ON TOTAL VERTEX-EDGE DOMINATION

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ABSTRACT. In this paper we obtain an improved upper bound of total vertex edge-domination number of a tree. If T is a connected tree with order n, then $\gamma_{ve}^t(T) \leq \dfrac{m}{3}$ with $m = 6\lceil \dfrac{n}{6} \rceil$ and we characterize the trees attaining this upper bound. Furthermore we provide a characterization of trees T with $\gamma_{ve}^t(T) = \gamma^t(T)$.

Keywords: Domination, vertex-edge domination, total vertex-edge domination, total domination.

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1. INTRODUCTION

Let $G = (V, E)$ be a simple connected graph whose vertex set V and the edge set E. For the open neighborhood of a vertex v in a graph G, the notation $N_G(v)$ is used as $N_G(v) = \{u|(u,v) \in E(G)\}$ and the closed neighborhood of v is used as $N_G[v] = N_G(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighborhood of S is $N(S) = \bigcup_{v \in S} N(v)$ and the closed neighborhood of S is $N[S] = N(S) \cup S$.

In this paper, if a vertex adjacent to a support vertex different from a leaf, we name it with parent support vertex. We denote path and star of order n, with P_n and S_n respectively. The diameter of a tree is denoted with $diam(T)$.

A subset $S \subseteq V$ is a dominating set, if every vertex in G either is element of S or is adjacent to at least one vertex in S. The domination number of a graph G is denoted with $\gamma(G)$ and it is equal to the minimum cardinality of a dominating set in G. By a similar definition, a subset $S \subseteq V$ is a total domination set if every vertex of S has a neighbor in S. The total domination number of a graph G is denoted with $\gamma^t(G)$ and it is equal to the minimum cardinality of a total dominating set in G. Fundamental notions of domination theory are outlined in the book [3] and studied in thesis [6].

A vertex v ve-dominates an edge e which is incident to v, as well as every edge adjacent to e. A set $S \subseteq V$ is a ve-dominating set if every edges of a graph G are ve-dominated

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by at least one vertex of S [2, 4, 5]. The minimum cardinality of a ve-dominating set is named with ve-domination number and denoted with $\gamma_{ve}(G)$.

A subset $S \subseteq V$ is a total vertex-edge dominating set (in simply, total ve-dominating set) of G, if S is a ve-dominating set and every vertex of S has a neighbor in S [1]. The total ve-domination number of a graph G is denoted with $\gamma_{ve}^t(G)$ and it is equal to the minimum cardinality of a total ve-dominating set.

Let T be a tree and u be a vertex of T. If there exists a neighbor vertex x of u as one of the subtree of $T - ux$ is a path P_n with x a leaf, it is said that u is adjacent to the P_n [7].

In this paper, we attain a new upper bound for a connected tree with order n such taht $\gamma_{ve}^t(T) \leq \dfrac{m}{3}$ for $m = 6\lceil \dfrac{n}{6} \rceil$ and we construct the family tree $\mathcal{F}$ attaining the upper bound.

2. The Upper Bound

Observation 2.1. *For every connected graph G, $\gamma_{ve}(G) \leq \gamma_{ve}^t(G) \leq \gamma^t(G)$ [1].*

Observation 2.2. *For every connected graph G with diameter at least three, there is a $\gamma_{ve}^t(G)$-set that contains no leaf of G [1].*

Theorem 2.1. *If T is a tree with order $n \geq 4$ and $diam(T) \geq 3$ with l leaves and s support vertices, then*

$$\gamma_{ve}^t(G) \leq \frac{n - l + s}{2}$$

with equality if and only if $T^* = H \circ P_3$ for some tree H [1].

Observation 2.3. *For every connected graph G, every support vertex is contained by every total domination set [7].*

Observation 2.4. *For every connected graph with diameter at least three, there is a total domination set contains no leaf [7].*

Observation 2.5. *For every connected graph with diameter at least four, every parent support vertex is contained by every total vertex-edge domination set.*

Lemma 2.1. *For P_n the path graph with n vertex, the total domination number is obtained by [8],*

$$\gamma^t(P_n) = \begin{cases} \frac{n}{2}, & n \equiv 0 \ (mod\,4) \\ \lfloor \frac{n}{2} \rfloor + 1, & otherwise \end{cases}$$

There is no total domination set in one vertex graph, so we interest the trees which has at least two vertices.

Definition 2.1. *We introduce an integer value help us to obtain the upper bound of total vertex-edge domination number. Let m is an integer which is calculated by least integer value function such that $m = 6\lceil \frac{n}{6} \rceil$ with n is order of a tree. It is clear that if $n \equiv 0 \ (mod\,6)$, then $m = n$.*

Now we show that if T is a tree of order n, then $\gamma_{ve}^t(T) \leq \dfrac{m}{3}$ where m is introduced in Definition 2.1. In order to characterize the trees attaining the upper bound, we construct a family tree $\mathcal{F}$ of trees $T = T_k$. Let $T_1 = P_6$ and for a k positive integer, T_{k+1} is a tree recursively obtained from T_k by attaching a path P_6 by joining one of its leaves to a vertex of T_k.

Theorem 2.2. *If $T \in \mathcal{F}$, then $\gamma_{ve}^t(T) = \dfrac{n}{3}$*

Proof. We use induction by using k operations to obtain the tree T. If $T = T_1 = P_6$, then $\gamma_{ve}^t(P_6) = \dfrac{6}{3} = 2$. Now let k is a positive integer. It is assumed that the result is true for every $T' = T_k$ which is an element of $\mathcal{F}$ obtained by $k - 1$ operations. So $n' = n - 6$. Let x a leaf of $T' = T_k$ which is a path P_6 $v_1 v_2 v_3 v_4 v_5 v_6$ is attached by joining one of its leaves to it. Let D' is $\gamma_{ve}^t(T')$-set. It is easy to see that $D' \cup \{v_3, v_4\}$ is a TVEDS of T. Thus, $\gamma_{ve}^t(T) \leq \gamma_{ve}^t(T') + 2$. Furtherly, if D is a $\gamma_{ve}^t(T)$-set, $D \setminus \{v_3, v_4\}$ is a TVEDS of T'. So that, $\gamma_{ve}^t(T') \leq \gamma_{ve}^t(T) - 2$. Consequently,

$$\gamma_{ve}^t(T) = \gamma_{ve}^t(T') + 2 = \frac{n'}{3} + 2 = \frac{n-6}{3} + 2 = \frac{n}{3}.$$

Now assume that a path P_6 $v_1 v_2 v_3 v_4 v_5 v_6$ is attached to a support vertex. Let D' is $\gamma_{ve}^t(T')$-set. It is clear that $D' \cup \{v_3, v_4\}$ is a TVEDS of T. Thus, $\gamma_{ve}^t(T) \leq \gamma_{ve}^t(T') + 2$. Inversely, $D \setminus \{v_3, v_4\}$ is a TVEDS of T'. Therefore,

$$\gamma_{ve}^t(T) = \gamma_{ve}^t(T') + 2 = \frac{n'}{3} + 2 = \frac{n-6}{3} + 2 = \frac{n}{3}.$$

Now assume that a path P_6 $v_1 v_2 v_3 v_4 v_5 v_6$ is attached to a parent support vertex and this vertex is named with x. x ve-dominates the edges xv_1, v_1v_2. If D' is a $\gamma_{ve}^t(T')$-set, $D \setminus \{v_4\}$ is a vertex-edge domination set of T but it is not total. Thus we add one of the vertex of $\{v_3 v_5\}$ for obtaining the TVEDS of T. Therefore, $\gamma_{ve}^t(T) \leq \gamma_{ve}^t(T') + 2$ and inversely, $\gamma_{ve}^t(T') \leq \gamma_{ve}^t(T) - 2$. Consequently,

$$\gamma_{ve}^t(T) = \gamma_{ve}^t(T') + 2 = \frac{n'}{3} + 2 = \frac{n-6}{3} + 2 = \frac{n}{3}.$$

$\square$

Theorem 2.3. *If T is a tree of order n, then $\gamma_{ve}^t(T) \leq \dfrac{m}{3}$ such that $m = 6\lceil \dfrac{n}{6} \rceil$ with equality if and only if $T \in \mathcal{F}$.*

Proof. Let diameter of T 2. So T is a star graph and $\gamma_{ve}^t(S_n) = 2$. It is clear that if diameter of T is smaller than 5, then $\gamma_{ve}^t(T) = 2$. We assume that $diam(T) \geq 5$. In this situation number of the vertices is at least 6. We use induction and it is assumed that the result is true for every tree $T' = T_k$ with order $n' < n$ and $m' < m$.

First assume some support vertex of T, for example x, is strong. Let y be a leaf adjacent to x and $T' = T - y$. Let D' is a $\gamma_{ve}^t(T')$-set and by Observation 2.2. Let D' is also a TVEDS of T. Thus, $\gamma_{ve}^t(T) \leq \gamma_{ve}^t(T') \leq \dfrac{m'}{3} \leq \dfrac{m}{3}$. So we can assume that every support vertex is weak.

We root T at a vertex of maximum eccentricity $diam(T)$. Let t be a leaf at maximum distance from p, v be parent of t, u be parent of v, w be parent of u, s be parent of w and r be parent of s in the rooted tree. The subtree induced by a vertex x and its descendants in the rooted tree T is denoted by T_x.

Assume that some child of u is a leaf and it is denoted with x. Let $T' = T - x$. If D' is a $\gamma_{ve}^t(T')$-set, D' is also a TVEDS of T by Observation 2.2. Thus, $\gamma_{ve}^t(T) \leq \gamma_{ve}^t(T') \leq \dfrac{m'}{3} \leq \dfrac{m}{3}$.

Now assume in the children of u there is a support vertex other than v, for example x. We take $T' = T - T_v$. Let D' is a $\gamma_{ve}^t(T')$-set. D' must contain the vertex u and D' is also a TVEDS of T by Observation 2.2. Thus $\gamma_{ve}^t(T) \leq \gamma_{ve}^t(T') \leq \dfrac{m'}{3} \leq \dfrac{m}{3}$.

Now assume that $d_T(u) = 2$. First assume that w is adjacent to a leaf, say x. Let $T' = T - x$ and D' is a $\gamma_{ve}^t(T')$-set. D' is also a TVEDS of T by Observation 2.2. Thus $\gamma_{ve}^t(T) \leq \gamma_{ve}^t(T') \leq \dfrac{m'}{3} \leq \dfrac{m}{3}$.

Now assume that a P_2 or P_3 is attached by joined one of its leaves to w. Let $T' = T - T_u$ and $n' = n - 3$. If D' is a $\gamma_{ve}^t(T')$-set, w must be contained by D'. So that $D' \cup \{u\}$ is a TVEDS of T. Thus

$$\gamma_{ve}^t(T) \leq \gamma_{ve}^t(T') + 1 \leq \frac{m'}{3} + 1 = \frac{6\lceil \frac{n'}{6} \rceil}{3} + 1 = \frac{6\lceil \frac{n-3}{6} \rceil}{3} + 1 \leq \frac{6\lceil \frac{n}{6} \rceil}{3} - 2\lceil \frac{3}{6} \rceil + 1 < 6\lceil \frac{n}{6} \rceil = \frac{m}{3}.$$

Now assume that $d_T(w) = 2$. In this case first, let s is adjacent to a leaf, say x. If D' is a $\gamma_{ve}^t(T')$-set contains no leaf, it is a TVEDS of T. Therefore $\gamma_{ve}^t(T) \leq \gamma_{ve}^t(T') \leq \dfrac{m'}{3} \leq \dfrac{m}{3}$.

Now assume that a path P_2, P_3 or P_4 is attached to w by an edge. Let $T' = T - T_w$ and D' is a $\gamma_{ve}^t(T')$-set. Thus $n' = n - 4$ and

$$\gamma_{ve}^t(T) \leq \gamma_{ve}^t(T') + \gamma_{ve}^t(P_4) = \frac{m'}{3} + \gamma_{ve}^t(P_4) = \frac{6\lceil \frac{n'}{6} \rceil}{3} + \frac{6\lceil \frac{4}{6} \rceil}{3} = \frac{6\lceil \frac{n-4}{6} \rceil}{3} + \frac{6\lceil \frac{4}{6} \rceil}{3} < 6\lceil \frac{n}{6} \rceil = \frac{m}{3}.$$

Now assume that $d_T(s) = 2$. Let $T' = T - T_s$. So we have $n' = n - 5$. If D' is a $\gamma_{ve}^t(T')$-set, the total vertex-edge domination number of T is

$$\gamma_{ve}^t(T) \leq \gamma_{ve}^t(T') + \gamma_{ve}^t(P_5) = \frac{6\lceil \frac{n'}{6} \rceil}{3} + \frac{6\lceil \frac{5}{6} \rceil}{3} = \frac{6\lceil \frac{n-5}{6} \rceil}{3} + \frac{6\lceil \frac{5}{6} \rceil}{3} < 6\lceil \frac{n}{6} \rceil = \frac{m}{3}.$$

Now assume that $d_T(r) = 2$ and we take $T' = T - T_r$. We have $n' = n - 6$. If $n' = 1$, then $T = P_7$ and we obtain $\gamma_{ve}^t(P_7) = 3 \leq 4$. We assume that $n' \geq 2$. If D' is a $\gamma_{ve}^t(T')$-set, $D' \cup \{w, u\}$ is be a $\gamma_{ve}^t(T)$-set. Thus,

$$\gamma_{ve}^t(T) \leq \gamma_{ve}^t(T') + 2 = \frac{m'}{3} + 2 = \frac{6\lceil \frac{n'}{6} \rceil}{3} + 2 = \frac{6\lceil \frac{n-6}{6} \rceil}{3} + 2 = \frac{m}{3}.$$

Our upper bound is sharp and best possible not only trees but also other graphs. We use H graphs which are consisted from two n vertex paths connected with an edge, to see this fact with two cases.

In the first case, we use corona product of H graphs with P_2. For second case we use 2-corona of the H graphs for every $x \in H$ we add two vertices u and v with the edges xu and uv.

For the first case we obtain a polycyclic graph G has n triangular and $6n$ vertices. Thus $\gamma_{ve}^t(G) = \dfrac{6n}{3} = 2n$. In the second case $\gamma_{ve}^t(G) = 2n$. Furthermore the number of the leaves is equal to the number of support vertices. Thus $\gamma_{ve}^t(G) = \dfrac{6n}{2} = 3n$ by upper bound defined in [1]. $\gamma_{ve}^t(G) = \dfrac{6n}{3} = 2n$ by our upper bound.

This fact is current for paths too. For the paths the number of the leaves is equal to the number of support vertices. Thus $\gamma_{ve}^t(P_n) = \dfrac{n}{2}$ by upper bound defined in [1] and by

our upper bound $\gamma_{ve}^t(P_n) = \dfrac{6\lceil\dfrac{n}{6}\rceil}{3} = 2\lceil\dfrac{n}{6}\rceil$. If these bounds are checked, it was seen that our bound is efficient and best possible.

$\square$

3. The trees with $\gamma_{ve}^t(T) = \gamma^t(T)$

Now we find a partial answer for trees which is mentioned in [1] with Problem 4.2 for the graphs which are characterized by the equation $\gamma_{ve}^t(G) = \gamma^t(G)$.

First we found the paths P_n of order n attaining the equality $\gamma_{ve}^t(P_n) = \gamma^t(P_n)$ and construct a family $\mathcal{T}$ of these paths. Because these paths are the first members of the family T. We use Theorem 2.3 and Lemma 2.1. We have to look into four situations;

   i) $n \equiv 0 \,(mod4, \; mod6)$,

  ii) $n \equiv 0 \,(\text{only } mod4)$

 iii) $n \equiv 0 \,(\text{only } mod6)$

 iv) n is not a multiple 4 and 6.

For the first situation,

$$\frac{n}{3} = \frac{n}{2} \Rightarrow n = 0.$$

For (ii) $n \equiv 0 \,(mod4)$,

$$2\lceil\frac{n}{6}\rceil = \frac{n}{2} \Rightarrow \frac{n}{4} = \lceil\frac{n}{6}\rceil \Rightarrow \frac{n}{4} - 1 < \frac{n}{6} \leq \frac{n}{4} \Rightarrow 0 \leq n < 12.$$

n can be 4 and 8 for this situation and P_4 and P_8 attain the equality such that $\gamma_{ve}^t(P_4) = \gamma^t(P_4) = 2$, $\gamma_{ve}^t(P_8) = \gamma^t(P_8) = 4$.

For (iii) $n \equiv 0 \,(mod6)$,

$$\frac{n}{3} = \lfloor\frac{n}{2}\rfloor + 1 \Rightarrow \frac{n}{3} - 1 = \lfloor\frac{n}{2}\rfloor \Rightarrow \frac{n}{3} - 1 \leq \frac{n}{2} < \frac{n}{3} \Rightarrow -6 \leq n < 0.$$

and there is no positive solve.

For the last situation,

$$2\lceil\frac{n}{6}\rceil = \lfloor\frac{n}{2}\rfloor + 1.$$

If it is checked, this equation is attained for $n = 2, 3, 7$ by using the upper bound for total vertex-edge domination. But for $n = 7$ $\gamma_{ve}^t(P_7) = 3 \neq \gamma^t(P_7) = 4$.

Consequently the equation $\gamma_{ve}^t(P_n) = \gamma^t(P_n)$ is attained for only the paths P_2, P_3, P_4, P_8.

Now we construct a family tree $\mathcal{T}$ of trees $T = T_k$. Let $T_1 \in \{P_2, P_3, P_4, P_8\}$ and for a k positive integer, T_{k+1} is a tree recursively obtained from T_k by one of the following two operations,

Operation O_1: Add a vertex with an edge to any support vertex of $T = T_k$.

Operation O_2: Add a vertex with an edge to a vertex of $T = T_k$ adjacent to a path P_2.

Theorem 3.1. *Let T be a tree. If $T \in \mathcal{T}$, then $\gamma_{ve}^t(T) = \gamma^t(T)$.*

Proof. We use induction on the number of k operations which are used to construct the tree T. If $T_1 \in \{P_2, P_3, P_4, P_8\}$, then $\gamma_{ve}^t(P_2) = \gamma^t(P_2) = 2$, $\gamma_{ve}^t(P_3) = \gamma^t(P_3) = 2$, $\gamma_{ve}^t(P_4) = \gamma^t(P_4) = 2$ and $\gamma_{ve}^t(P_8) = \gamma^t(P_8) = 4$.

Assume that the argument is true for every $T' = T_k$ of the family $\mathcal{T}$ obtained by $k-1$ operations and we want to show $T = T_{k+1} \in \mathcal{T}$.

First assume that T is obtained from T' by operation O_1. Let D' is a TDS of T'. It is easy to see that D' is also a TDS of T by observation 2.4. Thus, $\gamma^t(T) \leq \gamma^t(T')$. Obviously, $\gamma_{ve}^t(T') \leq \gamma_{ve}^t(T)$. By induction hypothesis, $\gamma^t(T) \leq \gamma^t(T') = \gamma_{ve}^t(T') \leq \gamma_{ve}^t(T)$ and by Observation 2.1 $\gamma^t(T) \geq \gamma_{ve}^t(T)$ it is obtained that $\gamma^t(T) = \gamma_{ve}^t(T)$.

Now First assume that T is obtained from T' by operation O_2. Let x be a vertex of $T' = T_k$ which is adjacent to a P_2 and the P_2 be yz as y a neighbor of x with degree two.

Let $D^{'}$ is a TDS of $T^{'}$. If we attach a vertex to $x, y \in D^{'}$. y has to be dominated, thus $x \in D^{'}$. Therefore $D^{'}$ is a TDS of T and $\gamma^t(T) \leq \gamma^t(T^{'})$. Obviously, $\gamma_{ve}^t(T^{'}) \leq \gamma_{ve}^t(T)$. Thus $\gamma^t(T) \leq \gamma^t(T^{'}) = \gamma_{ve}^t(T^{'}) \leq \gamma_{ve}^t(T)$ and using the fact $\gamma^t(T) \geq \gamma_{ve}^t(T)$ we obtain $\gamma^t(T) = \gamma_{ve}^t(T)$.

$\square$

Remark 3.1. *If $T \in \mathcal{T}$, then T becomes a star graph, a bistar graph or a combination of two bistar graph by an edge between any two leaves of these bistars which we name it double bistar graph.*

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