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SPLINE SOLUTIONS OF LINEAR FRACTIONAL BVPS WITH TWO CAPUTOS APPROACHES

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Abstract. In this paper, an efficient numerical methods based on cubic polynomial spline functions are proposed for the linear fractional boundary value problems (FBVPs) with Caputos left and right fractional operator. In computing the approximation to the solutions of FBVPs, consistency relations have been derived with the help of spline functions. For convergence analysis of this method, it is assumed that the exact solu- tion of FBVP belongs to a class of C^6-functions. Numerical examples are considered to illustrate the accuracy and efficiency of this method and compare the results with other methods developed by Akram and Tariq in [18] and Zahra and Elkholy in [28-30].

Keywords: Linear FBVPs, Cubic Spline Function, Caputos Fractional Operators, Error Bound.

AMS Subject Classification: 34A08, 65L10, 26A33.

1. Introduction

Fractional calculus has been used to model physical and engineering processes that are found to be best described by fractional differential equations [1, 2]. The theory of frac- tional calculus has two definitions of fractional derivatives as its base: a left derivative, which is nonlocal by looking to the past/left of the current time/space, and a right deriva- tive, which is nonlocal by looking to the future/right of the current instant/position. Both perspectives (left and right, causal and anti-causal) make all sense in many applications, like signal processing, where bilateral operators, like the bilateral Laplace transform, and right and left functions have a central role.
In general, it is a difficult task to find the exact solutions to most of the differential equations of fractional order. Therefore, some efficient and reliable schemes are developed to solve fractional differential equations and many researchers gave their considerable to the numeri- cal solution of fractional differential equations. In this context, Moghaddam and Mostaghim established finite difference method to solve fractional differential equation [3]. Aleroev de- veloped the solution of the Strum-Liouville problem for fractional boundary value problem [4]. Jafari and Daftardar-Gejji established the Adomian decomposition method to compute approximate solutions of fractional boundary value problems with fractional derivative in Caputos sense [5].

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Odibat and Momani developed the variational iteration method in order to solve the differential equation of fractional order [6]. Abu Arqub deveolped the residual power series method to solve the differential equations of fractional order [7,8]. Recently, some powerful and efficient methods have been proposed to obtain numerical solutions of fractional differential equations [9-11].
Many authors used the spline technique to establish the accurate and efficient numerical schemes for the solution of differential equations [12-14]. For example, Siddiqi and Akram constructed many numerical schemes with the help of different spline functions such as polynomial splines and non-polynomial splines for the solution of sixth, eighth and tenth order BVPs [15 - 17]. Akram and Tariq established the spline methods to compute approx- imate solution for the fractional boundary value problem [18-21]. The theory of fractional boundary value problems (FBVPs) has received considerable interest in recent years. The interest towards the theory of existence and uniqueness of solutions to FBVPs is apparent from the recent publications [22-24]. FBVPs occur in the explanation of many physical stochastic-transport processes and in the inspection of liquid filtration, which arises in a strongly porouss medium [25]. Also, boundary value problems with integral boundary con- ditions establish a very fascinating and predominant class of problems. Two, three, four, multi point and nonlocal boundary value problems are the special cases of such problems. Cellular systems and population dynamic are some phenomenon in which boundary con- ditions of integral type occur [26]. Analysis and representation of many physical systems demand solutions of fractional boundary value problems. In the present paper, the new numerical scheme is developed for obtaining an approximation to the solution of linear fractional differential equation:

$$y''(x) + (\eta D^{\alpha} + \mu)y(x) = f(x), \qquad x \in [a,\ b],\ n-1 < \alpha < n,\ n \in \mathrm{N}, \tag{1}$$

subject to

$$y(a) = 0, \qquad y(b) = 0, \tag{2}$$

where η and μ are real constants and$f(x)$ is a continuous function on the interval $[a, b]$ and D^{α} denotes fractional derivative in Caputo's sense.
The existence result for the solution of concerned fractional two-point boundary value prob- lem can be seen in [27]. The paper is organized as follows: some preliminaries of fractional calculus are given in section 2. In section 3, cubic spline method is developed for the solution of FBVP with left and right fractional operators. The matrix form of the proposed scheme is discussed in section 4. In section 5, the convergence analysis of numerical method is given and this method is of $O(h^{2-\alpha})$. In section 6, three examples are given to compare and illustrate the efficiency of the method and it has been shown that numerical method perform better than [18] and [28-30].

2. Preliminaries

2.1. **Background Notions of Fractional Calculus.** Let $y(x)$ be a function defined on finite interval (a, b), then

Definition 2.1. [31, 32] *The Riemann Liouville fractional derivative of order α is defined as*

$$^{R}D^{\alpha}y(x) = \frac{1}{\Gamma(m-\alpha)}\frac{d^m}{ds^m}\int_0^x (x-s)^{m-\alpha-1}y(s)ds,\ \alpha > 0,\ m-1 < \alpha < m, \tag{3}$$

where $\Gamma(.)$ is the gamma function.

Definition 2.2. $[31-34]$ *The right and left sided Caputo's fractional derivative of order α is defined as*

$$D^{\alpha}_{-b}y(x) = \begin{cases} I^{m-\alpha}_{-b}D^m y(x), & m-1<\alpha<m, m \in \mathrm{N}, \\ \frac{D^m y(x)}{Dx^m}, & \alpha = m \end{cases}$$

and

$$D^{\alpha}_{a+}y(x) = \begin{cases} I^{m-\alpha}_{a+}D^m y(x), & m-1<\alpha<m, m \in \mathrm{N}, \\ \frac{D^m y(x)}{Dx^m}, & \alpha = m, \end{cases}$$

respectively, where D^m is ordinary differential operator.

Properties of Fractional Integrals and Fractional Derivatives [31-34]

(1) If $y(x)$ is continuous function and $\alpha,\ \beta > 0$, then the following results hold:

(i) $I_a^\alpha I_a^\beta y(x) = I_a^\beta I_a^\alpha y(x) = I_a^{\alpha+\beta} y(x)$

(ii) $I^\alpha x^m = \frac{\Gamma(m+1)}{\Gamma(m+1+\alpha)} x^{m+\alpha}$

(iii) ${}^R D_a^\alpha (I_a^\beta y(x)) = I_a^{\beta-\alpha} y(x), \quad \alpha - \beta < 0$

(2) If $y(x)$ is continuous and $\alpha < 1,\ \beta > 0$, then the right Riemann Liouville fractional operators follow the following properties:

(i) $I_{b-}^\alpha I_{b-}^\beta y(x) = I_{b-}^\beta I_{b-}^\alpha y(x) = I_{b-}^{\alpha+\beta} y(x)$

(ii) ${}^R D_{b-}^\alpha I_{b-}^\alpha y(x) = y(x)$.

(iii) $I_{b-}^\alpha (b-x)^m = \frac{\Gamma(m+1)}{\Gamma(m+1+\alpha)} (b-x)^{m+\alpha}, \quad m \in \mathrm{N}.$

(3) $D^\alpha C = 0, \qquad C$ *is constant.*

(4) $D^\alpha(\lambda y(x) + \mu q(x)) = \lambda D^\alpha y(x) + \mu D^\alpha q(x)$.

(5) $D^\alpha y(x) =^R D^\alpha [y(x) - \sum_{k=0}^{m-1} \frac{1}{k!}(x-a)^k y^k(a)]$.

3. Cubic Spline Method for FBVPs

Consider the following FBVP:

$$y''(x) + (\eta D^\alpha + \mu) y = f(x), \qquad x \in [a,b], \quad m-1 < \alpha \le m, \tag{4}$$

subject to

$$y(a) = 0, \qquad y(b) = 0. \tag{5}$$

Let $x_i = a + ih$ $(i = 0, 1, ..., n,\quad h = \frac{b-a}{n},\ n > 0)$ be grid points of the uniform partition of $[a,b]$ into the subintervals $[x_{i-1}, x_i]$. Let $y(x)$ be the exact solution of Eq.(6) and S_i be an approximation to $y_i = y(x_i)$ obtained by the cubic spline function passing through the points (x_i, S_i) and (x_{i+1}, S_{i+1}).

The numerical solution of given FBVP is discussed with left differential operator (first case) and secondly with right differential operator (second case).

Case 1 : Numerical Solution of FBVP with Left Fractional Operator In this case, consider that cubic spline segment has the following form:

$$\widehat{\Upsilon}_i(x) = \widehat{a}_i (x - x_{i-1})^3 + \widehat{b}_i (x - x_{i-1})^2 + \widehat{c}_i (x - x_{i-1}) + \widehat{d}_i, \qquad i = 1, 2, ..., n,$$

where $\widehat{a}_i$, $\widehat{b}_i$, $\widehat{c}_i$ and $\widehat{d}_i$ are undetermined coefficients. These coefficients are expressed in terms of S_i and M_i as

$$\widehat{\Upsilon}_i(x_{i-1}) = S_{i-1}, \qquad \widehat{\Upsilon}_i(x_i) = S_i, \qquad \widehat{\Upsilon}_i''(x_{i-1}) = M_{i-1}, \qquad \widehat{\Upsilon}_i''(x_i) = M_i,$$

and are calculated, as

$$\widehat{a}_i = \frac{1}{6h}(M_i - M_{i-1}), \quad \widehat{b}_i = \frac{M_{i-1}}{2}, \quad \widehat{c}_i = \frac{S_i}{h} - \frac{S_{i-1}}{h} - \frac{h}{6}(M_i - M_{i-1}) - \frac{M_{i-1}}{2}h, \quad \widehat{d}_i = S_{i-1}.$$

By the derivative continuities of order up to the maximum of 2 and values of the constants, the following recurrence relation is obtained, as

$$S_{i+1} - 2S_i + S_{i-1} = \frac{h^2}{6}(M_{i+1} + 4M_i + M_{i-1}), \qquad i = 1, 2, ...n-1. \tag{6}$$

The approximations of M_0 and M_n in terms of functional values are defined as

$$M_0 \cong \frac{2S_0 - 5S_1 + 4S_2 - S_3}{h^2}$$

and

$$M_n \cong \frac{2S_n - 5S_{n-1} + 4S_{n-2} - S_{n-3}}{h^2}.$$

For $i = 1$, the consistency relation can be taken as

$$\frac{1}{6}S_3 + \frac{1}{3}S_2 + \frac{-7}{6}S_1 + \frac{2}{3}S_0 = \frac{h^2}{6}(M_2 - 4M_1) \tag{7}$$

and for $i = n-1$, the consistency relation can be taken as

$$\frac{1}{6}S_{n-3} + \frac{1}{3}S_{n-2} + \frac{-7}{6}S_{n-1} + \frac{2}{3}S_n = \frac{h^2}{6}(M_{n-2} - 4M_{n-1}). \tag{8}$$

Also M_i are taken from Eq. (6), as

$$M_i + \mu S_i + \eta D^{\alpha}_{x_{i-1}} \widehat{\Upsilon_i}(x) \mid_{x=x_i} = f_i, \qquad i = 0, 1, ...n, \tag{9}$$

where $f_i = f(x_i)$.

Case 2 : Numerical Solution of FBVP with Right Fractional Operator

In this case, consider that in each subinterval the cubic spline segment is defined as:

$$\Upsilon_i(x) = a_i(x_{i+1} - x)^3 + b_i(x_{i+1} - x)^2 + c_i(x_{i+1} - x) + d_i, \qquad i = 0, 1, ..., n-1,$$

where a_i, b_i, c_i and d_i are undetermined coefficients. These coefficients are expressed in terms of S_i and M_i as

$$\Upsilon_i(x_i) = S_i, \qquad \Upsilon_i(x_{i+1}) = S_{i+1}, \qquad \Upsilon''_i(x_i) = M_i, \qquad \Upsilon''_i(x_{i+1}) = M_{i+1},$$

and are calculated, as

$$a_i = \frac{1}{6h}(M_i - M_{i+1}), \quad b_i = \frac{M_{i+1}}{2}, \quad c_i = \frac{S_i}{h} - \frac{S_{i+1}}{h} - \frac{h}{6}(M_i - M_{i+1}) - \frac{M_{i+1}}{2}h, \quad d_i = S_{i+1}.$$

By derivative continuities of order up to the maximum of 2 and values of the constants, same relations Eq. (8), Eq. (9) and Eq. (10) are obtained. Where M_i are taken from Eq. (6), as

$$M_i + \mu S_i + \eta D^{\alpha}_{x_{i+1}} \Upsilon_i(x) \mid_{x=x_i} = f_i, \quad i = 0, 1, ...n. \tag{10}$$

Lemma 3.1. *Let $y \in C^6[a, b]$ then the local trucation errors $\widetilde{t_i}$, $i = 0, 1, ..., n-1$ associated with the Eq. (9), Eq. (8) and Eq. (10) are*

$$\widetilde{t_i} = \begin{cases} \frac{5}{72}h^4 y_1^{(4)} + O(h^5), & i = 1, \\ \frac{-1}{12}h^4 y_i^{(4)} + O(h^6), & i = 2, 3, ..., n-2, \\ \frac{5}{72}h^4 y_{n-1}^{(4)} + O(h^5), & i = n-1. \end{cases}$$

Moreover,

$$||T||_\infty = c_1 h^4 Z_4, \qquad Z_4 = max_{x \in [0,1]}|y^{(4)}(x)|, \tag{11}$$

where c_1 is a constant and also independent of h.

4. The matrix form of the suggested scheme

The matrices X and Q are obtained with the help of system (8) for $i = 1, 2, ..., n-1$. Let $Y = [y_1, y_2, ..., y_{n-1}]^T$, $S = [S_1, S_2, ..., S_{n-1}]^T$, $M = [M_1, M_2, ..., M_{n-1}]^T$, $E = (e_i)$ and $T = (\widetilde{t_i})$ for $i = 1, 2, ..., n-1$ are $(n-1)$ dimensional column vectors.

From system (8) – (10), it can be written as

$$XS = h^2 QM, \tag{12}$$

where $X = (x_{ij})$, $Q = (q_{ij})$ are $(n-1) \times (n-1)$ matrices and

$$q_{ij} = \begin{cases} 4, & i = j = 1, n-1, \\ 1, & i = 1, j = 2, \\ 1, & i = n-1, j = n-2, \\ 4, & i = j = 2, 3, ..., n-2, \\ 1, & |i-j| = 1, i, j = 2, 3, ..., n-2 \\ 0, & otherwise, \end{cases}$$

$$x_{ij} = \begin{cases} \frac{-7}{6}, & i = j = 1, n-1, \\ \frac{1}{3}, & i = 1, j = 2, \\ \frac{1}{6}, & i = 1, j = 3, \\ \frac{1}{3}, & i = n-1, j = n-2, \\ \frac{1}{6}, & i = n-1, j = n-3, \\ -2, & i = j = 2, 3, ..., n-2, \\ 1, & |i-j| = 1, i, j = 2, 3, ..., n-2 \\ 0, & otherwise, \end{cases}$$

The system (12) in matrix form can be written as

$$PS + HM = F, \tag{13}$$

where $P = (p_{ij})$, $H = (h_{ij})$ are matrices of order $(n-1) \times (n-1)$ and

$$p_{ij} = \begin{cases} p_1, & i = j = 1, 2, ..., n-2, \\ p_2, & j - i = 1, i, j = 1, 2, ..., n-2 \\ p_{13}, & i = n-1, j = n-1, \\ p_{12}, & i = n-1, j = n-2, \\ p_{11}, & i = n-1, j = n-3, \\ 0, & otherwise, \end{cases}$$

$$h_{ij} = \begin{cases} h_1, & i = j = 1, 2, ..., n-2, \\ h_2, & j - i = 1, i, j = 1, 2, ..., n-2 \\ h_1, & i = n-1, j = n-1, \\ 0, & otherwise, \end{cases}$$

where

$$p_1 = \mu + \frac{\eta h^{-\alpha}}{\Gamma(2-\alpha)},\ p_2 = \frac{-\eta h^{-\alpha}}{\Gamma(2-\alpha)},\ h_1 = 1 + \frac{\eta h^{2-\alpha}}{6\Gamma(4-\alpha)}(5\alpha - \alpha^2),\ h_2 = \frac{\eta h^{2-\alpha}}{3\Gamma(4-\alpha)}(2\alpha - \alpha^2),$$

$$p_{11} = \frac{-h_2}{h^2},\ \ p_{12} = \frac{4h_2}{h^2}\ \ and\ \ p_{13} = \frac{-5h_2}{h^2} + p_1.$$

Moreover $F = (f_i)$ is $(n-1)$ dimensional column vector such that

$$F = \begin{cases} f_i, & i = 1, 2, ..., n-2, \\ f_{n-1} + \frac{\eta h^{-\alpha}}{\Gamma(2-\alpha)} S_n, & i = n-1. \end{cases}$$

The Eq.(15) can be written as

$$M = H^{-1}F - H^{-1}PS,$$

From Eq. (14) and Eq. (15), it can be written, as

$$(X + h^2 Q H^{-1} P)S = h^2 Q H^{-1} F. \tag{14}$$

5. Convergence of the method

In order to get a bound on $\|E\|_\infty$, consider

$$(X + h^2QH^{-1}P)Y = h^2QH^{-1}F + T. \tag{15}$$

From Eq. (16) and Eq. (17),

$$(X + h^2QH^{-1}P)E = T. \tag{16}$$

From Eq. (18), E can be expressed as

$$E = (I + h^2X^{-1}QH^{-1}P)^{-1}X^{-1}T. \tag{17}$$

Lemma 5.1. [35] *If Z is a matrix of order n and $\|Z\| < 1$, then $(I + Z)^{-1}$ exists and*

$$\|(I + Z)^{-1}\| < \frac{1}{1 - \|Z\|}.$$

Lemma 5.2. *The infinite norm of H^{-1} satisfies the inequality*

$$\|H^{-1}\|_\infty \leq \frac{6\Gamma(4-\alpha)}{6\Gamma(4-\alpha) - 3\eta h^{2-\alpha}(3\alpha - \alpha^2)}, \tag{18}$$

provided that $\frac{3\eta h^{2-\alpha}(3\alpha-\alpha^2)}{6\Gamma(4-\alpha)} \leq 1$.

Proof. The matrix H can be written, as

$$H = I + \frac{\eta h^{2-\alpha}}{6\Gamma(4-\alpha)}\widetilde{H},$$

where matrix $\widetilde{H}$ is

$$\begin{pmatrix} 5\alpha-\alpha^2 & 4\alpha-2\alpha^2 & & & & & \\ & 5\alpha-\alpha^2 & 4\alpha-2\alpha^2 & & & & \\ & & 5\alpha-\alpha^2 & 4\alpha-2\alpha^2 & & & \\ & & \ddots & \ddots & & & \\ & & & 5\alpha-\alpha^2 & 4\alpha-2\alpha^2 & & \\ & & & & 5\alpha-\alpha^2 & 4\alpha-2\alpha^2 & \\ & & & & & 5\alpha-\alpha^2 & \end{pmatrix}.$$

The matrix H^{-1} can be expressed, as

$$H^{-1} = (I + \frac{\eta h^{2-\alpha}}{6\Gamma(4-\alpha)}\widetilde{H})^{-1},$$

Using the Lemma 5.1, if

$$\|\frac{\eta h^{2-\alpha}}{6\Gamma(4-\alpha)}\widetilde{H}\|_\infty < 1, \tag{19}$$

then

$$\|H^{-1}\|_\infty \leq \frac{1}{1 - \|\frac{\eta h^{2-\alpha}}{6\Gamma(4-\alpha)}\widetilde{H}\|_\infty},$$

where

$$\|\frac{\eta h^{2-\alpha}}{6\Gamma(4-\alpha)}\widetilde{H}\|_\infty = \frac{3\eta h^{2-\alpha}}{6\Gamma(4-\alpha)}(3\alpha - \alpha^2). \tag{20}$$

In this case, at $\alpha = 1$, maximum value of Eq. (20) is

$$\|\frac{\eta h^{2-\alpha}}{6\Gamma(4-\alpha)}\widetilde{H}\|_\infty \leq \eta\frac{h}{2}.$$

In order to satisfy the Lemma 5.2, the parameter η must satisfy the following condition:

$$\eta_{max} < \frac{2}{h}$$

and

$$\|H^{-1}\|_\infty \leq \frac{6\Gamma(4-\alpha)}{6\Gamma(4-\alpha)-3\eta h^{2-\alpha}(3\alpha-\alpha^2)}.$$

□

Lemma 5.3. *The matrix* $(X+h^2QH^{-1}P)$ *in Eq. (18) is nonsingular, provided that:*

$$h^2\eta\lambda_1+\xi\lambda<1,$$

where $\lambda_1=\frac{h^{-\alpha}\alpha(3-\alpha)}{2\Gamma(4-\alpha)}$, $\lambda=\mu+\frac{\eta h^{-\alpha}}{3\Gamma(4-\alpha)}(5\alpha^2-19\alpha+18)$ *and* $\xi=\frac{1}{8}((b-a)^2+h^2)$.
Then

$$\|E\|_\infty \leq \frac{\|X^{-1}\|_\infty\|T\|_\infty}{1-h^2\|X^{-1}\|_\infty\|Q\|_\infty\|H^{-1}\|_\infty\|P\|_\infty} \cong O(h^{2-\alpha}). \tag{21}$$

Proof. From Eq. (19) and Lemma 5.1,

$$\|E\|_\infty = max_{1\leq i\leq n-1}|e_i| \leq \frac{\|X^{-1}\|_\infty\|T\|_\infty}{1-h^2\|X^{-1}\|_\infty\|Q\|_\infty\|H^{-1}\|_\infty\|P\|_\infty}, \tag{22}$$

provided that $h^2\|X^{-1}\|_\infty\|Q\|_\infty\|H^{-1}\|_\infty\|P\|_\infty<1$. From [36],

$$\|X^{-1}\|_\infty = \frac{1}{8h^2}((b-a)^2+h^2).$$

Also,

$$\|Q\|_\infty = 1 \text{ and } \|P\|_\infty = \mu+\frac{\eta h^{-\alpha}}{3\Gamma(4-\alpha)}(5\alpha^2-19\alpha+18).$$

By substituting the values of $\|X^{-1}\|_\infty, \|Q\|_\infty, \|H^{-1}\|_\infty$ and $\|P\|_\infty$ in Eq. (24) and using Eq. (13), it can be written as

$$\|E\|_\infty \leq \frac{c_1h^2Z_4(\xi(1-h^2\eta\lambda_1)}{(1-(h^2\eta\lambda_1+\xi\lambda))} \cong O(h^{2-\alpha}), \tag{23}$$

where $Z_4 = max_{a\leq x\leq b} \mid y^4(x)|$. □

Theorem 5.1. *Let* $y(x)$ *be the exact solution of the Bagley-Torvik FBVP Eq. (6) with boundary condition Eq. (7) and* $y_i, i=0,1,2,...,n-1$, *satisfy the discrete BVP Eq. (17). Moreover, if* $e_i=y_i-S_i$, *then*

$$\|E\|_\infty = O(h^{2-\alpha}).$$

6. Numerical Results

In this section, to check the accuracy, efficiency and validity of the method, some examples of suggested method are given and also compare the results with other methods.

Example 6.1 Consider the following FBVPs:

$$y''(x)+\eta D^\alpha y(x)+\mu y(x)=f(x), \qquad x\in[0,1],$$

with

$$y(0)=0, \quad y(1)=0,$$

where $f(x)=30x^4-(5-\alpha)(4-\alpha)x^{3-\alpha}+\frac{\Gamma(7)\eta x^{6-\alpha}}{\Gamma(7-\alpha)}-\frac{\Gamma(6-\alpha)\eta x^{5-2\alpha}}{\Gamma(6-2\alpha)}+\mu x^6-\mu x^{5-\alpha}$. Also $\eta=0.05$ and $\mu=0.01$. The exact solution of this problem is $x^6-x^{5-\alpha}$. The present scheme is applied with different values of α and results are shown in Table 1 and Figure 1.

Example 6.2 Consider the following FBVP:

$$y''(x)+(\eta D^\alpha+\mu)y=f(x), \qquad x\in[0,1],$$

TABLE 1. Maximum absolute error for different values of α.

h	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$
$1/8$	$4.5E-03$	$5.0E-03$	$5.3E-03$
$1/16$	$1.8E-03$	$1.8E-03$	$1.7E-03$
$1/32$	$2.6202E-04$	$1.9092E-04$	$4.0731E-04$

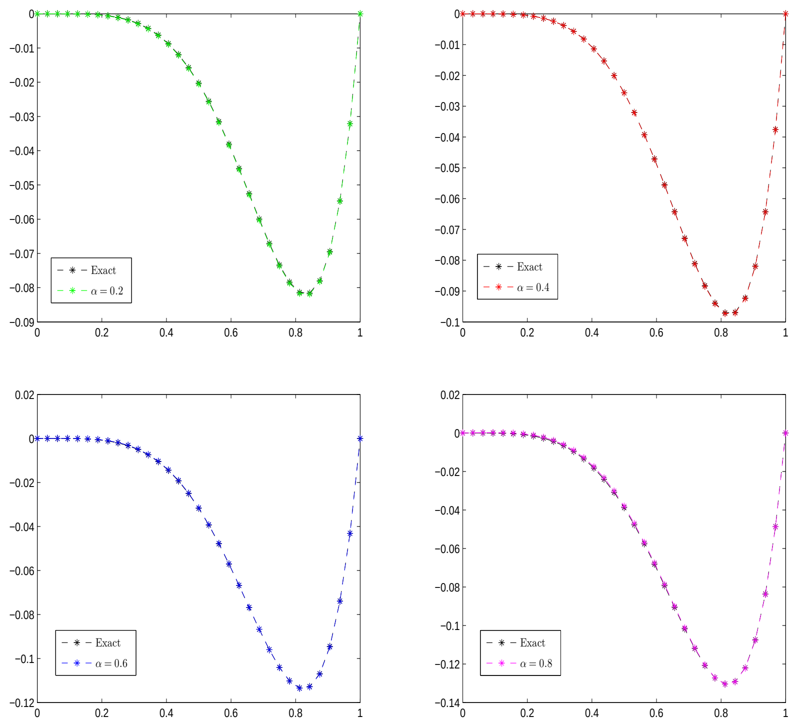

FIGURE 1. Exact and Approximate Solutions of Example 6.1 with different values of α.

TABLE 2. For $\alpha = 0$, comparison of maximum absolute errors of presented method with [18] and [28].

n	Presented Method	*G. Akram* [18]	*W. K. Zahra* [28]
8	$1.09E-02$	$1.50E-02$	$9.29E-02$
16	$1.1E-03$	$5.6E-03$	$2.578E-02$
32	$3.1E-03$	$4.5E-03$	$7.15E-03$

$$y(0) = 0, \quad y(1) = 0.$$

The exact solution of this problem is $x^6(1-x^2)$. Also, for $\eta = 1$ and $\mu = 0$, $f(x) = \frac{720x^4}{\Gamma(5)} - \frac{40320x^6}{\Gamma(7)} + \frac{\Gamma(7)x^{6-\alpha}}{\Gamma(7-c)} - \frac{\Gamma(9)x^{(8-\alpha)}}{\Gamma(9-c)}$. For different values of α, numerical results are shown in Table 2-4 and Figure 2. Also, the results of same problem are compared with the numerical schemes in [18] and [28], and found that results of suggested method are more accurate than [18] and [28].

Example 6.3 Consider the following FBVPs:

TABLE 3. For $\alpha = 0.2$, comparison of maximum absolute errors of presented method with [18] and [28].

n	Presented Method	*G. Akram* [18]	*W. K. Zahra* [28]
8	$1.18E-02$	$1.72E-02$	$1.06E-01$
16	$3.3E-03$	$7.9E-03$	$2.91E-02$
32	$5.2E-03$	$6.6E-03$	$8.05E-03$

TABLE 4. For $\alpha = 0.4$, comparison of maximum absolute errors of presented method with [18] and [28].

n	Presented Method	*G. Akram* [18]	*W. K. Zahra* [28]
8	$1.30E-02$	$2.05E-02$	$1.43E-01$
16	$6.7E-03$	$1.14E-02$	$4.11E-02$
32	$8.4E-03$	$9.8E-03$	$1.10E-02$

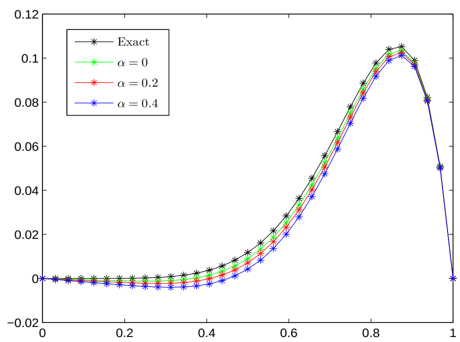

FIGURE 2. Exact and Approximate Solutions of Example 6.2 with different values of α.

$$y''(x) + (\eta D^{\alpha} + \mu)y = f(x), \qquad x \in [0, 1],$$

with

$$y(0) = 0, \quad y(1) = 0,$$

where $f(x) = 4x^2(5x-3) + \frac{120\eta x^{5-\alpha}}{\Gamma(6-\alpha)} - \frac{24\eta x^{4-\alpha}}{\Gamma(5-\alpha)} + \mu x^5 - \mu x^4$. The exact solution of this problem is $x^4(x-1)$.

The present scheme is applied with $n = 8$, $\eta = 0.5$, $\alpha = 0.3$ and $\mu = 1$ and the numerical results which are obtained from Spline technique (ST) fo FBVP are shown in Table 5. Furthermore in the limit, as α goes to zero, the method provides solution for the integer order system. Also, the results of same problem are compared with the numerical schemes in [29] and [30]. Also founds that results of suggested method are more accurate than [29] and [20].

7. Conclusion

Numerical method is established for the approximate solution of FBVP, using cubic polynomial spline. The suggested method also utilize the properties of fractional derivatives in order to solve this problem. This numerical scheme is computationally captivate and also descriptive

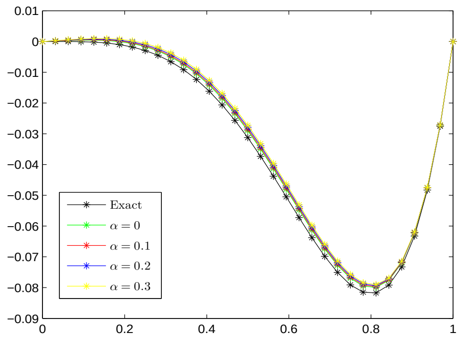

Figure 3. Exact and Approximate Solutions of Example 6.3 with different values of α.

Table 5. For $\alpha = 0.3$, Comparison of the numerical results of presented method with [29] and [30].

x	Exact Solution	Approx. Solution	Error[26]	Error[27]	Presented Method Error
0	0	0	0	0	0
0.125	-0.0002	-0.0001	$2.00E-03$	$1.73E-04$	$1.00E-04$
0.250	-0.0029	-0.0026	$4.08E-03$	$5.35E-04$	$4.00E-04$
0.375	-0.0124	-0.0119	$5.83E-03$	$7.98E-04$	$5.00E-04$
0.500	-0.0313	-0.0307	$6.85E-03$	$6.74E-04$	$5.00E-04$
0.625	-0.0572	-0.0567	$6.81E-03$	$9.50E-05$	$5.00E-04$
0.750	-0.0791	-0.0786	$5.57E-03$	$1.78E-03$	$5.00E-04$
0.875	-0.0733	-0.0721	$3.26E-03$	$3.42E-03$	$1.20E-03$
1	0	0	0	$9.44E-04$	0

examples show applications of this problem. It is proved that the method is of $O(h^{2-\alpha})$ which shows that if h is reduced by factor $1/2$ then $\|E\|_\infty$ is reduced by factor $1/2^{2-\alpha}$. Fast convergence and simple applicability of the cubic splines provide a solid foundation for using these functions in the context of numerical approximation of ordinary differential equations, partial differential equations and integral equations. The extension of these methods to fractional nonlinear boundary value problems is under process.

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