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BIGEOMETRIC INTEGRAL CALCULUS

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Abstract. Objective of this paper is to discuss about the properties of indefinite and definite bigeometric integration. We also discuss about some applications of bigeometric integration.

Bigeometric differentiation; G-derivative; geometric real numbers; geometric arithmetic.

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1. Introduction

In 1967, Michael Grossman and Robert Katz introduced first system of non-Newtonian calculus which we call 'Multiplicative Calculus'. Later, they had created infinite family of non-Newtonian calculi and published the book "Non-Newtonian Calculus" [15] in 1972. After then, further development of non-Newtonian calculi and their applications are seen in Tekin and Başar [24], Çakmak and Başar [10], Stanley [23], Jane Grossman [16], Grosssman et al. [17], Grossman et al. [18], Grossman [19], Campbell [12], Michael Coco [13], Córdova-Lepe [21], Bashirov et al. [2, 3], Spivey [22] etc. . An extension of geometric calculus to functions of complex variables is handled by Bashirov and Rıza [1], Çakmak and Başar [8], Kadak and Özlük [20], Uzer [26], Türkmen and Başar [25]. In [9], Çakmak and Başar constructed the field $\mathbb{C}^*$ of $*$-complex numbers and the concept of $*$-metric. Based on multiplicative calculus matrix transformations in sequence spaces are studied and characterized by Çakmak and Başar [11].

Bigeometric-calculus is one type of non-Newtonian calculus in which deviations are measured by ratios instead of differences. In this article we discuss about bigeometric integration, but often the term 'G-integration' will be used instead of 'bigeometric integration'. Reason for using the term G-integration is that many researchers have been developing bigeometric calculus taking arithmetic increment, i.e. linear increments to the argument as $a, a+h, a+2h, a+3h$ etc, but we are trying to develop it, taking geometric increment, i.e. product increments to the argument as a, ah, ah^2, ah^3 etc. That is why we are using the term 'G-calculus' to remove the confusion among readers. And the reason

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for taking geometric increments is to develop Grossman and Katz's [15] bigeometric calculus with the aid of geometric arithmetic independently. If we take linear increments to the argument, sometimes it is troublesome to deduce some expressions in terms of geometric arithmetic independently. Of course, often we use products instead of geometric sum in practice, still all the formulae of G-calculus can be expressed in terms of geometric arithmetic.

2. Geometric arithmetic and geometric real field

Bigeometric calculus is based on the geometric arithmetic $(\mathbb{R}(G), \oplus, \ominus, \odot, \oslash, <)$, where $\mathbb{R}(G)$ is the set of geometric real numbers defined by C. Türkmen and F. Başar [25] as

$$\mathbb{R}(G) = \{e^x : x \in \mathbb{R}\} = \mathbb{R}^+ \backslash \{0\}.$$

We refer Grossman and Katz [15] or Boruah and Hazarika [6] for better knowing about geometric arithmetic and its generator.

Remark 2.1. $(\mathbb{R}(G), \oplus, \odot)$ *is a field with geometric zero* 1 *and geometric identity* e. *But* $(\mathbb{C}(G), \oplus, \odot)$ *is not a field, however, geometric binary operation* $\odot$ *is not associative in* $\mathbb{C}(G)$. *For, we take* $x = e^{1/4}, y = e^4$ *and* $z = e^{(1+i\pi/2)} = ie$. *Then*

$$(x \odot y) \odot z = e \odot z = z = ie \text{ but } x \odot (y \odot z) = x \odot e^4 = e.$$

3. Definitions and results

3.1. **G-derivative.** In [6] we defined G-differentiation of a bi-positive function f as

$$\frac{d^G f}{dx^G} = f^G(x) = {}_G\lim_{h \to 1} \frac{f(x \oplus h) \ominus f(x)}{h}{}_G = \lim_{h \to 1} \left[\frac{f(hx)}{f(x)}\right]^{\frac{1}{\ln h}} \quad \text{for } h \in \mathbb{R}(\mathbb{G}).$$

The n^{th} G-derivative of $f(x)$ is denoted by $f^{[n]}(x)$. The relation between G-derivative and classical derivative is

$$f^G(x) \text{ or } f^{[1]}(x) = e^{x\frac{f'(x)}{f(x)}}.$$

3.2. Some standard G-derivatives:

(1) $\frac{d^G}{dx^G}(c) = 1.$
(2) $\frac{d^G}{dx^G}(cf(x)) = f^G(x).$
(3) $\frac{d^G}{dx^G}([f(x)]^c) = (f^G(x))^c.$
(4) $\frac{d^G}{dx^G}(f(x).g(x)) = f^G(x).g^G(x).$
(5) $\frac{d^G}{dx^G}\left(\frac{f(x)}{g(x)}\right) = f^G(x) \Big/ g^G(x).$
(6) $\frac{d^G}{dx^G}(\sin x) = e^{x\cot x}.$
(7) $\frac{d^G}{dx^G}(\csc x) = e^{-x\cot x}.$
(8) $\frac{d^G}{dx^G}(\cos x) = e^{-x\tan x}.$
(9) $\frac{d^G}{dx^G}(\sec x) = e^{x\tan x}.$
(10) $\frac{d^G}{dx^G}(\tan x) = e^{x\sec x\csc x}.$
(11) $\frac{d^G}{dx^G}(\cot x) = e^{-x\sec x\csc x}.$
(12) $\left(f^{\ln g}\right)^G = \left(f^G\right)^{\ln g}.\left(g^G\right)^{\ln f}.$
(13) $\left(e^f\right)^G = e^{xf'}.$
(14) $(f+g)^G = \left(f^G\right)^{\frac{f}{f+g}}.\left(g^G\right)^{\frac{g}{f+g}}.$
(15) $\frac{d^G}{dx^G}(x^n) = e^n$ and so $\frac{d^G}{dx^G}\left(\frac{1}{x}\right) = \frac{1}{e}.$
(16) $\frac{d^G}{dx^G}(f \circ g)(x) = \left[f^G(x)\right]^{g'(x)}.$

4. G-Integration

G-integration is the inverse operation of the G-differentiation. We'll denote the G-integration by the symbol ${}_G\!\int$ and G-integration of $f(x)$ is represented by ${}_G\!\int f(x)\odot dx^G$ or ${}_G\!\int [f(x)]^{\ln(dx^G)}$. That is, if $\frac{d^G}{dx^G}(f(x)) = F(x)$, then we write ${}_G\!\int F(x)\odot dx^G = f(x)$. We have,

$$\frac{d^G}{dx^G}(cf(x)) = \frac{d^G}{dx^G}(f(x)) = F(x) \text{ (say)}.$$

So, we write

$${}_G\!\int F(x)\odot dx^G = cf(x) \text{ where } c \text{ is called constant of G-integration.}$$

Theorem 4.1. *If $\int$ is the ordinary integration and ${}_G\!\int$ is the G-integration, then*

$${}_G\!\int f(x)\odot dx^G = c\, e^{\int \frac{\ln(f(x))}{x} dx},$$

where c is a constant, which will be called G-integrating constant.

Proof. We know that, if $f(x) = e^{\frac{xF'(x)}{F(x)}}$ then

$${}_G\!\int f(x)\odot dx^G = c_1 F(x).$$

Now, taking natural logarithm to both sides and then ordinary integration gives

$$\int \frac{\ln(f(x))}{x}\, dx = \ln c_2 + \int \frac{F'(x)}{F(x)}\, dx \text{ where, } \ln c_2 \text{ is the constant of integration.}$$
$$= \ln c_2 + \ln\{F(x)\}.$$
$$\Rightarrow \int \frac{\ln(f(x))}{x}\, dx = \ln\{c_2.F(x)\}.$$
$$\Rightarrow F(x) = \frac{1}{c_2} e^{\int \frac{\ln(f(x))}{x}\, dx}.$$
$$\Rightarrow {}_G\!\int f(x)\odot dx^G = c_1 F(x) = \frac{c_1}{c_2}\, e^{\int \frac{\ln(f(x))}{x}\, dx}.$$
$$\Rightarrow {}_G\!\int f(x)\odot dx^G = c\, e^{\int \frac{\ln(f(x))}{x}\, dx} \quad \text{where } c = \frac{c_1}{c_2}.$$

□

4.1. Some standard G-integrals:

Following are some standard results derived directly from standard results of G-differential calculus. Throughout the paper, we shall not multiply the constant of integration except some special cases. It will remain understood that we have multiplied it.

(1) ${}_G\!\int 1\odot dx = c.$

(2) ${}_G\!\int e^n\odot dx = x^n.$

(3) ${}_G\!\int e^x\odot dx = e^x.$

(4) ${}_G\!\int x^n\odot dx = e^{\frac{n\ln^2 x}{2}}.$

(5) ${}_G\int e^{x\cot x} \odot dx^G = \sin x.$

(6) ${}_G\int e^{-x\cot x} \odot dx^G = \csc x.$

(7) ${}_G\int e^{-x\tan x} \odot dx^G = \cos x.$

(8) ${}_G\int e^{x\tan x} \odot dx^G = \sec x.$

(9) ${}_G\int e^{x\sec x\csc x} \odot dx^G = \tan x.$

(10) ${}_G\int e^{-x\sec x\csc x} \odot dx^G = \cot x.$

4.2. Integration by transforming the function to the form $e^{x\frac{f'(x)}{f(x)}}$:
We know that ${}_G\int e^{x\frac{f'(x)}{f(x)}} \odot dx^G = f(x)$. So, if we can transform a given function to the form $e^{x\frac{f'(x)}{f(x)}}$, we can find its integral easily. For example, we have

(1) ${}_G\int e^{\left(\frac{1}{\ln x}\right)} \odot dx^G = \ln x$ by taking $f(x) = \ln x$.

(2) ${}_G\int e^{\left(\frac{6x^2-7x}{3x^2-7x+2}\right)} \odot dx^G$ by taking $f(x) = f(x) = 3x^2 - 7x + 2$.

4.3. Integration by the relation between G-integral and ordinary integral:
In Theorem 4.1, we have established that

$$ {}_G\int f(x) \odot dx^G = e^{\int \frac{\ln(f(x))}{x} dx}. \tag{1}$$

With the help of this relation, we can find the G-integration of a given function.

Example 4.1. *Evaluate* ${}_G\int x^x \odot dx^G$.

Sol.

$$ {}_G\int x^x \odot dx^G = e^{\int \frac{\ln x^x}{x} dx} = e^{\int \ln x\, dx} = e^{(x\ln x - x)} = x^x e^{-x}.$$

Example 4.2. *Evaluate* ${}_G\int \ln x \odot dx^G$.

Sol.

$$ {}_G\int \ln x \odot dx^G = e^{\int \frac{\ln(\ln x)}{x}\, dx} = e^{[\{\ln(\ln x)-1\}.\ln x]} = x^{\{\ln(\ln x)-1\}}.$$

4.4. Properties of G-integration:

Property 1: ${}_G\int [f(x)]^c \odot dx^G = \left[{}_G\int f(x) \odot dx^G\right]^c.$

Property 2: ${}_G\int f(x).g(x) \odot dx^G = {}_G\int f(x) \odot dx^G \, . \, {}_G\int g(x) \odot dx^G.$

Property 3: ${}_G\int \frac{f(x)}{g(x)} \odot dx^G = {}_G\int f(x) \odot dx^G \Big/ {}_G\int g(x) \odot dx^G.$

These properties are obvious from standard derivatives. With the help of these properties we can integrate complicated functions easily. For, we have ${}_G\int x \odot dx^G = e^{\frac{\ln^2 x}{2}}$. Hence by the first property,

(i) ${}_G\int x^n \odot dx^G = \left[{}_G\int x \odot dx^G\right]^n = e^{\frac{n\ln^2 x}{2}}.$

(ii) ${}_G\int \sqrt{x} \odot dx^G = \left[{}_G\int x \odot dx^G\right]^{\frac{1}{2}} = e^{\frac{\ln^2 x}{4}}.$

Similarly, as ${}_G\!\int e^x \odot dx^G = e^x$, we have ${}_G\!\int e^{ax} \odot dx^G = \left[{}_G\!\int e^x \odot dx^G\right]^a = e^{ax}$.

Property 4: ${}_G\!\int e^{xf(x)} \odot dx^G = e^{\int f(x)dx}$.

In particular,

$$ {}_G\!\int e^{f(x)} \odot dx^G = e^{\int \frac{f(x)}{x} dx}. \tag{2}$$

Similarly, if a is any positive constant, then

$$ {}_G\!\int a^{f(x)} \odot dx^G = a^{\int \frac{f(x)}{x} dx}.$$

Using this property, we can find G-integration of exponential functions easily. Some uses of this property are given below.

Example 4.3. *Evaluate* ${}_G\!\int e^{(3x^2+5x+2)} \odot dx^G$.

Sol.

$$ {}_G\!\int e^{(3x^2+5x+2)} \odot dx^G = e^{\int \frac{3x^2+5x+2}{x} dx} = e^{\int (3x+5+\frac{2}{x})\, dx} = e^{(\frac{3}{2}x^2+5x+2\ln x)}.$$

Similarly, using the Property 4, we have

$$ {}_G\!\int e^{x\cos x} \odot dx^G = e^{\sin x}. \qquad {}_G\!\int e^{x\sin x} \odot dx^G = e^{-\cos x}.$$

$$ {}_G\!\int e^{x\tan x} \odot dx^G = \sec x. \qquad {}_G\!\int e^{x\cot x} \odot dx^G = \sin x.$$

$$ {}_G\!\int e^{x\sec x} \odot dx^G = \tan\left(\frac{x}{2}+\frac{\pi}{4}\right). \qquad {}_G\!\int e^{x\csc x} \odot dx^G = \tan\left(\frac{x}{2}\right).$$

5. Definite Bigeometric Integral

If ${}_G\!\int f(x) \odot dx^G = F(x)$, then

$$ {}_G\!\int_a^b f(x) \odot dx^G = [F(x)]_a^b = \frac{F(b)}{F(a)}$$

will be called definite G-integral of $f(x)$ between the limits a and b. Usually, we define definite G-integral for function f which is positive valued in an positive interval $[a, b]$. Of course, for some functions definite G-integrals exist in negative intervals too.

If we replace $F(x)$ by $cF(x)$ as the value of the integral, we have

$$ {}_G\!\int_a^b f(x) \odot dx^G = [cF(x)]_a^b = \frac{cF(b)}{cF(a)} = \frac{F(b)}{F(a)}.$$

Thus the value does not depend on the constant in evaluation of definite G-integral.

Note: It is obvious from the definition that

$$ {}_G\!\int_a^b f(x) \odot dx^G = e^{\int_a^b \frac{\ln f(x)}{x}\, dx}. \tag{3}$$

Example 5.1. ${}_G\!\int_2^5 e^{x^2} \odot dx = e^{\frac{21}{2}}$.

Example 5.2. ${}_G\!\int_0^{\frac{\pi}{2}} e^{x\sin x} \odot dx^G = e$.

5.1. Properties of definite G-integral:

Property 1: We have

$$ {}_G\int_a^b f(x) \odot dx^G = {}_G\int_a^b f(t) \odot dt^G. $$

Proof. Let ${}_G\int f(x) \odot dx^G = F(x)$. Then ${}_G\int f(t) \odot dt^G = F(t)$. Now

$$ {}_G\int_a^b f(x) \odot dx^G = [F(x)]_a^b = \frac{F(b)}{F(a)}. $$

$$ {}_G\int_a^b f(t) \odot dt^G = [F(t)]_a^b = \frac{F(b)}{F(a)} = {}_G\int_a^b f(x) \odot dx^G . $$

□

Property 2: We have

$$ {}_G\int_a^b f(x) \odot dx^G = 1 \Big/ {}_G\int_b^a f(x)\odot dx^G. $$

Proof. Let ${}_G\int f(x) \odot dx^G = F(x)$. Then

$$ {}_G\int_a^b f(x) \odot dx^G = \frac{F(b)}{F(a)} = 1 \Big/ \frac{F(a)}{F(b)} = 1 \Big/ {}_G\int_b^a f(x)\odot dx^G. $$

□

Property 3: We have

$$ {}_G\int_a^b f(x) \odot dx^G = {}_G\int_a^c f(x) \odot dx^G \quad . {}_G\int_c^b f(x) \odot dx^G \text{ where } a < c < b. $$

Proof. Let ${}_G\int f(x) \odot dx^G = F(x)$. Then

$$ R.H.S. = {}_G\int_a^c f(x) \odot dx^G \quad . {}_G\int_c^b f(x) \odot dx^G $$

$$ = [F(x)]_a^c . [F(x)]_c^b = \frac{F(c)}{F(a)} . \frac{F(b)}{F(c)} = \frac{F(b)}{F(a)} = {}_G\int_a^b f(x) \odot dx^G = L.H.S. $$

□

Property 4: We have

$$ {}_G\int_1^a f(x) \odot dx^G = {}_G\int_1^a f\left(\frac{a}{x}\right) \odot dx^G . $$

Proof. Let ${}_G\int f(x) \odot dx^G = F(x)$. Then

$$ {}_G\int_1^a f(x) \odot dx^G = \frac{F(a)}{F(1)}. $$

We know that,

$\left(\frac{a}{b}\right) \odot y = \frac{a\odot y}{b\odot y}$, so $\left(\frac{1}{e}\right) \odot dx^G = \frac{1\odot dx^G}{e\odot dx^G} = \frac{1}{dx^G}$. Also,

$$ \frac{d^G}{dx^G}(x^n) = e^n \text{ and so } \frac{d^G}{dx^G}\left(\frac{1}{x}\right) = \frac{1}{e}. $$

Now let $\frac{a}{x} = y$. By G-differentiation, we get $\frac{1}{dx^G} = dy^G$. Also, $x = a \Rightarrow y = 1$ and $x = 1 \Rightarrow y = a$. Therefore

$$\begin{aligned} {}_G\!\int_1^a f\left(\frac{a}{x}\right) \odot dx^G &= {}_G\!\int_a^1 f(y) \odot \frac{1}{dy^G} \\ &= {}_G\!\int_a^1 \frac{1}{f(y)} \odot dy^G, \quad \text{since } \frac{a}{b} \odot \frac{c}{d} = \frac{b}{a} \odot \frac{d}{c} \\ &= {}_G\!\int_1^a f(y) \odot dy^G = {}_G\!\int_1^a f(x) \odot dx^G. \end{aligned}$$

□

Note: Since ${}_G\!\int 1 \odot dy^G = c$ (constant) and hence ${}_G\!\int_a^1 1 \odot dy^G = \frac{c}{c} = 1$.

Property 5: We have

$$ {}_G\!\int_a^1 f(x) \odot \frac{1}{dx^G} = {}_G\!\int_1^a f(x) \odot dx^G. $$

Proof.

$$ {}_G\!\int_a^1 f(x) \odot \frac{1}{dx^G} = {}_G\!\int_a^1 \frac{1}{f(x)} \odot dx^G = 1 \Big/ {}_G\!\int_a^1 f(x) \odot dx^G = {}_G\!\int_1^a f(x) \odot dx^G. $$

□

Definition 5.1 (Geometric odd function and even function)**.** *A positive valued function f defined in a positive interval is said to be geometric odd function if*

$$ f\left(\frac{1}{x}\right) = \frac{1}{f(x)} $$

and is said to be geometric even function if

$$ f\left(\frac{1}{x}\right) = f(x). $$

For example, the function $f(x) = x^n, n \in \mathbb{N}$ is a geometric odd function, since

$$ f\left(\frac{1}{x}\right) = \left(\frac{1}{x}\right)^n = \frac{1}{x^n} = \frac{1}{f(x)}. $$

The function $g(x) = x + \frac{1}{x}$ is a geometric even function, since

$$ g\left(\frac{1}{x}\right) = \frac{1}{x} + x = g(x). $$

Property 6: We have

$$ {}_G\!\int_{1/a}^a f(x) \odot dx^G = 1 \text{ or } \left[{}_G\!\int_1^a f(x) \odot dx^G \right]^2. $$

according as $f(x)$ is geometric odd or geometric even function, respectively.

Proof. We have

$$ {}_G\!\int_{1/a}^a f(x) \odot dx^G = {}_G\!\int_{1/a}^1 f(x) \odot dx^G \,.\, {}_G\!\int_1^a f(x) \odot dx^G. \tag{4} $$

Let I$= {}_G\int_{1/a}^{1} f(x)\odot dx^G = {}_G\int_{1/a}^{1} f(x)\odot dx^G$. In the integral, we put $x=\frac{1}{t}$. Then $dx^G=\frac{1}{dt^G}$. Hence

$$\mathrm{I} = {}_G\int_{a}^{1} f\left(\frac{1}{t}\right)\odot \frac{1}{dt^G} = {}_G\int_{1}^{a} f\left(\frac{1}{t}\right)\odot dt^G = {}_G\int_{1}^{a} f\left(\frac{1}{x}\right)\odot dx^G.$$

Now, two cases arise

Case 1: If f is odd, then

$$I = {}_G\int_{1}^{a} \frac{1}{f(x)}\odot dx^G = 1\Big/ {}_G\int_{1}^{a} f(x)\odot dx^G.$$

Then, equation (4) gives

$${}_G\int_{1/a}^{a} f(x)\odot dx^G = 1.$$

Case 2: If f is even, then

$$I = {}_G\int_{1}^{a} f(x)\odot dx^G.$$

Then, equality (4) gives

$${}_G\int_{1/a}^{a} f(x)\odot dx^G = {}_G\int_{1}^{a} f(x)\odot dx^G \,.\, {}_G\int_{1}^{a} f(x)\odot dx^G = \left[{}_G\int_{1}^{a} f(x)\odot dx^G\right]^2.$$

□

Property 7: We have

$${}_G\int_{1}^{a^2} f(x)\odot dx^G = \begin{cases} \left[{}_G\int_{1}^{a} f(x)\odot dx^G\right]^2 & \text{if } f\left(\frac{a^2}{x}\right)=f(x) \\ 1 & \text{if } f\left(\frac{a^2}{x}\right)=\frac{1}{f(x)}. \end{cases} \tag{5}$$

Proof.

$${}_G\int_{1}^{a^2} f(x)\odot dx^G = {}_G\int_{1}^{a} f(x)\odot dx^G \,.\, {}_G\int_{a}^{a^2} f(x)\odot dx^G. \tag{6}$$

Case 1: Let $f\left(\frac{a^2}{x}\right)=f(x)$. Then the second integral of (6) becomes

$${}_G\int_{a}^{a^2} f(x)\odot dx^G = {}_G\int_{a}^{a^2} f\left(\frac{a^2}{x}\right)\odot dx^G = {}_G\int_{1}^{a} f(x)\odot dx^G.$$

Then (6) gives

$${}_G\int_{1}^{a^2} f(x)\odot dx^G = \left[{}_G\int_{1}^{a} f(x)\odot dx^G\right]^2.$$

Case 2: Let $f\left(\frac{a^2}{x}\right)=\frac{1}{f(x)}$. Then the second integral of (6) becomes

$${}_G\int_{a}^{a^2} f(x)\odot dx^G = {}_G\int_{a^2}^{a} \frac{1}{f(x)}\odot dx^G = 1\Big/ {}_G\int_{1}^{a} f(x)\odot dx^G.$$

Then (6) gives

$$ {}_G\int_{1}^{a^2} f(x) \odot dx^G = 1. $$

□

5.2. Definite Bigeometric Integral as a Limit of geometric Sum: So far G-integration has been defined as the inverse process of G-differentiation. But it is possible to regard a definite G-integral as the limit of the product(or geometric sum) of certain number of terms, when the number of terms tends to infinite and each term tends to 1.

Definition 5.2. *Let f be a single valued continuous bipositive function defined in the positive interval $[a,b]$, where $b > a$. Let the interval $[a,b]$ be divided into a geometric progression of $n+1$ terms with common ratio h to get the partition $P = \{a = x_0, x_1 = ah, x_2 = ah^2, \cdots, x_n = ah^n = b\}$, where $h = \left(\frac{b}{a}\right)^{1/n}$. Then we define*

$$ {}_G\int_{a}^{b} f(x) \odot dx^G = {}_G\lim h \odot \left[f(a) \oplus f(ah) \oplus f(ah^2) \oplus \cdots \oplus f(ah^{n-1})\right] \tag{7} $$

$$ = \lim \left[f(a).f(ah).f(ah^2) \cdots f(ah^{n-1})\right]^{\ln h} \tag{8} $$

when $n \to \infty, h \to 1$ and $h^n \to \frac{b}{a}$.

$$ \therefore \; {}_G\int_{a}^{b} f(x) \odot dx^G = \lim_{h\to 1} \left[\prod_{r=0}^{n-1} f(ah^r)\right]^{\ln h} \text{ or } \lim_{n\to\infty} \left[\prod_{r=0}^{n-1} f(ah^r)\right]^{\ln h} $$

where $n \to \infty$ as $h \to 1$ and h^n remains equal to $\frac{b}{a}$.

It is to be noted that this definition is equivalent to the definition of bigeometric integral given by Grossman in [14, p. 31].

Example 5.3. *Find* ${}_G\int_{a}^{b} e^{x^2} \odot dx^G$ *using definition of G-integral as the limit of a product.*

Sol. From the definition of a definite G-integral as the limit of a sum, we know that

$$ {}_G\int_{a}^{b} e^{x^2} \odot dx^G = \lim_{h\to 1} \left[f(a).f(ah).f(ah^2).\cdots.f(ah^{n-1})\right]^{\ln h} $$

where $n \to \infty$ as $h \to 1$ and $h^n \to \frac{b}{a}$.

Here $f(x) = e^{x^2}$; $\therefore f(a), f(ah), f(ah^2)$, etc. will be $e^{a^2}, e^{a^2h^2}, e^{a^2h^4}, \cdots$, respectively.

$$ {}_G\int_{a}^{b} e^{x^2} \odot dx^G = \lim_{h\to 1} \left[e^{a^2}.e^{a^2h^2}.e^{a^2h^4} \cdots e^{a^2h^{2(n-1)}}\right]^{\ln h} \text{ where } n \to \infty \text{ as } h \to 1 \text{ and } h^n \to \frac{b}{a} $$

$$ = \lim_{h\to 1} \left[e^{a^2(1+h^2+h^4+\cdots+h^{2(n-1)})}\right]^{\ln h} $$

$$ = \lim_{h\to 1} \left[e^{a^2.\frac{h^{2n}-1}{h^2-1}}\right]^{\ln h} = \lim_{h\to 1} \left[e^{a^2.\frac{(b/a)^2-1}{h^2-1}}\right]^{\ln h} \quad \because h^n = \frac{b}{a} $$

$$ = \lim_{h\to 1} \left[e^{\frac{b^2-a^2}{h^2-1}}\right]^{\ln h} = \lim_{h\to 1} \left[h^{\frac{b^2-a^2}{h^2-1}}\right] \quad \because (e^m)^{\ln x} = x^m $$

$$ = \lim_{h\to 1} \left[h^{\frac{1}{h-1}}\right]^{\frac{b^2-a^2}{h+1}} = e^{\frac{b^2-a^2}{2}}, \qquad \because \lim_{h\to 1} h^{\frac{1}{h-1}} = e. $$

Example 5.4. *Evaluate* ${}_G\int_a^b e^m \odot dx^G$ *using G-integral as the limit of a product.*

Sol. From the definition of a definite G-integral as the limit of a sum, we know that

$$ {}_G\int_a^b e^m \odot dx^G = \lim_{h\to 1}\left[f(a).f(ah).f(ah^2).\cdots.f(ah^{n-1})\right]^{\ln h} $$

Here $f(x) = e^m$(a constant); so $f(a) = f(ah) = f(ah^2) = \cdots = f(ah^{n-1}) = e^m$. Hence

$$ {}_G\int_a^b e^m \odot dx^G = \lim_{h\to 1}\left[\underbrace{e^m.e^m.e^m\cdots e^m}_{\text{n times}}\right]^{\ln h} \quad \text{where } n\to\infty \text{ as } h\to 1 \text{ and } h^n \to \frac{b}{a} $$

$$ = \lim_{h\to 1}\left[e^{mn}\right]^{\ln h} = \lim_{h\to 1} e^{mn\ln h} = \lim_{h\to 1} h^{mn} = \lim_{h\to 1}\left[h^n\right]^m = \left[\frac{b}{a}\right]^m = \frac{b^m}{a^m}. $$

Thus, ${}_G\int_a^b e^m \odot dx^G = \frac{b^m}{a^m}$. Also for verification, we can solve the problem by direct method without using the limit of product as follows:

$$ {}_G\int_a^b e^m \odot dx^G = [x^m]_a^b = \frac{b^m}{a^m} \quad \because {}_G\int e^m \odot dx^G = x^m. $$

Example 5.5. *Find* ${}_G\int_a^b x \odot dx^G$ *from definition of G-integral as the limit of a product.*

Sol. From the definition of a definite G-integral as the limit of a sum, we have

$$ {}_G\int_a^b x \odot dx^G = \lim_{h\to 1}\left[f(a).f(ah).f(ah^2).\cdots.f(ah^{n-1})\right]^{\ln h} $$

Here $f(x) = x$; $\therefore f(ah^r) = ah^r$, for $r = 0, 1, 2, \cdots, n-1$. Hence

$$ {}_G\int_a^b x \odot dx^G = \lim_{n\to\infty}\left[\prod_{r=0}^{n-1} f(ah^r)\right]^{\ln h} \quad \text{where } n\to\infty \text{ as } h\to 1 \text{ and } h^n \to \frac{b}{a} $$

$$ = \lim_{h\to 1}\left[\prod_{r=0}^{n-1} ah^r\right]^{\ln h} = \lim_{h\to 1}\left[a.ah.ah^2\cdots ah^{n-1}\right]^{\ln h} $$

$$ = \lim_{h\to 1}\left[a^n.h^{1+2+\cdots+n-1}\right]^{\ln h} = \lim_{h\to 1}\left[a^n.h^{\frac{n(n-1)}{2}}\right]^{\ln h} = \lim_{h\to 1}\left[a^{n\ln h}.h^{\frac{n(n-1)\ln h}{2}}\right] $$

$$ = \lim_{h\to 1}\left[a^{\ln h^n}.\left(h^n\right)^{1/2\ln h^{(n-1)}}\right] = \lim_{h\to 1}\left[a^{\ln h^n}.\left(h^n\right)^{1/2\ln(h^n/h)}\right] $$

$$ = a^{\ln\left(\frac{b}{a}\right)}.\left(\frac{b}{a}\right)^{1/2\ln\left(\frac{b}{a}\right)} \qquad \because h^n \to \tfrac{b}{a} \text{ as } h\to 1 $$

$$ = a^{1/2\ln\left(\frac{b}{a}\right)}.b^{1/2\ln\left(\frac{b}{a}\right)} = a^{\frac{\ln b-\ln a}{2}}.b^{\frac{\ln b-\ln a}{2}} = \frac{a^{\frac{\ln b}{2}}}{b^{\frac{\ln a}{2}}}\cdot\frac{b^{\frac{\ln b}{2}}}{a^{\frac{\ln a}{2}}} = \frac{b^{\frac{\ln b}{2}}}{a^{\frac{\ln a}{2}}} \quad \left(\because a^{\frac{\ln b}{2}} = b^{\frac{\ln a}{2}}\right) $$

$$ = \frac{e^{\frac{\ln^2 b}{2}}}{e^{\frac{\ln^2 a}{2}}} = e^{\left(\frac{\ln^2 b-\ln^2 a}{2}\right)}. $$

It is possible to apply the method of G-integration as limit of product to determine limits of different complicated products.

6. Conclusion

Numerical methods discussed in [7] for solving G-differential equations give more accuracy in comparison to classical calculus. It is expected that numerical methods for solving G-integral equations will give better approximations too. Keeping that aim, here we have just discussed about basic properties of G-integral calculus.We can apply the definition of G-integral as limit of products to determine the limits of complicated products.

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