we define $\mathbb{T}_{\kappa} = \mathbb{T}\setminus\{m\}$; otherwise $\mathbb{T}_{\kappa} = \mathbb{T}$.

The mappings $\mu, v : \mathbb{T} \to [0, \infty)$ defined by

$$\mu(t) = \sigma(t) - t$$

and

$$v(t) = t - \rho(t)$$

is called, respectively, *forward* and *backward grininess functions*.

Definition 1.2. *Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^{\kappa}$. Then we define $f^{\Delta}(t)$, to be the number (provided it exist) with the property that given any $\epsilon > 0$, there is a neighborhood U of t such that*

$$\left|(f(\sigma(t)) - f(s)) - f^{\Delta}(t)(\sigma(t) - s)\right| \leq \epsilon\,|\sigma(t) - s|$$

for all $s \in U$.

We call $f^{\Delta}(t)$ the delta derivative of f at t. We say that f is delta differentiable on $\mathbb{T}^{\kappa}$ provided $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}^{\kappa}$.

If $\mathbb{T} = \mathbb{R}$, then

$$f^{\Delta}(t) = f'(t).$$

If $\mathbb{T} = \mathbb{N}$, then

$$f^{\Delta}(t) = f(t+1) - f(t).$$

is the forward difference operator.

For a function $f : \mathbb{T} \to \mathbb{R}$ we define $f^{\sigma} : \mathbb{T} \to \mathbb{R}$ by $f^{\sigma}(t) = f(\sigma(t))$ for all $t \in \mathbb{T}$, (that is $f^{\sigma} = f \circ \sigma$). We also define $f^{\rho} : \mathbb{T} \to \mathbb{R}$ by $f^{\rho}(t) = f(\rho(t))$ for all $t \in \mathbb{T}$, (that is $f^{\rho} = f \circ \rho$).

For all $t \in \mathbb{T}^{\kappa}$, we have the following properties.

(i) If f is delta differentiable at t, then f is continuous at t.

(ii) If f is left continuous at t and t is right-scattered, then f is Δ differentiable at t with $f^{\Delta}(t) = \frac{f^{\sigma}(t) - f(t)}{\mu(t)}$.

(iii) If t is right-dense, then f is delta differentiable at t if and only if $\lim\limits_{s\to t} \frac{f(t)-f(s)}{t-s}$ exists as a finite number. In this case, $f^{\Delta}(t) = \lim\limits_{s\to t} \frac{f(t)-f(s)}{t-s}$.

(iv) If f is delta differentiable at t then $f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t)$.

In the same manner, for all $\mathbb{T}_{\kappa}$ we have the following properties:

Definition 1.3. *A function $f : \mathbb{T} \to \mathbb{R}$ is called $rd -$ continuous if it is continuous at all right-dense points in $\mathbb{T}$ and its left-sided limits are finite at all left-dense points in $\mathbb{T}$. We denote by C_{rd} the set of all rd-continuous functions. We say that f is rd-continuously delta differentiable (and write $f \in C^1_{rd}$ if $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}^{\kappa}$ and $f^{\Delta} \in C_{rd}$.*

Definition 1.4. *A function $F : \mathbb{T} \to \mathbb{R}$ is called a delta antiderivative of $f : \mathbb{T} \to \mathbb{R}$ if $F^{\Delta}(t) = f(t)$, for all $t \in \mathbb{T}^{\kappa}$, then the delta integral can be defined as $\int_a^t f(s)\Delta s = F(t) - F(a)$.*

Theorem 1.2. *(Theorem 1.77 in [2]). If $a, b, c \in \mathbb{T}$, $\beta \in \mathbb{R}$, and $f, g \in C_{rd}$, then*

1 $\int_a^b (f(t) + g(t))\Delta t = \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t$;

(1) $\int_a^b \beta f(t)\Delta t = \beta \int_a^b f(t)\Delta t$;

(2) $\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t$;

(3) $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t$;

(4) $\int_a^b f(\sigma(t)) g^{\Delta}(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t) g(t)\Delta t$;

(5) $\int_a^b f(t) g^{\Delta}(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t) g(\sigma(t))\Delta t$;

(6) $\int_a^a f(t)\Delta t = 0$;

(7) if $f(t) \geq 0$ for all t, then $\int_a^b f(t)\Delta t \geq 0$;

(8) if $|f(t)| \leq g(t)$ on $[a, b)$, then

$$\left|\int_a^b f(t)\Delta t\right| \leq \int_a^b g(t)\Delta t.$$

In [2] Bohner and Peterson gave the following integration rule for time scales:

Theorem 1.3. (*Substitution* [2]) *Assume that $v : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\tilde{\mathbb{T}} = v(\mathbb{T})$ is a time scale. If $f : \mathbb{T} \to \mathbb{R}$ is an rd-continuous function and v is differentiable with rd-continuous derivative, then for $a, b \in \mathbb{T}$,*

$$\int_a^b f(t) v^{\Delta}(t)\Delta t = \int_{v(a)}^{v(b)} \left(f \circ v^{-1}\right)(s)\tilde{\Delta} s.$$

The following analogue of Hölder's inequality for time scales had proved by Wong et al. in [6]:

Theorem 1.4. (*Hölder's Inequality* [6]) *Let $h, f, g \in C_{rd}([a,b],[0,\infty))$ with*

$$\int_a^b h(x)\, g^q(x)\, \Delta x > 0.$$

If $\frac{1}{p}+\frac{1}{q}=1$ with $p>1$, then

$$\int_a^b h(x) f(x) g(x)\, \Delta x \leq \left(\int_a^b h(x) f^p(x)\, \Delta x\right)^{\frac{1}{p}} \left(\int_a^b h(x) q^q(x)\, \Delta x\right)^{\frac{1}{q}}.$$

In [2] *Bohner and Peterson gave the following Ostrowski type results:*

Lemma 1.1. *Let $a,b,s,t \in \mathbb{T}$, $a<b$ and $f:[a,b]\to\mathbb{R}$ be differentiable. Then*

$$f(t) = \frac{1}{b-a}\int_a^b f^\sigma(t)\,\Delta t + \frac{1}{b-a}\int_a^b p(t,s)\, f^\Delta(s)\,\Delta s$$

where

$$p(t,s) = \begin{cases} s-a & a \leq s < t \\ s-b & t \leq s \leq b \end{cases}.$$

Theorem 1.5. *Let $a,b,s,t \in \mathbb{T}$, $a<b$ and $f:[a,b]\to\mathbb{R}$ be differentiable. Then*

$$\left| f(t) - \frac{1}{b-a}\int_a^b f^\sigma(t)\,\Delta t \right| \leq \frac{M}{b-a}\left(h_2(t,a) - h_2(t,b)\right),$$

where

$$M = \sup_{a<t<b} \left| f^\Delta(t) \right|$$

and time scales polynomial $h_k : \mathbb{T}^2 \to \mathbb{R}$, $k \in \mathbb{N}_0$ be defined by

$$h_0(t,s) = 1 \quad , h_{k+1}(t,s) = \int_s^t h_k(\tau,s)\,\Delta\tau$$

for all $s,t \in \mathbb{T}$.

2. Main Results

Lemma 2.1. *Let $\mathbb{T}$, $\tilde{\mathbb{T}}$ be time scales and $f : I_{\mathbb{T}} \to \mathbb{R}$ is a $\Delta-$differentiable mapping on $I_{\mathbb{T}}^{\kappa}$. Assume that $\tilde{\mathbb{T}} = v(\mathbb{T})$ with $v = \frac{t-a}{b-a}$. If $\left|f^\Delta\right|$ is convex on $I_{\mathbb{T}}$, $f^\Delta \in C_{rd}(I,\mathbb{R})$ and $a,b,t \in I_{\mathbb{T}}$ with $a<b$ then the following identity holds:*

$$f(t) - \frac{1}{b-a}\int_a^b f^\sigma(t)\,\Delta t = (b-a)\int_0^1 k(t,s)\, f^\Delta(sb+(1-s)a)\,\tilde{\Delta} s$$

where

$$k(t,s) = \begin{cases} s & 0 \leq s < \frac{t-a}{b-a} \\ s-1 & \frac{t-a}{b-a} \leq s \leq 1 \end{cases}.$$

Proof. First we recall the Montgomery identity in Lemma 1.1

$$f(t)=\frac{1}{b-a}\int_a^b f^{\sigma}(t)\,\Delta t+\frac{1}{b-a}\int_a^b p(t,s)\,f^{\Delta}(s)\,\Delta s, \tag{3}$$

where

$$p(t,s)=\begin{cases} s-a & a\le s<t\\ s-b & t\le s\le b\end{cases}.$$

If we use Theroem 1.3 in (3) we have

$$\begin{aligned}\frac{1}{b-a}\int_a^b p(t,s)\,f^{\Delta}(s)\,\Delta s &= \int_a^b p(t,s)\,f^{\Delta}(s)\,v^{\Delta}(s)\,\Delta s\\ &= \int_0^1 p\left(v^{-1}(t),v^{-1}(r)\right)f^{\Delta}\left(v^{-1}(r)\right)\tilde{\Delta}r\\ &= (b-a)\int_0^1 k(t,r)\,f^{\Delta}(rb+(1-r)a)\,\tilde{\Delta}r\end{aligned}$$

and that completes the proof. □

Theorem 2.1. *Let $\mathbb{T}$, $\tilde{\mathbb{T}}$ be time scales and $f:I_{\mathbb{T}}\to\mathbb{R}$ is a $\Delta-$differentiable mapping on $I_{\mathbb{T}}^{\kappa}$. Assume that $\tilde{\mathbb{T}}=v(\mathbb{T})$ with $v=\frac{t-a}{b-a}=c$. If $\left|f^{\Delta}\right|$ is convex on $I_{\mathbb{T}}$, $f^{\Delta}\in C_{rd}(I,\mathbb{R})$ and $a,b,t\in I_{\mathbb{T}}$ with $a<b$ then the following inequality holds:*

$$\begin{aligned}&\left|f(t)-\frac{1}{b-a}\int_a^b f^{\sigma}(u)\,\Delta u\right|\\ \le\ & (b-a)\left\{(\lambda_2+\mu_1-\mu_2)\left|f^{\Delta}(b)\right|+(\mu_2+\lambda_1-\lambda_2)\left|f^{\Delta}(a)\right|\right\}\end{aligned}$$

where

$$\begin{aligned}\lambda_1&=\int_0^c s\tilde{\Delta}s & \mu_1&=\int_c^1(1-s)\,\tilde{\Delta}s\\ \lambda_2&=\int_0^c s^2\tilde{\Delta}s & \mu_2&=\int_c^1(1-s)^2\,\tilde{\Delta}s\end{aligned} \tag{4}$$

for $c\in\tilde{\mathbb{T}}$.

Proof. By using the Lemma 2.1 and the property of modulus, for $s\in\tilde{\mathbb{T}}$ we have

$$\begin{aligned}&\left|f(t)-\frac{1}{b-a}\int_a^b f^{\sigma}(t)\,\Delta t\right| \qquad (5)\\ \le\ &(b-a)\int_0^c\left|sf^{\Delta}[sb+(1-s)a]\right|\tilde{\Delta}s+\int_c^1\left|(s-1)\,f^{\Delta}[sb+(1-s)a]\right|\tilde{\Delta}s.\end{aligned}$$

Since $\left|f^{\Delta}\right|$ is convex then

$$\begin{aligned}
& \int_0^c \left|sf^{\Delta}\left[sb+(1-s)a\right]\right| \tilde{\Delta}s + \int_c^1 \left|(s-1)f^{\Delta}\left[sb+(1-s)a\right]\right| \tilde{\Delta}s \\
\leq & \int_0^c s\left[s\left|f^{\Delta}(b)\right| + (1-s)\left|f^{\Delta}(a)\right|\right]\tilde{\Delta}s + \int_c^1 |s-1|\left[s\left|f^{\Delta}(b)\right| + (1-s)\left|f^{\Delta}(a)\right|\right]\tilde{\Delta}s \\
= & \left(\int_0^c s^2\tilde{\Delta}s + \int_c^1 (1-s)s\tilde{\Delta}s\right)\left|f^{\Delta}(b)\right| + \left(\int_0^c s(1-s)\tilde{\Delta}s + \int_c^1 (1-s)^2\tilde{\Delta}s\right)\left|f^{\Delta}(a)\right| \\
= & \left(\int_0^c s^2\tilde{\Delta}s + \int_c^1 \left[(1-s)-(1-s)^2\right]\tilde{\Delta}s\right)\left|f^{\Delta}(b)\right| + \left(\int_0^c \left[s-s^2\right]\tilde{\Delta}s + \int_c^1 (1-s)^2\tilde{\Delta}s\right)\left|f^{\Delta}(a)\right|
\end{aligned}$$

This implies the desired result. □

Remark 2.1. *Under the Assumptions of Theorem 2.1, assuming* $M = \sup_{a<t<b}\left|f^{\Delta}(t)\right|$, *we have Theorem 1.5.*

Remark 2.2. *(Continuous Case) Let* $\mathbb{T} = \mathbb{R}$. *Then we obtain the inequality* (2).

Corollary 2.1. *(Discrete Case). Let* $\mathbb{T} = \mathbb{Z}$. *Let* $a = 0$, $b = n$, $s = j$, $t = i$ *and* $f(k) = x_k$. *Then the following inequality holds:*

$$\begin{aligned}
& \left|x_i - \frac{1}{n}\sum_{j=1}^{n} x_j\right| \\
\leq & \frac{1}{6n}\left\{\left[g(i) + h(n-i+1)\right]\left|x_{n+1}-x_n\right| + \left[g(n-i+1)+h(i)\right]\left|x_1 - x_0\right|\right\}
\end{aligned}$$

where

$$\begin{aligned}
g(u) &= u(u-1)(2u-1) \\
h(u) &= u(u-1)(3n-2u-1)
\end{aligned}$$

Proof. Since the fact that

$$\begin{aligned}
\left|f^{\Delta}(b)\right| &= \left|x_{n+1}-x_n\right|, \\
\left|f^{\Delta}(a)\right| &= \left|x_1 - x_0\right|,
\end{aligned}$$

If we calculate the four integrals in (4), we deduce the desired result. □

Example 2.1. *Using Corollary 2.1, let* $f(x) = x^2$, $\mathbb{T} = \mathbb{Z}$ *and* $a = 0$, $b = 12$. *If we apply the formula we have*

$$\left|x_i - \frac{1}{n}\sum_{j=1}^{n} x_j\right| = \left|i^2 - \frac{1}{12}\frac{(12).(13)(25)}{2}\right| = \left|i^2 - 162,5\right| = 162,5 - i^2$$

and

$$\begin{aligned}
& \frac{1}{6n}\left\{\left[g\left(i\right)+h\left(n-i\right)\right]\left|x_{n+1}-x_{n}\right|+\left[g\left(n-i\right)+h\left(i\right)\right]\left|x_{1}-x_{0}\right|\right\} \\
= & \frac{1}{72}\left\{\left[g\left(i\right)+h\left(n-i\right)\right]\left(25\right)+\left[g\left(n-i\right)+h\left(i\right)\right]\right\} \\
= & \frac{1}{72}\left(39336-528i-844i^{2}+96i^{3}\right).
\end{aligned}$$

One can easily see that $LHS \leq RHS$.

Theorem 2.2. *Let $\mathbb{T}$, $\tilde{\mathbb{T}}$ be time scales and $f : I_{\mathbb{T}} \to \mathbb{R}$ is a $\Delta-$differentiable mapping on $I_{\mathbb{T}}^{\kappa}$. Assume that $\tilde{\mathbb{T}}$ $=v\left(\mathbb{T}\right)$ with $v=\frac{t-a}{b-a}=c$. If $\left|f^{\Delta}\right|^{q}$ is convex on $I_{\mathbb{T}}$, $f^{\Delta}\in C_{rd}\left(I,\mathbb{R}\right)$ and $a,b,t\in I_{\mathbb{T}}$ with $a<b$, $q>1$ then the following inequality holds:*

$$\begin{aligned}
& \left|f\left(t\right)-\frac{1}{b-a}\int_{a}^{b}f^{\sigma}\left(u\right)\Delta u\right| \\
\leq & \left(b-a\right)\left(\lambda_{1}+\mu_{1}\right)^{\frac{1}{p}}\left\{\left(\lambda_{2}+\mu_{1}-\mu_{2}\right)\left|f^{\Delta}\left(b\right)\right|^{q}+\left(\mu_{2}+\lambda_{1}-\lambda_{2}\right)\left|f^{\Delta}\left(a\right)\right|^{q}\right\}^{\frac{1}{q}}
\end{aligned}$$

where

$$\begin{aligned}
\lambda_{1} &= \int_{0}^{c}s\tilde{\Delta}s \qquad \mu_{1}=\int_{c}^{1}\left(1-s\right)\tilde{\Delta}s \\
\lambda_{2} &= \int_{0}^{c}s^{2}\tilde{\Delta}s \qquad \mu_{2}=\int_{c}^{1}\left(1-s\right)^{2}\tilde{\Delta}s
\end{aligned}$$

and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By using the Lemma 2.1 and the property of modulus, for $s\in\tilde{\mathbb{T}}$ we have

$$\begin{aligned}
& \left|f\left(t\right)-\frac{1}{b-a}\int_{a}^{b}f^{\sigma}\left(t\right)\Delta t\right| \\
\leq & \left(b-a\right)\int_{0}^{1}\left|k\left(t,s\right)\right|\left|f^{\Delta}\left[sb+\left(1-s\right)a\right]\right|\tilde{\Delta}s
\end{aligned} \tag{6}$$

By using Hölder Inequality in Theorem 1.4 we can state that

$$\begin{aligned}
& \int_{0}^{1}\left|k\left(t,s\right)\right|\left|f^{\Delta}\left[sb+\left(1-s\right)a\right]\right|\tilde{\Delta}s \\
\leq & \left(\int_{0}^{1}\left|k\left(t,s\right)\right|\tilde{\Delta}s\right)^{\frac{1}{p}}\left(\int_{0}^{1}\left|k\left(t,s\right)\right|\left|f^{\Delta}\left[sb+\left(1-s\right)a\right]\right|^{q}\tilde{\Delta}s\right)^{\frac{1}{q}}
\end{aligned}$$

Thus we have

$$\begin{aligned}
\int_{0}^{1}\left|k\left(t,s\right)\right|\tilde{\Delta}s &= \int_{0}^{c}s\tilde{\Delta}s+\int_{c}^{1}\left(1-s\right)\tilde{\Delta}s \\
&= \lambda_{1}+\mu_{1}
\end{aligned}$$

and since $\left|f^{\Delta}\right|^{q}$ is convex than we can write

$$\begin{aligned}&\int_{0}^{1}|k(t,s)|\left|f^{\Delta}[sb+(1-s)a]\right|^{q}\tilde{\Delta}s\\ \leq\ &\int_{0}^{1}|k(t,s)|\left[s\left|f^{\Delta}(b)\right|^{q}+(1-t)\left|f^{\Delta}(b)\right|^{q}\right]\tilde{\Delta}s.\end{aligned}$$

By using the same method as in the proof of previous Theorem we have

$$\begin{aligned}&\int_{0}^{1}|k(t,s)|\left[s\left|f^{\Delta}(b)\right|^{q}+(1-t)\left|f^{\Delta}(b)\right|^{q}\right]\tilde{\Delta}s\\ =\ &(\lambda_2+\mu_1-\mu_2)\left|f^{\Delta}(b)\right|^{q}+(\mu_2+\lambda_1-\lambda_2)\left|f^{\Delta}(a)\right|^{q}.\end{aligned}$$

Writing these results in (6) we deduce the desired result. □

Remark 2.3. *(Continuous Case) In Theorem 2.2, Let $\mathbb{T}=\mathbb{R}$. Then we obtain*

$$\begin{aligned}&\left|f(x)-\frac{1}{b-a}\int_{a}^{b}f(u)\,du\right|\\ \leq\ &(b-a)^{1-\frac{2}{p}}\left\{\frac{(x-a)^2+(b-x)^2}{2}\right\}^{\frac{1}{p}}\\ &\times\left\{\left[\frac{1}{6}+\frac{1}{3}\left(\frac{x-a}{b-a}\right)^3\right]\left|f'(b)\right|+\left[\frac{1}{6}+\frac{1}{3}\left(\frac{b-x}{b-a}\right)^3\right]\left|f'(a)\right|\right\}^{\frac{1}{q}}.\end{aligned}$$

Corollary 2.2. *(Discrete Case). In Theorem 2.2, Let $\mathbb{T}=\mathbb{Z}$. Let $a=0,\ b=n,\ t=i$ and $f(k)=x_k$. Then the following inequality holds*

$$\begin{aligned}&\left|x_i-\frac{1}{n}\sum_{j=1}^{n}x_j\right|\\ \leq\ &\frac{n^{\frac{-1}{q}}}{2}\left(\frac{1}{3}\right)^{\frac{1}{q}}\left\{[g(i)+h(n-i+1)]\left|x_{n+1}-x_n\right|^q+[g(n-i+1)+h(i)]\left|x_1-x_0\right|^q\right\}^{\frac{1}{q}}\\ &\times[r(i)+r(n-i+1)]^{\frac{1}{p}}.\end{aligned}$$

where

$$\begin{aligned}g(u)&=u(u-1)(2u-1)\\ h(u)&=u(u-1)(3n-2u+1)\\ r(u)&=u(u-1).\end{aligned}$$

Corollary 2.3. *Under the Assumptions of Theorem 2.2, assuming $M=\sup_{a<t<b}\left|f^{\Delta}(t)\right|^{q}$, we have the following inequality*

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_{a}^{b}f^{\sigma}(t)\,\Delta t\right|\\ \leq\ &(b-a)\{\lambda_1+\mu_1\}M.\end{aligned}$$

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