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INEQUALITIES GENERATED WITH RIEMANN-LIOUVILLE FRACTIONAL INTEGRAL OPERATOR

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Abstract. The primary objective of this study is to handle new generalized midpoint, trapezoid and Simpson's type inequalities with the help of Riemann-Liouville fractional integral operator. In order to do this, a new fractional integral identity is obtained. Then by using this identity, some inequalities for the class of functions whose derivatives in absolute values at certain powers are convex are derived. It is observed that the obtained inequalities are generalizations of some results exist in the literature.

Keywords: Riemann-Liouville fractional integrals, Hadamard type inequalities, Simpson's inequality, convex functions.

AMS Subject Classification: 26A33, 26D10, 26D15.

1. Introduction

It is a well known fact that inequalities have important role in the studies of inequality theory, linear programming, extremum problems, optimization, error estimates and game theory. Over the years, only integer real order integrals were taken into account while handling new results about integral inequalities. However, in the recent years fractional integral operator have been considered by many scientists (see [3], [4], [6]-[11]) and the references therein. There are some inequalities in the literature that accelerates studies on integral inequalities. One of the most famous and practical inequality in the literature is Hermite-Hadamard inequality given in the following:

Theorem 1.1. *Let f be defined from interval I (a nonempty subset of $\mathbb{R}$) to $\mathbb{R}$ be a convex function on I and $a, b \in I$ with $a < b$. Then the double inequality given in the following holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

Another well-known inequality namely Simpson's inequality is given in the following:

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Theorem 1.2. *Let $f:[a,b]\to\mathbb{R}$ be a four times continuously differentiable mapping on (a,b) and $\left\|f^{(4)}\right\|_\infty=\sup_{x\in(a,b)}\left|f^{(4)}(x)\right|<\infty$. Then the following inequality holds:*

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\le\frac{1}{2880}\left\|f^{(4)}\right\|_\infty(b-a)^2. \quad (2)$$

Now we will mention about Riemann-Liouville fractional integration operator (see [5]) which ables to integrate functions on fractional orders.

Definition 1.1. *Let $f\in L_1[a,b]$. $J_{a+}^{\alpha}f$ and $J_{b-}^{\alpha}f$ which are called left-sided and right-sided Riemann-Liouville integrals of order $\alpha>0$ with $0\le a\le x\le b$ are defined by*

$$J_{a+}^{\alpha}f=\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}f(t)\,dt,\qquad x>a \quad (3)$$

and

$$J_{b-}^{\alpha}f=\frac{1}{\Gamma(\alpha)}\int_x^b(t-x)^{\alpha-1}f(t)\,dt,\qquad x<b \quad (4)$$

respectively where $\Gamma(\alpha)=\int_0^\infty e^{-t}u^{\alpha-1}du$. Here $J_{a^+}^{0}f(x)=J_{b^-}^{0}f(x)=f(x)$.

Erdélyi et al. deeply involved in hypergeometric functions which Whittaker has discovered in 1904 and gave the definition of it in [1] as:

$${}_2F_1(a,b;c;z)=\frac{1}{\beta(b,b-c)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt,\ c>b>0,\ |z|<1 \quad (5)$$

Throughout the paper we will use $T_f(\alpha,\lambda,k;a,b)$ in the mean of following statement:

$$\begin{aligned} T_f(\alpha,\lambda,k;a,b) &= \lambda(\alpha f(a)+(1-\alpha)f(b)) \\ &\quad+\left((1-\alpha)^k+\alpha^k-\lambda\right)f(\alpha a+(1-\alpha)b) \\ &\quad-\frac{\Gamma(k+1)}{(b-a)^k}\left[J^k_{[\alpha a+(1-\alpha)b]^-}f(a)+J^k_{[\alpha a+(1-\alpha)b]^+}f(b)\right] \end{aligned} \quad (6)$$

and Γ is Euler Gamma function, i.e., $\Gamma(\alpha)=\int_0^\infty e^{-t}u^{\alpha-1}du$.

İşcan obtained new midpoint, trapezoid and Simpson's type inequalities by using the following lemma in 2012 in [2].

Lemma 1.1. *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I° such that $f'\in L[a,b]$ where $a,b\in I$ with $a<b$ and $\alpha,\lambda\in[0,1]$. Then the following equality holds:*

$$\begin{aligned} &\lambda(\alpha f(a)+(1-\alpha)f(b))+(1-\lambda)f(\alpha a+(1-\alpha)b)-\frac{1}{b-a}\int_a^b f(x)\,dx \\ &= (b-a)\left[\int_0^{1-\alpha}(t-\alpha\lambda)f'(tb+(1-t)a)\,dt\right. \\ &\quad\left.+\int_{1-\alpha}^1(t-1+\lambda(1-\alpha))f'(tb+(1-t)a)\,dt\right]. \end{aligned} \quad (7)$$

In this paper, a new kernel has been obtained and new theorems including Riemann-Liouville fractional integral operator have been gathered by inspiring from Lemma 1.1. By using these theorems; midpoint, trapezoid and Simpson's type inequalities can be obtained.

2. Main Results

Lemma 2.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a,b]$ where $a, b \in I$ with $a < b$ and $\alpha, \lambda \in [0,1]$. Then for $k > 0$, the following equality is valid:*

$$\begin{aligned} T_f(\alpha,\lambda,k;a,b) &= (b-a)\left[\int_0^{1-\alpha}\left(t^k-\alpha\lambda\right)f'(tb+(1-t)a)\,dt\right. \\ &\quad \left. -\int_{1-\alpha}^{1}\left[(1-t)^k-\lambda(1-\alpha)\right]f'(tb+(1-t)a)\,dt\right]. \end{aligned} \tag{8}$$

Proof. Integrating by parts then with the change of variable $x = tb + (1-t)a$ we get

$$\begin{aligned} I_1 &= \int_0^{1-\alpha}\left(t^k-\alpha\lambda\right)f'(tb+(1-t)a)\,dt \\ &= \frac{\left((1-\alpha)^k-\alpha\lambda\right)f((1-\alpha)b+\alpha a)+\alpha\lambda f(a)}{b-a} \\ &\quad -\frac{k}{b-a}\int_a^{\alpha a+(1-\alpha)b}\left(\frac{x-a}{b-a}\right)^{k-1}\frac{f(x)}{b-a}dx. \end{aligned} \tag{9}$$

With the use of Riemann-Liouville fractional integral we get

$$\begin{aligned} I_1 &= \frac{\left((1-\alpha)^k-\alpha\lambda\right)f((1-\alpha)b+\alpha a)+\alpha\lambda f(a)}{b-a} \\ &\quad -\frac{k\Gamma(k)}{(b-a)^{k+1}}\left[J^k_{[\alpha a+(1-\alpha)b]^-}f(a)\right]. \end{aligned} \tag{10}$$

Similarly we have

$$\begin{aligned} I_2 &= \int_{1-\alpha}^{1}\left[(1-t)^k-\lambda(1-\alpha)\right]f'(tb+(1-t)a)\,dt \\ &= \frac{-\lambda(1-\alpha)f(b)-\left[\alpha^k-\lambda(1-\alpha)\right]f((1-\alpha)b+\alpha a)}{b-a} \\ &\quad +\frac{k}{b-a}\int_{\alpha a+(1-\alpha)b}^{b}\left(\frac{b-x}{b-a}\right)^{k-1}\frac{f(x)}{b-a}dx. \end{aligned} \tag{11}$$

Definition of Rieman-Liouville fractional integral operator make it possible to write

$$\begin{aligned} I_2 &= \frac{-\lambda(1-\alpha) f(b) - \left[\alpha^k - \lambda(1-\alpha)\right] f((1-\alpha)b + \alpha a)}{b-a} \\ &+ \frac{k\Gamma(k)}{(b-a)^{k+1}} \left[J^k_{[\alpha a + (1-\alpha) b]^+} f(b) \right]. \end{aligned} \tag{12}$$

Subtracting (12) from (10) and multiplying with $(b-a)$, then rearranging the inequality, we get the desired result. □

Remark 2.1. *If we choose $k=1$ in Lemma 2.1, we get Lemma 1.1 proved in* [2].

Theorem 2.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a,b]$ where $a, b \in I$ with $a < b$ and $\alpha, \lambda \in [0,1]$. Then for $k > 0$, the following inequality is valid:*

$$\frac{|T_f(\alpha, \lambda, k; a, b)|}{b-a} \tag{13}$$

$$\leq \begin{cases} (K_1 + M_1)|f'(b)| + (L_1 + N_1)|f'(a)| & \text{for } \alpha\lambda \leq (1-\alpha)^k \text{ and } \lambda(1-\alpha) \leq \alpha^k \\ (K_1 + M_2)|f'(b)| + (L_1 + N_2)|f'(a)| & \text{for } \alpha\lambda \leq (1-\alpha)^k \text{ and } \lambda(1-\alpha) > \alpha^k \\ (K_2 + M_1)|f'(b)| + (L_2 + N_1)|f'(a)| & \text{for } \alpha\lambda > (1-\alpha)^k \text{ and } \lambda(1-\alpha) \leq \alpha^k \\ (K_2 + M_2)|f'(b)| + (L_2 + N_2)|f'(a)| & \text{for } \alpha\lambda > (1-\alpha)^k \text{ and } \lambda(1-\alpha) > \alpha^k \end{cases}$$

where

$$\begin{aligned} K_1 &= \frac{\alpha^2 \lambda^{\frac{2+k}{k}} - \alpha\lambda(1-\alpha)^2}{2} + \frac{(1-\alpha)^{2+k} - \alpha^{1+k}\lambda^{\frac{2+k}{k}}}{2+k} \\ K_2 &= \frac{\alpha\lambda(1-\alpha)^2}{2} - \frac{(1-\alpha)^{2+k}}{2+k} \\ L_1 &= \frac{(1-\alpha)^{1+k}}{(2+k)(1+k)} - \frac{(1-\alpha)(1+\alpha)\alpha\lambda + \alpha^2\lambda^{\frac{2+k}{k}} - 2\alpha^2\lambda^{\frac{1+k}{k}}}{2} \\ &\quad - \frac{\alpha^{1+k}\lambda^{\frac{1+k}{k}}}{1+k} + \frac{\alpha^{2+k}\lambda^{\frac{2+k}{k}} + (1-\alpha)^{1+k}\alpha}{2+k} \\ L_2 &= \frac{\left(1-\alpha^2\right)\alpha\lambda}{2} - \frac{(1-\alpha)^{k+1}(k\alpha+\alpha+1)}{(k+1)(k+2)} \\ M_1 &= \frac{(1-\alpha)^2 + 1 - \lambda(1-\alpha)\left[2\left(1-(\lambda(1-\alpha))^{\frac{1}{k}}\right)^2\right]}{2} \\ &\quad + \frac{\alpha^{k+1} - 2(\lambda(1-\alpha))^{\frac{k+1}{k}}}{k+1} + \frac{2(\lambda(1-\alpha))^{\frac{k+2}{k}} - \alpha^{k+2}}{k+2} \\ M_2 &= \frac{\alpha^{k+2}}{k+2} - \frac{\alpha^{k+1}}{k+1} + \frac{\lambda\alpha(1-\alpha)(2-\alpha)}{2} \\ N_1 &= \frac{\lambda(1-\alpha)\left(2(\lambda(1-\alpha))^{\frac{2}{k}} - \alpha^2\right)}{2} + \frac{\alpha^{k+2} - 2(\lambda(1-\alpha))^{\frac{k+2}{k}}}{k+2} \\ N_2 &= \frac{\lambda\alpha^2(1-\alpha)}{2} - \frac{\alpha^{k+2}}{k+2}. \end{aligned}$$

Proof. Using Lemma 2.1 and properties of absolute value we get

$$\begin{aligned}\frac{|T_f(\alpha,\lambda,k;a,b)|}{b-a} &\leq \int_0^{1-\alpha}\left|t^k-\alpha\lambda\right|\left|f'(tb+(1-t)a)\right|dt \\ &+\int_{1-\alpha}^{1}\left|(1-t)^k-\lambda(1-\alpha)\right|\left|f'(tb+(1-t)a)\right|dt. \qquad (14)\end{aligned}$$

Then by taking into account convexity of $|f'|$ we get

$$\frac{|T_f(\alpha,\lambda,k;a,b)|}{b-a} \leq \int_0^{1-\alpha}\left|t^k-\alpha\lambda\right|\left[t\left|f'(b)\right|+(1-t)\left|f'(a)\right|\right]dt \qquad (15)$$
$$+\int_{1-\alpha}^{1}\left|(1-t)^k-\lambda(1-\alpha)\right|\left[t\left|f'(b)\right|+(1-t)\left|f'(a)\right|\right]dt.$$

By necessary computations we get the result since

$$\int_0^{1-\alpha} t\left|t^k-\alpha\lambda\right|dt = \begin{cases} K_1, & \alpha\lambda\leq(1-\alpha)^k \\ K_2, & \alpha\lambda>(1-\alpha)^k\end{cases} \qquad (16)$$

$$\int_0^{1-\alpha} (1-t)\left|t^k-\alpha\lambda\right|dt = \begin{cases} L_1, & \alpha\lambda\leq(1-\alpha)^k \\ L_2, & \alpha\lambda>(1-\alpha)^k\end{cases} \qquad (17)$$

$$\int_{1-\alpha}^{1} t\left|(1-t)^k-\lambda(1-\alpha)\right|dt = \begin{cases} M_1, & \lambda(1-\alpha)\leq\alpha^k \\ M_2, & \lambda(1-\alpha)>\alpha^k\end{cases} \qquad (18)$$

and

$$\int_{1-\alpha}^{1} (1-t)\left|(1-t)^k-\lambda(1-\alpha)\right|dt = \begin{cases} N_1, & \lambda(1-\alpha)\leq\alpha^k \\ N_2, & \lambda(1-\alpha)>\alpha^k\end{cases}. \qquad (19)$$

□

Theorem 2.2. *Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I° such that $f'\in L[a,b]$ where $a,b\in I$ with $a<b$ and $\alpha,\lambda\in[0,1]$. Then for $k>0$, the following inequality is valid:*

$$\frac{|T_f(\alpha,\lambda,k;a,b)|}{b-a} \qquad (20)$$
$$\leq \begin{cases} (V_1)^{\frac{1}{p}}(Y)^{\frac{1}{q}}+(W_1)^{\frac{1}{p}}(Z)^{\frac{1}{q}} & \text{for } \alpha\lambda\leq(1-\alpha)^k \text{ and } \lambda(1-\alpha)\leq\alpha^k \\ (V_1)^{\frac{1}{p}}(Y)^{\frac{1}{q}}+(W_2)^{\frac{1}{p}}(Z)^{\frac{1}{q}} & \text{for } \alpha\lambda\leq(1-\alpha)^k \text{ and } \lambda(1-\alpha)>\alpha^k \\ (V_2)^{\frac{1}{p}}(Y)^{\frac{1}{q}}+(W_1)^{\frac{1}{p}}(Z)^{\frac{1}{q}} & \text{for } \alpha\lambda>(1-\alpha)^k \text{ and } \lambda(1-\alpha)\leq\alpha^k \\ (V_2)^{\frac{1}{p}}(Y)^{\frac{1}{q}}+(W_2)^{\frac{1}{p}}(Z)^{\frac{1}{q}} & \text{for } \alpha\lambda>(1-\alpha)^k \text{ and } \lambda(1-\alpha)>\alpha^k\end{cases}$$

where

$$\begin{aligned}
V_1 &= \frac{(\alpha-1)\left((1-\alpha)^k-\alpha\lambda\right)^{p+1}}{\alpha\lambda}\ {}_2F_1\left(1,1+p+\frac{1}{k};\frac{k+1}{k},\frac{(1-\alpha)^k}{\alpha\lambda}\right)\\
&\quad +\alpha\lambda^{\frac{1-k}{k}}\left(\alpha^k\lambda-\alpha\lambda\right)^{p+1}\ {}_2F_1\left(1,1+p+\frac{1}{k};\frac{k+1}{k},\alpha^{k-1}\right)\\
V_2 &= \frac{(1-\alpha)\left(\alpha\lambda-(1-\alpha)^k\right)^{1+p}}{\alpha\lambda}\ {}_2F_1\left(1,1+p+\frac{1}{k};\frac{k+1}{k},\frac{(1-\alpha)^k}{\alpha\lambda}\right)\\
W_1 &= \frac{\alpha\left(\alpha^k-\lambda+\alpha\lambda\right)^{1+p}\ {}_2F_1\left(1,1+p+\frac{1}{k};1+\frac{1}{k},\frac{\alpha^k}{(1-\alpha)\lambda}\right)}{(\alpha-1)\lambda}\\
&\quad -\frac{(\lambda-\alpha\lambda)^{\frac{1}{k}}\left((\alpha-1)\lambda+(\lambda-\alpha\lambda)^{1+p}\right)\ {}_2F_1\left(1,1+p+\frac{1}{k};1+\frac{1}{k},1\right)}{(\alpha-1)\lambda}\\
W_2 &= \frac{\alpha\left(\lambda-\alpha^k-\lambda\alpha\right)^{1+p}\ {}_2F_1\left(1,1+p+\frac{1}{k};1+\frac{1}{k},\frac{\alpha^k}{(1-\alpha)\lambda}\right)}{(1-\alpha)\lambda}\\
Y &= \frac{(1-\alpha)^2\left|f'(b)\right|^q+\left(1-\alpha^2\right)\left|f'(a)\right|^q}{2}\\
Z &= \frac{\alpha(2-\alpha)\left|f'(b)\right|^q+\alpha^2\left|f'(a)\right|^q}{2}
\end{aligned}$$

and for $p>1$, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Lemma 2.1 and properties of absolute value we get

$$\begin{aligned}
\frac{\left|T_f(\alpha,\lambda,k;a,b)\right|}{b-a} &\leq \int_0^{1-\alpha}\left|t^k-\alpha\lambda\right|\left|f'(tb+(1-t)a)\right|dt \qquad (21)\\
&\quad +\int_{1-\alpha}^{1}\left|(1-t)^k-\lambda(1-\alpha)\right|\left|f'(tb+(1-t)a)\right|dt.
\end{aligned}$$

By using Hölder's inequality we have

$$\begin{aligned}
&\frac{\left|T_f(\alpha,\lambda,k;a,b)\right|}{b-a} \qquad (22)\\
&\leq \left(\int_0^{1-\alpha}\left|t^k-\alpha\lambda\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^{1-\alpha}\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\\
&\quad +\left(\int_{1-\alpha}^{1}\left|(1-t)^k-\lambda(1-\alpha)\right|^p dt\right)^{\frac{1}{p}}\left(\int_{1-\alpha}^{1}\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}$$

Since $|f'|^q$ is convex on I we get

$$\begin{aligned} &\frac{|T_f(\alpha,\lambda,k;a,b)|}{b-a} \qquad (23)\\ &\le \left(\int_0^{1-\alpha}\left|t^k-\alpha\lambda\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^{1-\alpha}\left[t\left|f'(b)\right|^q+(1-t)\left|f'(a)\right|^q\right]dt\right)^{\frac{1}{q}}\\ &+\left(\int_{1-\alpha}^{1}\left|(1-t)^k-\lambda(1-\alpha)\right|^p dt\right)^{\frac{1}{p}}\left(\int_{1-\alpha}^{1}\left[t\left|f'(b)\right|^q+(1-t)\left|f'(a)\right|^q\right]dt\right)^{\frac{1}{q}}. \end{aligned}$$

With simple calculation we get

$$\int_0^{1-\alpha}\left|t^k-\alpha\lambda\right|^p dt = \begin{cases} V_1, & \alpha\lambda\le(1-\alpha)^k \\ V_2, & \alpha\lambda>(1-\alpha)^k \end{cases} \qquad (24)$$

$$\int_{1-\alpha}^{1}\left|(1-t)^k-\lambda(1-\alpha)\right|^p dt = \begin{cases} W_1, & \lambda(1-\alpha)\le\alpha^k \\ W_2, & \lambda(1-\alpha)>\alpha^k \end{cases} \qquad (25)$$

and

$$\int_0^{1-\alpha}\left[t\left|f'(b)\right|^q+(1-t)\left|f'(a)\right|^q\right]dt = Y \qquad (26)$$

$$\int_{1-\alpha}^{1}\left[t\left|f'(b)\right|^q+(1-t)\left|f'(a)\right|^q\right]dt = Z. \qquad (27)$$

By using (24)-(27) in (23) we get the desired result. □

Theorem 2.3. *Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I° such that $f'\in L[a,b]$ where $a,b\in I$ with $a<b$ and $\alpha,\lambda\in[0,1]$. Then for $k>0$ and $q\ge 1$, the following inequality is valid:*

$$\frac{|T_f(\alpha,\lambda,k;a,b)|}{b-a} \qquad (28)$$

$$\le \begin{cases} (G_1)^{1-\frac{1}{q}}\left(K_1|f'(b)|^q+L_1|f'(a)|^q\right)^{\frac{1}{q}} + (H_1)^{1-\frac{1}{q}}\left(M_1|f'(b)|^q+N_1|f'(a)|^q\right)^{\frac{1}{q}} & \text{for } \alpha\lambda\le(1-\alpha)^k \text{ and } \lambda(1-\alpha)\le\alpha^k \\[2ex] (G_1)^{1-\frac{1}{q}}\left(K_1|f'(b)|^q+L_1|f'(a)|^q\right)^{\frac{1}{q}} + (H_2)^{1-\frac{1}{q}}\left(M_2|f'(b)|^q+N_2|f'(a)|^q\right)^{\frac{1}{q}} & \text{for } \alpha\lambda\le(1-\alpha)^k \text{ and } \lambda(1-\alpha)>\alpha^k \\[2ex] (G_2)^{1-\frac{1}{q}}\left(K_2|f'(b)|^q+L_2|f'(a)|^q\right)^{\frac{1}{q}} + (H_1)^{1-\frac{1}{q}}\left(M_1|f'(b)|^q+N_1|f'(a)|^q\right)^{\frac{1}{q}} & \text{for } \alpha\lambda>(1-\alpha)^k \text{ and } \lambda(1-\alpha)\le\alpha^k \\[2ex] (G_2)^{1-\frac{1}{q}}\left(K_2|f'(b)|^q+L_2|f'(a)|^q\right)^{\frac{1}{q}} + (H_2)^{1-\frac{1}{q}}\left(M_2|f'(b)|^q+N_2|f'(a)|^q\right)^{\frac{1}{q}} & \text{for } \alpha\lambda>(1-\alpha)^k \text{ and } \lambda(1-\alpha)>\alpha^k \end{cases}$$

where

$$\begin{aligned}
G_1 &= \alpha\lambda(1-\alpha)-\frac{(1-\alpha)^{1+k}}{1+k}\\
G_2 &= \frac{2k(\alpha\lambda)^{\frac{1+k}{k}}+(1-\alpha)^{1+k}}{1+k}-\alpha\lambda(1-\alpha)\\
H_1 &= \frac{\alpha^{1+k}}{1+k}-\frac{2(\lambda(1-\alpha))^{\frac{1}{k}}}{1+k}+2\lambda(1-\alpha)(\lambda(1-\alpha))^{\frac{1}{k}}\\
H_2 &= \lambda(1-\alpha)\alpha-\frac{\alpha^{1+k}}{1+k}.
\end{aligned}$$

and K_1, K_2, L_1, L_2, M_1, M_2, N_1, N_2 defined in Theorem 2.1.

Proof. Using Lemma 2.1 and properties of absolute value we get

$$\begin{aligned}
\frac{|T_f(\alpha,\lambda,k;a,b)|}{b-a} &\leq \int_0^{1-\alpha}\left|t^k-\alpha\lambda\right|\left|f'(tb+(1-t)a)\right|dt \qquad (29)\\
&+\int_{1-\alpha}^{1}\left|(1-t)^k-\lambda(1-\alpha)\right|\left|f'(tb+(1-t)a)\right|dt.
\end{aligned}$$

With the help of power mean inequality we have

$$\begin{aligned}
\frac{|T_f(\alpha,\lambda,k;a,b)|}{b-a} &\leq \left(\int_0^{1-\alpha}\left|t^k-\alpha\lambda\right|dt\right)^{1-\frac{1}{q}}\\
&\times\left(\int_0^{1-\alpha}\left|t^k-\alpha\lambda\right|\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\\
&+\left(\int_{1-\alpha}^{1}\left|(1-t)^k-\lambda(1-\alpha)\right|dt\right)^{1-\frac{1}{q}}\\
&\times\left(\int_{1-\alpha}^{1}\left|(1-t)^k-\lambda(1-\alpha)\right|\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}} \qquad (30)
\end{aligned}$$

and by using convexity of $|f'|^q$ we have

$$\begin{aligned}
\frac{|T_f(\alpha,\lambda,k;a,b)|}{b-a} &\leq \left(\int_0^{1-\alpha}\left|t^k-\alpha\lambda\right|dt\right)^{1-\frac{1}{q}} \\
&\times\left(\int_0^{1-\alpha}\left|t^k-\alpha\lambda\right|\left[t\left|f'(b)\right|^q+(1-t)\left|f'(a)\right|^q\right]dt\right)^{\frac{1}{q}} \\
&+\left(\int_{1-\alpha}^{1}\left|(1-t)^k-\lambda(1-\alpha)\right|dt\right)^{1-\frac{1}{q}} \\
&\times\left(\int_{1-\alpha}^{1}\left|(1-t)^k-\lambda(1-\alpha)\right|\left[t\left|f'(b)\right|^q+(1-t)\left|f'(a)\right|^q\right]dt\right)^{\frac{1}{q}} \quad (31)
\end{aligned}$$

By simple computation we get

$$\int_0^{1-\alpha}\left|t^k-\alpha\lambda\right|dt = \begin{cases} G_1, & \alpha\lambda\leq(1-\alpha)^k \\ G_2, & \alpha\lambda>(1-\alpha)^k \end{cases} \quad (32)$$

$$\int_{1-\alpha}^{1}\left|(1-t)^k-\lambda(1-\alpha)\right|dt = \begin{cases} H_1, & \lambda(1-\alpha)\leq\alpha^k \\ H_2, & \lambda(1-\alpha)>\alpha^k \end{cases} \quad (33)$$

By replacing (32), (33) and K_1, K_2, L_1, L_2, M_1, M_2, N_1, N_2 defined in Theorem 2.1 in (31) we complete the proof. □

References

[1] Erdélyi A, Magnus W, Oberhettinger F, Tricomi F., (1981), Higher Transcendental Functions Vol. I-III. Melbourne, Florida, Krieger Pub.

[2] Iscan, İ., (2012), A new generalization of some integral inequalities and their applications. arXiv preprint arXiv:1207.1828.

[3] Katugampola UN., (2011), New approach to a generalized fractional integral. Appl Math Comput., 218 (3); 860-865.

[4] Katugampola UN., (2014), New approach to generalized fractional derivatives. Bull Math Anal Appl., 6 (4); 1-15.

[5] Kilbas AA, Srivastava HM, Trujillo JJ., (2014), Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies; 204 pp.

[6] Kilbas AA., (2001), Hadamard-type fractional calculus. J. Korean Mathematical Society, 38 (6); 1191–1204.

[7] Peng, C., Zhou, C., & Du, T. S., (2017), Riemann-Liouville fractional Simpson's inequalities through generalized (m, h1, h2)-preinvexity. Ital. J. Pure Appl. Math, 38, 345-367.

[8] Qiu K, Wang JR., (2018), A fractional integral identity and its application to fractional Hermite-Hadamard type inequalities. Jour Interdisciplinary Math, 21 (1); 1-16.

[9] Akdemir, A.O., Ekinci A. and Set, E., (2017), Conformable Fractıonal Integrals and Related New Integral Inequalıtıes, Journal of Nonlinear and Convex Analysis, Volume 18, Number 4, 661-674.

[10] Set, E., Gözpınar, A. and Ekinci, A., (2017), Hermite-Hadamard Type Inequalities via Conformable Fractional Integrals, Acta Mathematica Universitatis Comenianae, Vol. 86, Issue 2, pp. 309-320.

[11] Usta F, Sarıkaya MZ., (2017), Some Improvements of Conformable Fractional Integral Inequalities. International Journal of Analysis and Applications, 14 (2); 162-166.

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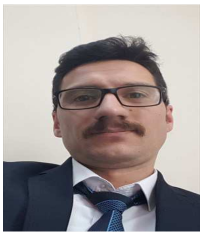

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