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G.GHORAI, M.PAL: APPLICATIONS OF BIPOLAR FUZZY SETS IN INTERVAL GRAPHS

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ABSTRACT. Currently, bipolar fuzzy graph is a growing research topic as it is the generalization of fuzzy graphs. In this paper, normal and convex bipolar fuzzy sets are defined and the notion of bipolar fuzzy interval is introduced as a generalization of fuzzy interval and described various methods of their construction. It is shown that intersection of two bipolar fuzzy intervals may not be a bipolar fuzzy interval. Finally, bipolar fuzzy interval graphs is introduced as the intersection graph of a finite family of bipolar fuzzy intervals. The relationship between the intersection graph of a $\{\alpha, \beta\}$-level family of bipolar fuzzy intervals and $\{\alpha, \beta\}$-cut of the intersection graph for that family have been established. It is proved that for every bipolar fuzzy interval graph G, the (α, β)-cut level graph G^{α}_{β} is an interval graph for each $(\alpha, \beta) \in (0, 1] \times [-1, 0)$. Also, some important hereditary properties of bipolar fuzzy interval graphs are presented.

Keywords: Bipolar fuzzy set, Normal and convex bipolar fuzzy set, Bipolar fuzzy interval, Bipolar fuzzy interval graph.

AMS Subject Classification: 05C72, 05C76

1. Introduction

The notion of fuzzy subset of a set was first introduced by Zadeh [33] in 1965. After that, fuzzy set theory has become an unavoidable research topic in different disciplines including medical and life sciences, management sciences, social sciences, engineering, statistics, graph theory, artificial intelligence, signal processing, multi agent system, pattern recognition, robotics, computer networks, expert systems, decision making and automata theory. As a generalization of fuzzy sets, Zhang [34, 35] initiated the concept of bipolar fuzzy sets in 1994. The bipolar fuzzy set constitute a generalization of Zadeh's fuzzy set theory whose membership degree range is $[-1, 1]$. In a bipolar fuzzy set, the membership degree 0 of an element means that the element is irrelevant to the corresponding property, the membership degree $(0, 1]$ of an element indicates that the element somewhat satisfies

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the property and the membership degree $[-1, 0)$ of an element indicates that the element somewhat satisfies the implicit counter-property.

In 1975, Rosenfeld [22] introduced the concept of fuzzy graphs considering the fuzzy relations between fuzzy sets and developed the structure of fuzzy graphs. Craine [4] first introduce the concept of fuzzy interval graphs in 1994. Mordeson and Nair [18] discussed about the properties of fuzzy graphs and hypergraphs. Fuzzy intersection graphs [17] was characterized by McAllister. Bhutani et al. [3] studied on degrees of end nodes and cut nodes in fuzzy graphs. In 2011, Akram [1, 2] introduced and studied the bipolar fuzzy graphs. Yang et al. [32] introduced generalized bipolar fuzzy graphs and Ghorai and Pal [12] introduced generalized regular bipolar fuzzy graphs. Samanta and Pal studied bipolar fuzzy hypergraphs [25], irregular bipolar fuzzy graphs [26], m-step fuzzy competition graphs [28]. Rashmanlou et al. [20] investigated some more properties of bipolar fuzzy graphs. Samanta et al. [29] introduced the bipolar fuzzy intersection graphs. Some more work on fuzzy graphs and bipolar fuzzy graphs cane be found on [23, 24, 27, 30, 31]. Later on, Ghorai and Pal [7, 8, 9, 10, 11] introduced and studied many variations of generalized m-polar fuzzy graphs. Intersection graphs are very important tool in many applications of our real world problems as well as theoretical points of view. Considering bipolar fuzzy models, one can get more precision, flexibility and comparability to the system as compared to the classical and fuzzy models. This motivates us to define bipolar fuzzy interval graph as a generalization of fuzzy interval graph. For this purpose, normal and convex bipolar fuzzy sets are defined to introduce the notion of bipolar fuzzy interval as a generalization of fuzzy interval. Different methods has been described for their formulation. It is shown by an example that intersection of two bipolar fuzzy intervals may not be a bipolar fuzzy interval. Finally, the notion of bipolar fuzzy interval graph is introduced for a finite family of bipolar fuzzy intervals and studied many results of it.

2. Preliminaries

In this section, we briefly recall some definitions of undirected graphs, the notions of fuzzy sets, fuzzy intersection graphs, fuzzy intervals, fuzzy interval graphs, bipolar fuzzy sets and bipolar fuzzy intersection graphs. For further details see references [6, 13, 14, 15, 16, 19].

Definition 2.1. [13] *A graph is an ordered pair $G^* = (V, E)$, where V is the set of vertices of G^* and E is the set of all edges of G^*. Two vertices x and y in an undirected graph G^* are said to be adjacent in G^* if xy is an edge of G^*. A simple graph is an undirected graph that has no loops and no more than one edge between any two different vertices.*

Definition 2.2. [13] *A subgraph of a graph $G^* = (V, E)$ is a graph $H = (W, F)$, where $W \subseteq V$ and $F \subseteq E$.*

Definition 2.3. [13] *Consider the cartesian product $G^* = G_1^* \times G_2^* = (V, E)$ of graphs G_1^* and G_2^*. Then $V = V_1 \times V_2$ and $E = \{(x, x_2)(x, y_2) : x \in V_1, x_2y_2 \in E_2\} \cup \{(x_1, z)(y_1, z) : z \in V_2, x_1y_1 \in E_1\}$.*

Definition 2.4. [13] *Let $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ be two simple graphs. Then the composition of the graph G_1^* with G_2^* is denoted by $G_1^*[G_2^*] = (V_1 \times V_2, E^0)$, where $E^0 = E \cup \{(x_1, x_2)(y_1, y_2) : x_1y_1 \in E_1, x_2 \neq y_2\}$ and E is defined in $G_1^* \times G_2^*$. Note that $G_1^*[G_2^*] \neq G_2^*[G_1^*]$.*

Definition 2.5. [13] *The union of two simple graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ is the simple graph with the vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. The union of G_1^* and G_2^* is denoted by $G^* = G_1^* \cup G_2^* = (V_1 \cup V_2, E_1 \cup E_2)$.*

Definition 2.6. [13] *The join of two simple graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ is the simple graph with the vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2 \cup E'$, where E' is the set of all edges joining the nodes of V_1 and V_2 and assume that $V_1 \cap V_2 = \emptyset$. The join of G_1^* and G_2^* is denoted by $G^* = G_1^* + G_2^* = (V_1 \cup V_2, E_1 \cup E_2 \cup E')$.*

Definition 2.7. [33] *A fuzzy subset μ on a set X is a map $\mu : X \to [0,1]$, called the membership function. A map $\rho : X \times X \to [0,1]$ is called a fuzzy relation on X if $\rho(x,y) \leq min\{\mu(x), \mu(y)\}$ for all $x, y \in X$. A fuzzy relation ρ is symmetric if $\rho(x,y) = \rho(y,x)$ for all $x, y \in X$.*

Definition 2.8. [15, 34] *Let X be a non empty set. A bipolar fuzzy set B in X is an object having the form $B = \{(x, \mu_B^P(x), \mu_B^N(x)) : x \in X\}$ where $\mu_B^P : X \to [0,1]$ and $\mu_B^N : X \to [-1,0]$ are mappings.*

For the sake of simplicity, we use the symbol $B = (\mu_B^P, \mu_B^N)$ for the bipolar fuzzy set $B = \{(x, \mu_B^P(x), \mu_B^N(x)) : x \in X\}$.

Definition 2.9. [34] *Let X be a non empty set. Then we call a mapping $A = (\mu_A^P, \mu_A^N) : X \times X \to [0,1] \times [-1,0]$ a bipolar fuzzy relation on X.*

Definition 2.10. [28] *Let $A = (\mu_A^P, \mu_A^N)$ be a bipolar fuzzy set on a non empty set X. Height of A is denoted by $h(A)$ and is defined as $h(A) = max\{\mu_A^P(x) : x \in X\}$ and depth of A is denoted by $d(A)$ and is defined by $d(A) = min\{\mu_A^N(x) : x \in X\}$.*

Definition 2.11. [18] *Let $\mathcal{F}$=$\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a finite family of fuzzy subsets of a set V and consider F as a crisp vertex set. The fuzzy intersection graph of $\mathcal{F}$ is the fuzzy graph $Int(\mathcal{F})$=(μ, ρ) where $\mu : \mathcal{F} \to [0,1]$ is defined by $\mu(\alpha_i) = h(\alpha_i)$ and $\rho : \mathcal{F} \times \mathcal{F} :\to [0,1]$ is defined by $\rho(\alpha_i, \alpha_j) = \begin{cases} h(\alpha_i \cap \alpha_j) & if \quad i \neq j \\ 0 & if \quad i = j. \end{cases}$*

Definition 2.12. [18] *Let V be a linearly ordered set. A fuzzy interval I on V is a normal, convex fuzzy subset of V. That is, there exists an $x \in V$ with $I(x) = 1$ and the ordering $w \leq y \leq z$ implies that $I(y) \geq min\{I(w), I(z)\}$.*

A fuzzy interval graph is the fuzzy intersection graph of a finite family of fuzzy intervals.

Definition 2.13. [28] *Let $\mathcal{F}$= $\{B_1 = (\mu_{B_1}^P, \mu_{B_1}^N), B_2 = (\mu_{B_2}^P, \mu_{B_2}^N), \ldots, B_n = (\mu_{B_n}^P, \mu_{B_n}^N)\}$ be a finite family of bipolar fuzzy sets defined on a non empty set X. Consider $\mathcal{F}$ as crisp vertex set V=$\{v_1, v_2, \ldots v_n\}$. The bipolar fuzzy intersection graph of $\mathcal{F}$ is the bipolar fuzzy graph (A, B) where $A = (\mu_A^P, \mu_A^N)$ is a bipolar fuzzy set on V and $B = (\mu_B^P, \mu_B^N)$ is a bipolar fuzzy set on $E = V \times V$. $\mu_A^P : V \to [0,1]$ is defined by $\mu_A^P(v_i) = h(B_i)$ and $\mu_A^N(v_i) = d(B_i)$. $\mu_B^P : V \times V \to [0,1]$ is defined by $\mu_B^P(v_i, v_j) = \begin{cases} h(B_i \cap B_j) & if \quad i \neq j \\ 0 & if \quad i = j. \end{cases}$*

Similarly, $\mu_B^N(v_i, v_j) = \begin{cases} d(B_i \cap B_j) & if \quad i \neq j \\ 0 & if \quad i = j. \end{cases}$

3. Bipolar fuzzy intervals

In this section, bipolar fuzzy intervals are defined on a linearly ordered set X as a generalization of fuzzy intervals. Before going into the bipolar fuzzy intervals, we first define normal and convex bipolar fuzzy sets.

Definition 3.1. *A bipolar fuzzy set $A = (\mu_A^P, \mu_A^N)$ on a non empty set X is said to be normal if $h(A) = 1$ and $d(A) = -1$.*

Definition 3.2. *Let $A = (\mu_A^P, \mu_A^N)$ be a bipolar fuzzy set on a linearly ordered set X. A is said to be convex bipolar fuzzy set if μ_A^P is a convex fuzzy set of X and μ_A^N is a concave fuzzy set of X, i.e. for $w, y, z \in X$ with $w \leq y \leq z$ implies $\mu_A^P(y) \geq min\{\mu_A^P(w), \mu_A^P(z)\}$ and $\mu_A^N(y) \leq max\{\mu_A^N(w), \mu_A^N(z)\}$.*

Theorem 3.1. *All $\{\alpha, \beta\}$-cut of a bipolar fuzzy set on a linearly ordered set X is convex for all $(\alpha, \beta) \in (0, 1] \times [-1, 0)$.*

Proof. Let $A = (\mu_A^P, \mu_A^N))$ be a bipolar fuzzy set on X and $(\alpha, \beta) \in (0, 1] \times [-1, 0)$.

We know that, $A_\beta^\alpha := \{x \in X : \mu_A^P(x) \geq \alpha$ and $\mu_A^N(x) \leq \beta\}$.

To show that A_β^α is convex for all $(\alpha, \beta) \in (0, 1] \times [-1, 0)$, we have to show that for $w, y, z \in A_\beta^\alpha$ with $w \leq y \leq z$, $\mu_A^P(y) \geq min\{\mu_A^P(w), \mu_A^P(z)\}$ and $\mu_A^N(y) \leq max\{\mu_A^N(w), \mu_A^N(z)\}$.

Now $w, y, z \in A_\beta^\alpha$ implies $\mu_A^P(w) \geq \alpha, \mu_A^P(y) \geq \alpha, \mu_A^P(z) \geq \alpha$ and
$\mu_A^N(w) \leq \beta, \mu_A^N(y) \leq \beta, \mu_A^N(z) \leq \beta$.
Therefore, $\mu_A^P(y) \geq \alpha = min\{\mu_A^P(w), \mu_A^P(z)\}$,
i.e. $\mu_A^P(y) \geq min\{\mu_A^P(w), \mu_A^P(z)\}$ and
$\mu_A^N(y) \leq \beta = max\{\mu_A^N(w), \mu_A^N(z)\}$, i.e. $\mu_A^N(y) \leq max\{\mu_A^N(w), \mu_A^N(z)\}$.
Hence the result. □

Definition 3.3. *Let X be a linearly ordered set. A bipolar fuzzy interval I on X is a normal, convex bipolar fuzzy subset of X. It is denoted by $I = (\mu_I^P, \mu_I^N)$ where $\mu_I^P : X \to [0, 1]$ and $\mu_I^N : X \to [-1, 0]$ are mappings.*

Example 3.1. *Examples of bipolar fuzzy intervals are shown in Fig. 1. In this Figure, (a) represents the bipolar fuzzy set expressing the proposition "close to 1.3", (b) represents the bipolar fuzzy set expressing the proposition "close to 1", (c) and (d) represent some basic types of bipolar fuzzy sets which are bipolar fuzzy intervals.*

Remark 3.1. *If $\mu_I^P(x) \neq 0$ and $\mu_I^N(x) = 0$ for all $x \in X$, it is the situation that the bipolar fuzzy interval becomes fuzzy interval on X.*

4. Operations on bipolar fuzzy intervals

In this section, Cartesian product, union and intersection on bipolar fuzzy intervals are defined. It is shown that Cartesian product and union of two bipolar fuzzy intervals is again a bipolar fuzzy interval. Also, an example is constructed to show that intersection of two bipolar fuzzy intervals may not be a bipolar fuzzy interval. Throughout this section, it is assumed that for any two ordered sets X and Y, the order on $X \times Y$ and $X \cup Y$ is the order which is defined in [5]. First of all, Cartesian product of two bipolar fuzzy intervals is defined and proved that it is again a bipolar fuzzy interval.

Definition 4.1. *Let $I_1 = (\mu_{I_1}^P, \mu_{I_1}^N)$ and $I_2 = (\mu_{I_2}^P, \mu_{I_2}^N)$ be two bipolar fuzzy intervals on the linearly ordered sets X and Y respectively. Then the cartesian product of I_1 and I_2 is a bipolar fuzzy set on $X \times Y$ denoted by $I_1 \times I_2 = (\mu_{I_1}^P \times \mu_{I_2}^P, \mu_{I_1}^N \times \mu_{I_2}^N)$ and is defined as:*
$(\mu_{I_1}^P \times \mu_{I_2}^P)(x, y) = min\{\mu_{I_1}^P(x), \mu_{I_2}^P(y)\}$ *and*
$(\mu_{I_1}^N \times \mu_{I_2}^N)(x, y) = max\{\mu_{I_1}^N(x), \mu_{I_2}^N(y)\}$ *for all* $(x, y) \in X \times Y$.

Proposition 4.1. *If $I_1 = (\mu_{I_1}^P, \mu_{I_1}^N)$ and $I_2 = (\mu_{I_2}^P, \mu_{I_2}^N)$ be two bipolar fuzzy intervals on the linearly ordered sets X and Y respectively, then $I_1 \times I_2$ is a bipolar fuzzy interval on $X \times Y$ in the product order.*

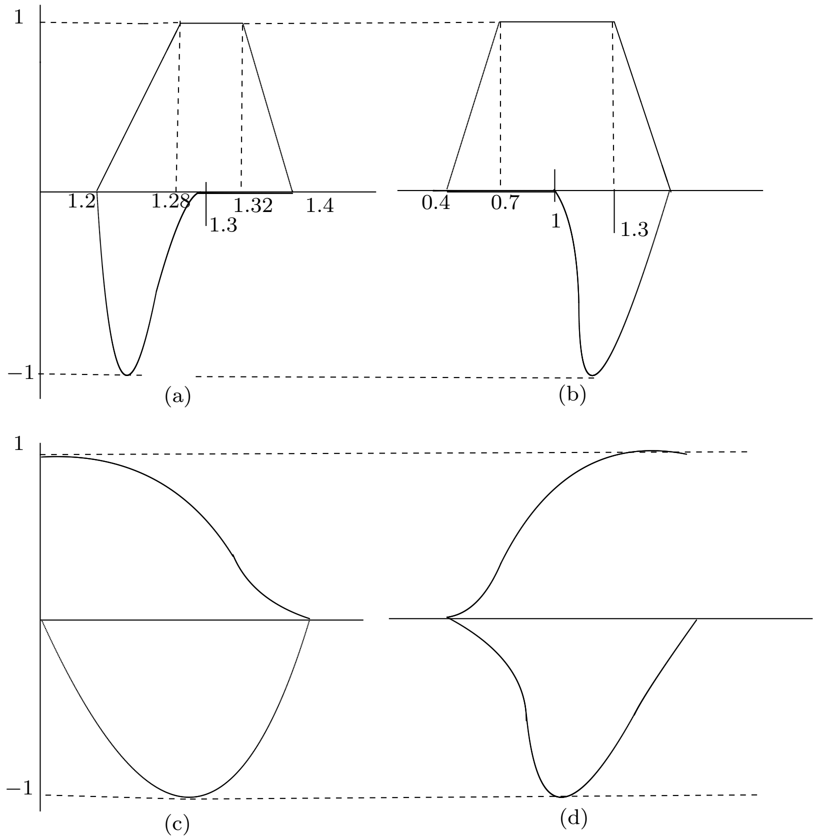


FIGURE 1. Different bipolar fuzzy intervals expressing their corresponding bipolar fuzzy sets

Proof. To show $I_1 \times I_2 = (\mu^P_{I_1} \times \mu^P_{I_2}, \mu^N_{I_1} \times \mu^N_{I_2})$ is a bipolar fuzzy interval, we show that it is a normal and convex bipolar fuzzy subset of $X \times Y$.

Normality of $I_1 \times I_2 = (\mu^P_{I_1} \times \mu^P_{I_2}, \mu^N_{I_1} \times \mu^N_{I_2})$ follows from the normality of $I_1 = (\mu^P_{I_1}, \mu^N_{I_1})$ and $I_2 = (\mu^P_{I_2}, \mu^N_{I_2})$.

For convexity of $I_1 \times I_2 = (\mu^P_{I_1} \times \mu^P_{I_2}, \mu^N_{I_1} \times \mu^N_{I_2})$, we show that for $\overline{w} = (w_1, w_2), \overline{y} = (y_1, y_2), \overline{z} = (z_1, z_2) \in X \times Y$ with $\overline{w} \leq \overline{y} \leq \overline{z}$

$\Rightarrow (\mu^P_{I_1} \times \mu^P_{I_2})(\overline{y}) \geq min\{(\mu^P_{I_1} \times \mu^P_{I_2})(\overline{w}), (\mu^P_{I_1} \times \mu^P_{I_2})(\overline{z})\}$ and

$(\mu^N_{I_1} \times \mu^N_{I_2})(\overline{y}) \leq max\{(\mu^N_{I_1} \times \mu^N_{I_2})(\overline{w}), (\mu^N_{I_1} \times \mu^N_{I_2})(\overline{z})\}$.

Now in product order on $X \times Y$, $\overline{w} \leq \overline{y} \leq \overline{z}$

$\Rightarrow w_1 \leq y_1 \leq z_1$ and $w_2 \leq y_2 \leq z_2$

Thus, $\mu^P_{I_1}(y_1) \geq min\{\mu^P_{I_1}(w_1), \mu^P_{I_1}(z_1)\}$, $\mu^N_{I_1}(y_1) \leq max\{\mu^N_{I_1}(w_1), \mu^N_{I_1}(z_1)\}$ and

$\mu^P_{I_2}(y_2) \geq min\{\mu^P_{I_2}(w_2), \mu^P_{I_2}(z_2)\}$, $\mu^N_{I_2}(y_2) \leq max\{\mu^N_{I_2}(w_2), \mu^N_{I_2}(z_2)\}$.

Therefore, $\{\mu^P_{I_1}(y_1), \mu^P_{I_2}(y_2)\} \geq min\{min\{\mu^P_{I_1}(w_1), \mu^P_{I_1}(z_1)\}, min\{\mu^P_{I_2}(w_2), \mu^P_{I_2}(z_2)\}\}$

$= \{min\{\mu^P_{I_1}(w_1), \mu^P_{I_2}(w_2)\}, min\{\mu^P_{I_1}(z_1), \mu^P_{I_2}(z_2)\}$

$= \{(\mu^P_{I_1} \times \mu^P_{I_2})(\overline{w}), (\mu^P_{I_1} \times \mu^P_{I_2})(\overline{z})\}$

i.e. $min\{\mu^P_{I_1}(y_1), \mu^P_{I_2}(y_2)\} \geq min\{(\mu^P_{I_1} \times \mu^P_{I_2})(\overline{w}), (\mu^P_{I_1} \times \mu^P_{I_2})(\overline{z})\}$,

i.e. $(\mu^P_{I_1} \times \mu^P_{I_2})(\overline{y}) \geq min\{(\mu^P_{I_1} \times \mu^P_{I_2})(\overline{w}), (\mu^P_{I_1} \times \mu^P_{I_2})(\overline{z})\}$.

Similarly, $(\mu^N_{I_1} \times \mu^N_{I_2})(\overline{y}) \leq max\{(\mu^N_{I_1} \times \mu^N_{I_2})(\overline{w}), (\mu^N_{I_1} \times \mu^N_{I_2})(\overline{z})\}$.

Hence the result. □

Union of bipolar fuzzy intervals is also a bipolar fuzzy set which is defined here. They are in fact, bipolar fuzzy interval proved in Proposition 4.2.

Definition 4.2. *Let* $I_1 = (\mu^P_{I_1}, \mu^N_{I_1})$ *and* $I_2 = (\mu^P_{I_2}, \mu^N_{I_2})$ *be two bipolar fuzzy intervals on the linearly ordered sets* X *and* Y *respectively. Then the union of* I_1 *and* I_2 *is a bipolar fuzzy set on* $X \cup Y$ *denoted by* $I_1 \cup I_2 = (\mu^P_{I_1} \cup \mu^P_{I_2}, \mu^N_{I_1} \cup \mu^N_{I_2})$ *and is defined as:*
$(\mu^P_{I_1} \cup \mu^P_{I_2})(x) = \mu^P_{I_1}(x)$ *if* $x \in X - Y$,
$(\mu^P_{I_1} \cup \mu^P_{I_2})(x) = \mu^P_{I_2}(x)$ *if* $x \in Y - X$ *and*
$(\mu^P_{I_1} \cup \mu^P_{I_2})(x) = max\{\mu^P_{I_1}(x), \mu^P_{I_2}\}$ *if* $x \in X \cap Y$;
$(\mu^N_{I_1} \cup \mu^N_{I_2})(x) = \mu^N_{I_1}(x)$ *if* $x \in X - Y$,
$(\mu^N_{I_1} \cup \mu^N_{I_2})(x) = \mu^N_{I_2}(x)$ *if* $x \in Y - X$ *and*
$(\mu^N_{I_1} \cup \mu^N_{I_2})(x) = min\{\mu^N_{I_1}(x), \mu^N_{I_2}\}$ *if* $x \in X \cap Y$.

Proposition 4.2. *If* $I_1 = (\mu^P_{I_1}, \mu^N_{I_1})$ *and* $I_2 = (\mu^P_{I_2}, \mu^N_{I_2})$ *be two bipolar fuzzy intervals on the disjoint linearly ordered sets* X *and* Y *respectively, then* $I_1 \cup I_2$ *is a bipolar fuzzy interval on the ordered set* $X \cup Y$.

Proof. To show that $I_1 \cup I_2 = (\mu^P_{I_1} \cup \mu^P_{I_2}, \mu^N_{I_1} \cup \mu^N_{I_2})$ is a bipolar fuzzy interval, we show that it is a normal and convex bipolar fuzzy subset of $X \cup Y$.

Normality of $I_1 \cup I_2 = (\mu^P_{I_1} \cup \mu^P_{I_2}, \mu^N_{I_1} \cup \mu^N_{I_2})$ follows from the normality of $I_1 = (\mu^P_{I_1}, \mu^N_{I_1})$ and $I_2 = (\mu^P_{I_2}, \mu^N_{I_2})$.

For convexity of $I_1 \cup I_2 = (\mu^P_{I_1} \cup \mu^P_{I_2}, \mu^N_{I_1} \cup \mu^N_{I_2})$, we show that for $w, y, z \in X \cup Y$ with $w \le y \le z$ implies $(\mu^P_{I_1} \cup \mu^P_{I_2})(y) \ge min\{(\mu^P_{I_1} \cup \mu^P_{I_2})(w), (\mu^P_{I_1} \cup \mu^P_{I_2})(z)\}$ and $(\mu^N_{I_1} \cup \mu^N_{I_2})(y) \le max\{(\mu^N_{I_1} \cup \mu^N_{I_2})(w), (\mu^N_{I_1} \cup \mu^N_{I_2})(z)\}$.

Now $w, y, z \in X \cup Y$, means $w, y, z \in X$ or $w, y, z \in Y$ (since $X \cap Y = \emptyset$).

Case (i): Let $w, y, z \in X$ be such that $w \le y \le z$. Since I_1 is a bipolar fuzzy interval, therefore I_1 is a bipolar fuzzy convex set.

Convexity of I_1 implies that $\mu^P_{I_1}(y) \ge min\{\mu^P_{I_1}(w), \mu^P_{I_1}(z)\}$ and $\mu^N_{I_1}(y) \le max\{\mu^N_{I_1}(w), \mu^N_{I_1}(z)\}$.

Now $\mu^P_{I_1}(y) = (\mu^P_{I_1} \cup \mu^P_{I_2})(y), \mu^P_{I_1}(w) = (\mu^P_{I_1} \cup \mu^P_{I_2})(w), \mu^P_{I_1}(z) = (\mu^P_{I_1} \cup \mu^P_{I_2})(z)$ and $\mu^N_{I_1}(y) = (\mu^N_{I_1} \cup \mu^N_{I_2})(y), \mu^N_{I_1}(w) = (\mu^N_{I_1} \cup \mu^N_{I_2})(w), \mu^N_{I_1}(z) = (\mu^N_{I_1} \cup \mu^N_{I_2})(z)$.

Hence, $(\mu^P_{I_1} \cup \mu^P_{I_2})(y) \ge min\{(\mu^P_{I_1} \cup \mu^P_{I_2})(w), (\mu^P_{I_1} \cup \mu^P_{I_2})(z)\}$
and $(\mu^N_{I_1} \cup \mu^N_{I_2})(y) \le max\{(\mu^N_{I_1} \cup \mu^N_{I_2})(w), (\mu^N_{I_1} \cup \mu^N_{I_2})(z)\}$.

Case (ii): Let $w, y, z \in Y$ be such that $w \le y \le z$. Similarly, convexity of I_2 implies the convexity of $I_1 \cup I_2$. Hence, $I_1 \cup I_2$ is a bipolar fuzzy interval. □

The intersection of bipolar fuzzy intervals is only a bipolar fuzzy set. An example is constructed below to support this.

Definition 4.3. *Let* $\{I_i\}_{i \in \Lambda}$ *be a family of bipolar fuzzy intervals on a linearly ordered set* X. *Then* $\bigcap_{i \in \Lambda} I_i$ *is a bipolar fuzzy set on* X *denoted by* $\bigcap_{i \in \Lambda} I_i = (\mu^P_{\cap I_i}, \mu^N_{\cap I_i})$ *and is defined as* $\mu^P_{\cap I_i}(x) = min\{\mu^P_{I_i}(x) : i \in \Lambda\}$ *and* $\mu^N_{\cap I_i}(x) = max\{\mu^N_{I_i}(x) : i \in \Lambda\}$ *for all* $x \in X$.

The intersection of a family of bipolar fuzzy intervals on a linearly ordered set may not be a bipolar fuzzy interval. For example, let us consider the bipolar fuzzy intervals I_1 and I_2 as shown in Fig. 2. Here, $h(I_1 \cap I_2) = 0.2$ and $d(I_1 \cap I_2) = -0.45$.

So, $I_1 \cap I_2$ is not normal. Hence $I_1 \cap I_2$ is not a bipolar fuzzy interval.

For a given family of bipolar fuzzy sets, it's $\{\alpha, \beta\}$-level family is defined as follows.

Definition 4.4. *Let* $(\alpha, \beta) \in (0, 1] \times [-1, 0)$. *For a family* $\mathcal{F}$ *of bipolar fuzzy subsets,* $\{\alpha, \beta\}$*-level family of* $\mathcal{F}$ *is defined as* $\mathcal{F}^\alpha_\beta = \{B^\alpha_\beta : B \in \mathcal{F}\}$.

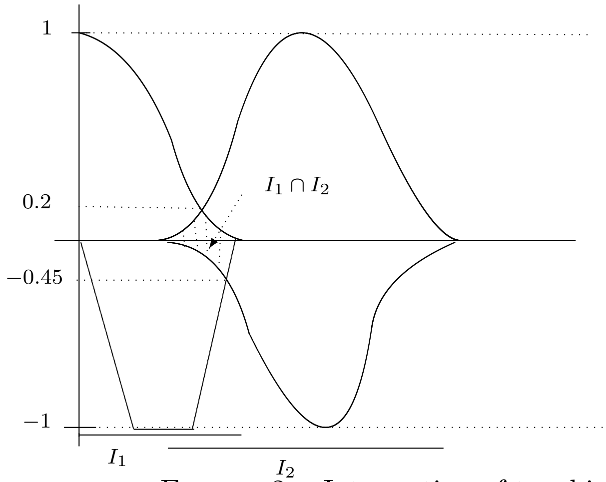


FIGURE 2. Intersection of two bipolar fuzzy intervals

5. Bipolar fuzzy interval graphs

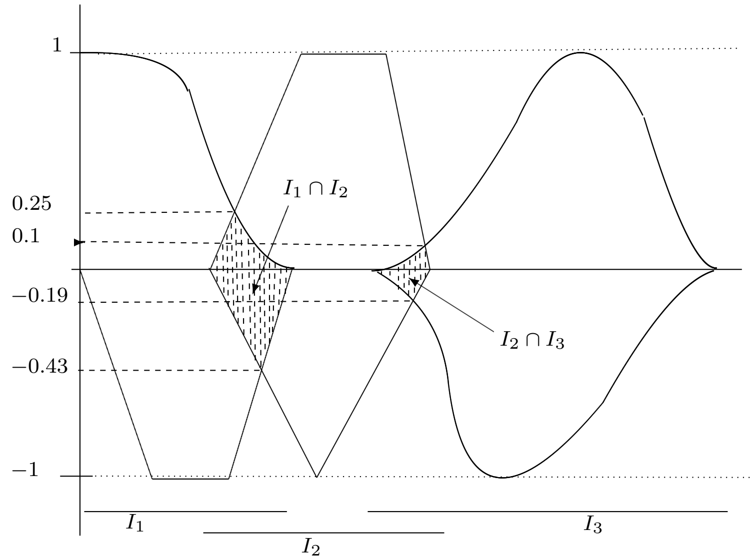
Fuzzy intersection graphs and fuzzy interval graphs were introduced in [18]. In 2014, Samanta et al. [28] introduced bipolar fuzzy intersection graphs of a finite family $\mathcal{F}$ of bipolar fuzzy sets. In this section, bipolar fuzzy interval graph is defined for a finite family $\mathcal{F}$ of bipolar fuzzy intervals defined in Section 3. In other words, bipolar fuzzy interval graphs are the intersection graphs of a finite family of bipolar fuzzy intervals. Also, the relationship between the intersection graph of a $\{\alpha,\beta\}$-level family of bipolar fuzzy intervals and $\{\alpha,\beta\}$-cut of the intersection graph of that family is established.

Definition 5.1. *Let $\mathcal{F}$=$\{I_1 = (\mu^P_{I_1}, \mu^N_{I_1}), I_2 = (\mu^P_{I_2}, \mu^N_{I_2}), \ldots, I_m = (\mu^P_{I_m}, \mu^N_{I_m})\}$ be a finite family of bipolar fuzzy intervals on a linearly ordered set X and consider $\mathcal{F}$ as a crisp vertex set V. The bipolar fuzzy interval graph of $\mathcal{F}$ is the bipolar fuzzy intersection graph G= $Int(\mathcal{F})$=(A,B) where $A = (\mu^P_A, \mu^N_A)$ is a bipolar fuzzy set of V defined by $\mu^P_A(I_i) = h(I_i) = 1$, $\mu^N_A(I_i) = d(I_i) = -1$ for all $i = 1, 2, \ldots, m$ and $B = (\mu^P_B, \mu^N_B)$ is a bipolar fuzzy set in $E = V \times V$ defined by $\mu^P_B(I_i, I_j) = h(I_i \cap I_j)$ if $i \neq j$ and 0 if $i = j$ and $\mu^N_B(I_i, I_j) = d(I_i \cap I_j)$ if $i \neq j$ and 0 if $i = j$; $i, j = 1, 2, \ldots, m$.*

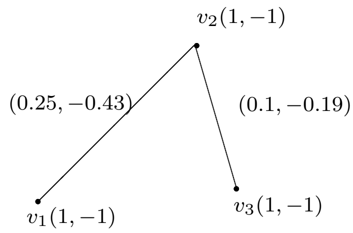
Example 5.1. *Let us consider three bipolar fuzzy intervals I_1, I_2, I_3 (see Fig. 3). Here $h(I_1 \cap I_2) = 0.25$, $d(I_1 \cap I_2) = -0.43$, $h(I_2 \cap I_3) = 0.1$, $d(I_2 \cap I_3) = -0.19$. The corresponding bipolar fuzzy interval graph is shown in Fig. 3.*

Remark 5.1. *If $\mathcal{F}$=$\{I_1 = (\mu^P_{I_1}, \mu^N_{I_1}), I_2 = (\mu^P_{I_2}, \mu^N_{I_2}), \ldots, I_m = (\mu^P_{I_m}, \mu^N_{I_m})\}$ is a family of bipolar fuzzy intervals on a linearly ordered set X and $(\alpha, \beta) \in (0, 1] \times [-1, 0)$, then $Int(\mathcal{F}^\alpha_\beta)$= $(Int(\mathcal{F}))^\alpha_\beta$. The graph $Int(\mathcal{F}^\alpha_\beta)$ has one vertex for each $I_i \in \mathcal{F}$ such that $(I_i)^\alpha_\beta \neq \phi$ or equivalently such that $h(I_i) \geq \alpha$ and $d(I_i) \leq \beta$. The pair $\{(I_i)^\alpha_\beta, (I_j)^\alpha_\beta\}$ is an edge of $Int(\mathcal{F}^\alpha_\beta)$ if and only if $i \neq j$ and $(I_i)^\alpha_\beta \cap (I_j)^\alpha_\beta \neq \phi$; equivalently if and only if $h(I_i \cap I_j) \geq \alpha$ and $d(I_i \cap I_j) \leq \beta$. Similarly, the graph $(Int(\mathcal{F}))^\alpha_\beta$ has one vertex for each $I_i \in \mathcal{F}$ such that $h(I_i) \geq \alpha$ and $d(I_i) \leq \beta$. The pair I_i, I_j is an edge of $(Int(\mathcal{F}))^\alpha_\beta$ if and only if $i \neq j$ and $h(I_i \cap I_j) \geq \alpha$ and $d(I_i \cap I_j) \leq \beta$. As graphs, two structures are equivalent.*

Theorem 5.1. *Let $G = Int(\mathcal{F})$ be a bipolar fuzzy interval graph. Then for each $(\alpha, \beta) \in (0, 1] \times [-1, 0)$, the $\{\alpha, \beta\}$-cut level graph G^α_β is an interval graph.*



(a) Three bipolar fuzzy intervals I_1, I_2 and I_3



(b) Bipolar fuzzy interval graphs corresponding to the bipolar fuzzy intervals of (a)

FIGURE 3. Bipolar fuzzy interval graph

Proof. Let $G = Int(\mathcal{F})$ for a family of bipolar fuzzy intervals $\mathcal{F}$=$\{I_1 = (\mu^P_{I_1}, \mu^N_{I_1}), I_2 = (\mu^P_{I_2}, \mu^N_{I_2}), \ldots, I_m = (\mu^P_{I_m}, \mu^N_{I_m})\}$.

For each $(\alpha, \beta) \in (0,1] \times [-1, 0)$, convexity implies that $I^\alpha_\beta \in \mathcal{F}^\alpha_\beta$ is a crisp interval. By Remark 5.1, $G^\alpha_\beta = (Int(\mathcal{F}))^\alpha_\beta$=$Int(\mathcal{F}^\alpha_\beta)$ and G^α_β is an interval graph. □

6. Operations on bipolar fuzzy interval graphs

In this section, some important hereditary properties of bipolar fuzzy interval graphs is discussed using the operations of Cartesian product, composition, union and join. New bipolar fuzzy interval graphs is constructed by applying the above operations on any two bipolar fuzzy interval graphs. Throughout this section the edge between two vertices u and v is denoted by uv rather than (u, v) because when we take Cartesian product, a vertex of the graph is, in fact, an ordered pair.

Definition 6.1. *Let $G_1 = Int(V_1) = (A_1, B_1)$ and $G_2 = Int(V_2) = (A_2, B_2)$ be two bipolar fuzzy interval graphs, where $A_1 = (\mu^P_{A_1}, \mu^N_{A_1})$ is a bipolar fuzzy subset of $V_1 = \{I^1_1 = (\mu^P_{I^1_1}, \mu^N_{I^1_1}), I^1_2 = (\mu^P_{I^1_2}, \mu^N_{I^1_2}), \ldots, I^1_m = (\mu^P_{I^1_m}, \mu^N_{I^1_m})\}$ and $A_2 = (\mu^P_{A_2}, \mu^N_{A_2})$ is a bipolar fuzzy subset of $V_2 = \{I^2_1 = (\mu^P_{I^2_1}, \mu^N_{I^2_1}), I^2_2 = (\mu^P_{I^2_2}, \mu^N_{I^2_2}), \ldots, I^2_n = (\mu^P_{I^2_n}, \mu^N_{I^2_n})\}$; $I^1_i = (\mu^P_{I^1_i}, \mu^N_{I^1_i}), i = 1, 2, \ldots, m$ and $I^2_j = (\mu^P_{I^2_j}, \mu^N_{I^2_j}); j = 1, 2, \ldots, n$ are the bipolar fuzzy intervals on the linearly ordered sets X and Y respectively. $B_1 = (\mu^P_{B_1}, \mu^N_{B_1})$ and $B_2 = (\mu^P_{B_2}, \mu^N_{B_2})$ are bipolar* *subsets of* $E_1 = V_1 \times V_1$ *and* $E_2 = V_2 \times V_2$ *respectively. Then the Cartesian product of two bipolar fuzzy interval graphs* G_1 *and* G_2 *is denoted by* $G_1 \times G_2 = (A_1 \times A_2, B_1 \times B_2)$ *and is defined as:*

(i) *For all* $(I_i^1, I_j^2) \in V_1 \times V_2$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$

$(\mu_{A_1}^P \times \mu_{A_2}^P)(I_i^1, I_j^2) = min\{h(I_i^1) = 1, h(I_j^2) = 1\} = 1$

$(\mu_{A_1}^N \times \mu_{A_2}^N)(I_i^1, I_j^2) = max\{d(I_i^1) = -1, d(I_j^2) = -1\} = -1.$

(ii) *For all* $I_i^1 \in V_1$, *for all* $I_j^2 I_k^2 \in E_2$, $i = 1, 2, \ldots, m$, $j, k = 1, 2, \ldots, n$

$(\mu_{B_1}^P \times \mu_{B_2}^P)((I_i^1, I_j^2)(I_i^1, I_k^2)) = min\{h(I_i^1) = 1, h(I_j^2 \cap I_k^2)\}$

$(\mu_{B_1}^N \times \mu_{B_2}^N)((I_i^1, I_j^2)(I_i^1, I_k^2)) = max\{d(I_i^1) = -1, d(I_j^2 \cap I_k^2)\}.$

(iii) *For all* $I_i^2 \in V_2$, *for all* $I_j^1 I_k^1 \in E_1$, $i = 1, 2, \ldots, n$, $j, k = 1, 2, \ldots, m$

$(\mu_{B_1}^P \times \mu_{B_2}^P)((I_j^1, I_i^2)(I_k^1, I_i^2)) = min\{h(I_j^1 \cap I_k^1), h(I_i^2) = 1\}$

$(\mu_{B_1}^N \times \mu_{B_2}^N)((I_j^1, I_i^2)(I_k^1, I_i^2)) = max\{d(I_j^1 \cap I_k^1), d(I_i^2) = -1\}.$

Proposition 6.1. *If* G_1 *and* G_2 *are the two bipolar fuzzy interval graphs, then* $G_1 \times G_2$ *is a bipolar fuzzy interval graph.*

Proof. Let $I_i^1 \in V_1$, $I_j^2 I_k^2 \in E_2$ where $i = 1, 2, \ldots, m$; $j, k = 1, 2, \ldots, n$.

Then

$(\mu_{B_1}^P \times \mu_{B_2}^P)((I_i^1, I_j^2)(I_i^1, I_k^2))$
$= min\{h(I_i^1) = 1, h(I_j^2 \cap I_k^2)\}$
$\leq 1 = min\{min\{h(I_i^1), h(I_j^2)\}, min\{h(I_i^1), h(I_k^2)\}\}$
$= min\{(\mu_{A_1}^P \times \mu_{A_2}^P)(I_i^1, I_j^2), (\mu_{A_1}^P \times \mu_{A_2}^P)(I_i^1, I_k^2)\}.$

$(\mu_{B_1}^N \times \mu_{B_2}^N)((I_i^1, I_j^2)(I_i^1, I_k^2))$
$= max\{d(I_i^1) = -1, d(I_j^2 \cap I_k^2)\}$
$\geq -1 = max\{max\{d(I_i^1), d(I_j^2)\}, max\{d(I_i^1), d(I_k^2)\}\}$
$= max\{(\mu_{A_1}^N \times \mu_{A_2}^N)(I_i^1, I_j^2), (\mu_{A_1}^N \times \mu_{A_2}^N)(I_i^1, I_k^2)\}.$

Let $I_i^2 \in V_2, I_j^1 I_k^1 \in E_1$ where $i = 1, 2, \ldots, n$, $j, k = 1, 2, \ldots, m$.

Then

$(\mu_{B_1}^P \times \mu_{B_2}^P)((I_j^1, I_i^2)(I_k^1, I_i^2))$
$= min\{h(I_j^1 \cap I_k^1), h(I_i^2) = 1\}$
$\leq 1 = min\{min\{h(I_j^1), h(I_i^2)\}, min\{h(I_k^1), h(I_i^2)\}\}$
$= min\{(\mu_{A_1}^P \times \mu_{A_2}^P)(I_j^1, I_i^2), (\mu_{A_1}^P \times \mu_{A_2}^P)(I_k^1, I_i^2)\}.$

$(\mu_{B_1}^N \times \mu_{B_2}^N)((I_j^1, I_i^2)(I_k^1, I_i^2))$
$= max\{d(I_j^1 \cap I_k^1), d(I_i^2) = -1\}$
$\geq -1 = max\{max\{d(I_j^1), d(I_i^2)\}, max\{d(I_k^1), d(I_i^2)\}\}$
$= max\{(\mu_{A_1}^N \times \mu_{A_2}^N)(I_j^1, I_i^2), (\mu_{A_1}^N \times \mu_{A_2}^N)(I_k^1, I_i^2)\}.$

By Proposition 4.1, $V_1 \times V_2 = \{I_i^1 \times I_j^2 : I_i^1 \in V_1, I_j^2 \in V_2, i = 1, 2, \ldots, m; j = 1, 2, \ldots, n\}$ is a family of bipolar fuzzy intervals on $X \times Y$. Therefore, $G_1 \times G_2 = Int(V_1 \times V_2)$.

This completes the proof. □

Now the composition of two bipolar fuzzy interval graphs is considered.

Definition 6.2. *Let* $G_1 = Int(V_1) = (A_1, B_1)$ *and* $G_2 = Int(V_2) = (A_2, B_2)$ *be two bipolar fuzzy interval graphs, where* $A_1 = (\mu_{A_1}^P, \mu_{A_1}^N)$ *is a bipolar fuzzy subset of* $V_1 = \{I_1^1 = (\mu_{I_1^1}^P, \mu_{I_1^1}^N), I_2^1 = (\mu_{I_2^1}^P, \mu_{I_2^1}^N), \ldots, I_m^1 = (\mu_{I_m^1}^P, \mu_{I_m^1}^N)\}$ *and* $A_2 = (\mu_{A_2}^P, \mu_{A_2}^N)$ *is a bipolar fuzzy subset of* $V_2 = \{I_1^2 = (\mu_{I_1^2}^P, \mu_{I_1^2}^N), I_2^2 = (\mu_{I_2^2}^P, \mu_{I_2^2}^N), \ldots, I_n^2 = (\mu_{I_n^2}^P, \mu_{I_n^2}^N)\}$*;* $I_i^1 = (\mu_{I_i^1}^P, \mu_{I_i^1}^N), i = 1, 2, \ldots, m$ *and* $I_j^2 = (\mu_{I_j^2}^P, \mu_{I_j^2}^N); j = 1, 2, \ldots, n$ *are the bipolar fuzzy intervals on the linearly* *ordered sets X and Y respectively. $B_1 = (\mu^P_{B_1}, \mu^N_{B_1})$ and $B_2 = (\mu^P_{B_2}, \mu^N_{B_2})$ are bipolar subsets of $E_1 = V_1 \times V_1$ and $E_2 = V_2 \times V_2$ respectively. Then the composition of G_1 and G_2 is denoted by $G_1[G_2] = (A_1 \circ A_2, B_1 \circ B_2)$ and is defined as:*

(i) For all $(I^1_i, I^2_j) \in V_1 \times V_2$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$

$$(\mu^P_{A_1} \circ \mu^P_{A_2})(I^1_i, I^2_j) = min\{h(I^1_i) = 1, h(I^2_j) = 1\} = 1$$
$$(\mu^N_{A_1} \circ \mu^N_{A_2})(I^1_i, I^2_j) = max\{d(I^1_i) = -1, d(I^2_j) = -1\} = -1.$$

(ii) For all $I^1_i \in V_1$, for all $I^2_j I^2_k \in E_2$, $i = 1, 2, \ldots, m$, $j, k = 1, 2, \ldots, n$

$$(\mu^P_{B_1} \circ \mu^P_{B_2})((I^1_i, I^2_j)(I^1_i, I^2_k)) = min\{h(I^1_i) = 1, h(I^2_j \cap I^2_k)\}$$
$$(\mu^N_{B_1} \circ \mu^N_{B_2})((I^1_i, I^2_j)(I^1_i, I^2_k)) = max\{d(I^1_i) = -1, d(I^2_j \cap I^2_k)\}.$$

(iii) For all $I^2_i \in V_2$, for all $I^1_j I^1_k \in E_1$, $i = 1, 2, \ldots, n$, $j, k = 1, 2, \ldots, m$

$$(\mu^P_{B_1} \circ \mu^P_{B_2})((I^1_j, I^2_i)(I^1_k, I^2_i)) = min\{h(I^1_j \cap I^1_k), h(I^2_i) = 1\}$$
$$(\mu^N_{B_1} \circ \mu^N_{B_2})((I^1_j, I^2_i)(I^1_k, I^2_i)) = max\{d(I^1_j \cap I^1_k), d(I^2_i) = -1\}.$$

(iv) For all $(I^1_i, I^2_j)(I^1_k, I^2_l) \in E^0 - E$, $i, k = 1, 2, \ldots, m$, $j, l = 1, 2, \ldots, n$

$$(\mu^P_{B_1} \circ \mu^P_{B_2})((I^1_i, I^2_j)(I^1_k, I^2_l)) = min\{h(I^2_j) = 1, h(I^2_l) = 1, h(I^1_i \cap I^1_k)\}$$
$$(\mu^N_{B_1} \circ \mu^N_{B_2})((I^1_i, I^2_j)(I^1_k, I^2_l)) = max\{d(I^2_j) = -1, d(I^2_l) = -1, d(I^1_i \cap I^1_k)\}$$

where $E^0 = \{(I^1_i, I^2_j)(I^1_i, I^2_k) : I^1_i \in V_1; I^2_j I^2_k \in E_2; i = 1, 2, \ldots, m; j, k = 1, 2, \ldots, n\} \cup \{(I^1_j, I^2_i)(I^1_k, I^2_i) : I^2_i \in V_2; I^1_j I^1_k \in E_1; i = 1, 2, \ldots, n; j, k = 1, 2, \ldots, m\} \cup \{(I^1_i, I^2_j)(I^1_k, I^2_l) : I^1_i I^1_k \in E_1; I^2_j \neq I^2_l; i, k = 1, 2, \ldots, m; j, l = 1, 2, \ldots, n\}$ *and* $E = \{(I^1_i, I^2_j)(I^1_i, I^2_k) : I^1_i \in V_1; I^2_j I^2_k \in E_2; i = 1, 2, \ldots, m; j, k = 1, 2, \ldots, n\} \cup \{(I^1_j, I^2_i)(I^1_k, I^2_i) : I^2_i \in V_2; I^1_j I^1_k \in E_1; i = 1, 2, \ldots, n; j, k = 1, 2, \ldots, m\}$.

Proposition 6.2. *If G_1 and G_2 are two bipolar fuzzy interval graphs, then $G_1[G_2]$ is a bipolar fuzzy interval graph.*

Proof. Let $I^1_i \in V_1$, $I^2_j I^2_k \in E_2$; where $i = 1, 2, \ldots, m$; $j, k = 1, 2, \ldots, n$.

Then

$(\mu^P_{B_1} \circ \mu^P_{B_2})((I^1_i, I^2_j)(I^1_i, I^2_k))$
$= min\{h(I^1_i) = 1, h(I^2_j \cap I^2_k)\} \leq 1 = min\{min\{h(I^1_i), h(I^2_j)\}, min\{h(I^1_i), h(I^2_k)\}\}$
$= min\{(\mu^P_{A_1} \circ \mu^P_{A_2})(I^1_i, I^2_j), (\mu^P_{A_1} \circ \mu^P_{A_2})(I^1_i, I^2_k)\}$.

$(\mu^N_{B_1} \circ \mu^N_{B_2})((I^1_i, I^2_j)(I^1_i, I^2_k))$
$= max\{d(I^1_i) = -1, d(I^2_j \cap I^2_k)\} \geq -1 = max\{max\{d(I^1_i), d(I^2_j)\}, max\{d(I^1_i), d(I^2_k)\}\}$
$= max\{(\mu^N_{A_1} \circ \mu^N_{A_2})(I^1_i, I^2_j), (\mu^N_{A_1} \circ \mu^N_{A_2})(I^1_i, I^2_k)\}$.

Let $I^2_i \in V_2, I^1_j I^1_k \in E_1$ where $i = 1, 2, \ldots, n$; $j, k = 1, 2, \ldots, m$.

In this case,

$(\mu^P_{B_1} \circ \mu^P_{B_2})((I^1_j, I^2_i)(I^1_k, I^2_i))$
$= min\{h(I^1_j \cap I^1_k), h(I^2_i) = 1\} \leq 1 = min\{min\{h(I^1_j), h(I^2_i)\}, min\{h(I^1_k), h(I^2_i)\}\}$
$= min\{(\mu^P_{A_1} \circ \mu^P_{A_2})(I^1_j, I^2_i), (\mu^P_{A_1} \circ \mu^P_{A_2})(I^1_k, I^2_i)\}$.

$(\mu^N_{B_1} \circ \mu^N_{B_2})((I^1_j, I^2_i)(I^1_k, I^2_i))$
$= max\{d(I^1_j \cap I^1_k), d(I^2_i) = -1\} \geq -1 = max\{max\{d(I^1_j), d(I^2_i)\}, max\{d(I^1_k), d(I^2_i)\}\}$
$= max\{(\mu^N_{A_1} \circ \mu^N_{A_2})(I^1_j, I^2_i), (\mu^N_{A_1} \circ \mu^N_{A_2})(I^1_k, I^2_i)\}$.

Let $(I^1_i, I^2_j)(I^1_k, I^2_l) \in E^0 - E$; $i, k = 1, 2, \ldots, m$; $j, l = 1, 2, \ldots, n$.

So, $I^1_i I^1_k \in E_1, I^2_j \neq I^2_l$.

Now,

$(\mu^P_{B_1} \circ \mu^P_{B_2})((I^1_i, I^2_j)(I^1_k, I^2_l))$
$= min\{h(I^2_j) = 1, h(I^2_l) = 1, h(I^1_i \cap I^1_k)\} \leq 1 = min\{min\{h(I^1_i), h(I^2_j)\}, min\{h(I^1_k), h(I^2_l)\}\}$
$= min\{(\mu^P_{A_1} \circ \mu^P_{A_2})(I^1_i, I^2_j), (\mu^P_{A_1} \circ \mu^P_{A_2})(I^1_k, I^2_l)\}$.

$(\mu^N_{B_1} \circ \mu^N_{B_2})((I^1_i, I^2_j)(I^1_k, I^2_l))$
$= max\{d(I^2_j) = -1, d(I^2_l) = -1, d(I^1_i \cap I^1_k)\} \geq -1 = max\{max\{d(I^1_i), d(I^2_j)\}, max\{d(I^1_k), d(I^2_l)\}\}$
$= max\{(\mu^N_{A_1} \circ \mu^N_{A_2})(I^1_i, I^2_j), (\mu^N_{A_1} \circ \mu^N_{A_2})(I^1_k, I^2_l)\}$.

By Proposition 4.1, $V_1 \times V_2 = \{I^1_i \times I^2_j : I^1_i \in V_1, I^2_j \in V_2, i = 1, 2, \ldots, m; j = 1, 2, \ldots, n\}$ is a family of bipolar fuzzy intervals on $X \times Y$.

Therefore, $G_1 \circ G_2 = Int(V_1 \times V_2)$.

Hence $G_1[G_2]$ is a bipolar fuzzy interval graph. □

Below, the union of two bipolar fuzzy interval graphs is considered.

Definition 6.3. *Let $G_1 = Int(V_1) = (A_1, B_1)$ and $G_2 = Int(V_2) = (A_2, B_2)$ be two bipolar fuzzy interval graphs, where $A_1 = (\mu^P_{A_1}, \mu^N_{A_1})$ is a bipolar fuzzy subset of $V_1 = \{I^1_1 = (\mu^P_{I^1_1}, \mu^N_{I^1_1}), I^1_2 = (\mu^P_{I^1_2}, \mu^N_{I^1_2}), \ldots, I^1_m = (\mu^P_{I^1_m}, \mu^N_{I^1_m})\}$ and $A_2 = (\mu^P_{A_2}, \mu^N_{A_2})$ is a bipolar fuzzy subset of $V_2 = \{I^2_1 = (\mu^P_{I^2_1}, \mu^N_{I^2_1}), I^2_2 = (\mu^P_{I^2_2}, \mu^N_{I^2_2}), \ldots, I^2_n = (\mu^P_{I^2_n}, \mu^N_{I^2_n})\}$; $I^1_i = (\mu^P_{I^1_i}, \mu^N_{I^1_i}), i = 1, 2, \ldots, m$ and $I^2_j = (\mu^P_{I^2_j}, \mu^N_{I^2_j}); j = 1, 2, \ldots, n$ are the bipolar fuzzy intervals on the disjoint linearly ordered sets X and Y respectively. $B_1 = (\mu^P_{B_1}, \mu^N_{B_1})$ and $B_2 = (\mu^P_{B_2}, \mu^N_{B_2})$ are bipolar subsets of $E_1 = V_1 \times V_1$ and $E_2 = V_2 \times V_2$ respectively where it is assumed that $V_1 \cap V_2 = \emptyset$. Then the union of G_1 and G_2 is denoted by $G_1 \cup G_2 = (A_1 \cup A_2, B_1 \cup B_2)$ and is defined as:*

(i) $(\mu^P_{A_1} \cup \mu^P_{A_2})(I^1_i) = h(I^1_i) = 1$ *for all* $I^1_i \in V_1$, $i = 1, 2, \ldots, m$
$(\mu^P_{A_1} \cup \mu^P_{A_2})(I^2_i) = h(I^2_i) = 1$ *for all* $I^2_i \in V_2$; $j = 1, 2, \ldots, n$
$(\mu^P_{A_1} \cup \mu^P_{A_2})(I) = h(I) = 1$ *if* $I \in V_1 \cap V_2$
$(\mu^N_{A_1} \cup \mu^N_{A_2})(I^1_i) = d(I^1_i) = -1$ *for all* $I^1_i \in V_1$, $i = 1, 2, \ldots, m$
$(\mu^N_{A_1} \cup \mu^N_{A_2})(I^2_i) = d(I^2_i) = -1$ *for all* $I^2_i \in V_2$, $j = 1, 2, \ldots, n$
$(\mu^N_{A_1} \cup \mu^N_{A_2})(I) = d(I) = -1$ *if* $I \in V_1 \cap V_2$.

(ii) $(\mu^P_{B_1} \cup \mu^P_{B_2})(I^1_i I^1_j) = h(I^1_i \cap I^1_j)$ *for all* $I^1_i I^1_j \in E_1$, $i, j = 1, 2, \ldots, m$
$(\mu^P_{B_1} \cup \mu^P_{B_2})(I^2_i I^2_j) = h(I^2_i \cap I^2_j)$ *for all* $I^2_i I^2_j \in E_2$, $i, j = 1, 2, \ldots, n$
$(\mu^P_{B_1} \cup \mu^P_{B_2})(I_1 I_2) = h(I_1 \cap I_2)$ *if* $I_1 I_2 \in E_1 \cap E_2$
$(\mu^N_{B_1} \cup \mu^N_{B_2})(I^1_i I^1_j) = d(I^1_i \cap I^1_j)$ *for all* $I^1_i I^1_j \in E_1$, $i, j = 1, 2, \ldots, m$
$(\mu^N_{B_1} \cup \mu^N_{B_2})(I^2_i I^2_j) = d(I^2_i \cap I^2_j)$ *for all* $I^2_i I^2_j \in E_2$, $i, j = 1, 2, \ldots, n$
$(\mu^N_{B_1} \cup \mu^N_{B_2})(I_1 I_2) = d(I_1 \cap I_2)$ *if* $I_1 I_2 \in E_1 \cap E_2$.

Example 6.1. *Consider the bipolar fuzzy interval graphs $G_1 = Int(V_1) = (A_1, B_1)$ and $G_2 = Int(V_2) = (A_2, B_2)$, where $V_1 = \{I_1, I_2, I_3, I_4\}$ and $V_2 = \{I_2, I_4, I_5, I_6, I_7\}$(see Fig. 4). A_i is a bipolar fuzzy set on V_i given by $\mu^P_{A_i}(I_j) = h(I_j) = 1$ and $\mu^N_{A_i}(I_j) = d(I_j) = -1$ for all $i = 1, 2; j = 1, 2, \ldots, 7$. B_i is a bipolar fuzzy subset on $V_i \times V_i$, $i = 1, 2$ given by $\mu^P_{B_1}(I_1 I_2) = h(I_1 \cap I_2) = 0.2$, $\mu^N_{B_1}(I_1 I_2) = d(I_1 \cap I_2) = -0.5$, $\mu^P_{B_1}(I_1 I_3) = h(I_1 \cap I_3) = 0.3$, $\mu^N_{B_1}(I_1 I_3) = d(I_1 \cap I_3) = -0.3$, $\mu^P_{B_1}(I_2 I_4) = h(I_2 \cap I_4) = 0.4$, $\mu^N_{B_1}(I_2 I_4) = d(I_2 \cap I_4) = -0.3$, $\mu^P_{B_1}(I_3 I_4) = h(I_3 \cap I_4) = 0.1$, $\mu^N_{B_1}(I_3 I_4) = d(I_3 \cap I_4) = -0.2$.*

$\mu^P_{B_2}(I_2 I_5) = h(I_2 \cap I_5) = 0.2$, $\mu^N_{B_2}(I_2 I_5) = d(I_2 \cap I_5) = -0.8$, $\mu^P_{B_2}(I_5 I_6) = h(I_5 \cap I_6) = 0.1$, $\mu^N_{B_2}(I_5 I_6) = d(I_5 \cap I_6) = -0.4$, $\mu^P_{B_2}(I_6 I_7) = h(I_6 \cap I_7) = 0.3$, $\mu^N_{B_2}(I_6 I_7) = d(I_6 \cap I_7) = -0.5$, $\mu^P_{B_2}(I_4 I_5) = h(I_4 \cap I_5) = 0.31$, $\mu^N_{B_2}(I_4 I_5) = d(I_4 \cap I_5) = -0.52$, $\mu^P_{B_2}(I_5 I_7) = h(I_5 \cap I_7) = 0.6$, $\mu^N_{B_2}(I_5 I_7) = d(I_5 \cap I_7) = -0.5$. Then by Definition 6.3, we have constructed $G_1 \cup G_2$ which is shown in Fig. 4.

Proposition 6.3. *If G_1 and G_2 are two bipolar fuzzy interval graphs, then $G_1 \cup G_2$ is a bipolar fuzzy interval graph.*

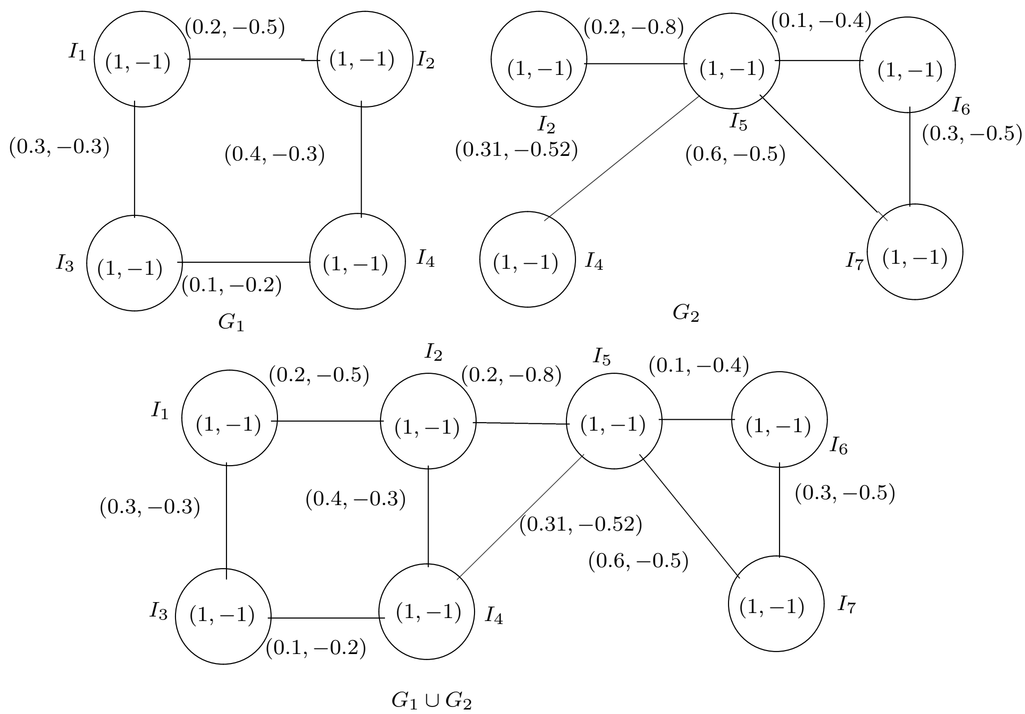


FIGURE 4. Union of two bipolar fuzzy interval graphs

Proof. Follows from the definition. □

Finally, the join of two bipolar fuzzy interval graphs is considered. In joining the graphs, it additionally contains more edges compared to the union of graphs.

Definition 6.4. *Let* $G_1 = Int(V_1) = (A_1, B_1)$ *and* $G_2 = Int(V_2) = (A_2, B_2)$ *be two bipolar fuzzy interval graphs, where* $A_1 = (\mu^P_{A_1}, \mu^N_{A_1})$ *is a bipolar fuzzy subset of* $V_1 = \{I^1_1 = (\mu^P_{I^1_1}, \mu^N_{I^1_1}), I^1_2 = (\mu^P_{I^1_2}, \mu^N_{I^1_2}), \ldots, I^1_m = (\mu^P_{I^1_m}, \mu^N_{I^1_m})\}$ *and* $A_2 = (\mu^P_{A_2}, \mu^N_{A_2})$ *is a bipolar fuzzy subset of* $V_2 = \{I^2_1 = (\mu^P_{I^2_1}, \mu^N_{I^2_1}), I^2_2 = (\mu^P_{I^2_2}, \mu^P_{I^2_2}), \ldots, I^2_n = (\mu^P_{I^2_n}, \mu^P_{I^2_n})\}$*;* $I^1_i = (\mu^P_{I^1_i}, \mu^N_{I^1_i}), i = 1, 2, \ldots, m$ *and* $I^2_j = (\mu^P_{I^2_j}, \mu^N_{I^2_j}); j = 1, 2, \ldots, n$ *are the bipolar fuzzy intervals on the disjoint linearly ordered sets X and Y respectively.* $B_1 = (\mu^P_{B_1}, \mu^N_{B_1})$ *and* $B_2 = (\mu^P_{B_2}, \mu^N_{B_2})$ *are bipolar subsets of* $E_1 = V_1 \times V_1$ *and* $E_2 = V_2 \times V_2$ *respectively. Then the union of* G_1 *and* G_2 *is denoted by* $G_1 + G_2 = (A_1 + A_2, B_1 + B_2)$ *and is defined as:*

(i) $(\mu^P_{A_1} + \mu^P_{A_2})(I) = (\mu^P_{A_1} \cup \mu^P_{A_2})(I)$
$(\mu^N_{A_1} + \mu^N_{A_2})(I) = (\mu^N_{A_1} \cup \mu^N_{A_2})(I)$ *if* $I \in V_1 \cup V_2$.

(ii) $(\mu^P_{B_1} + \mu^P_{B_2})(I_1I_2) = (\mu^P_{B_1} \cup \mu^P_{B_2})(I_1I_2)$
$(\mu^N_{B_1} + \mu^N_{B_2})(I_1I_2) = (\mu^N_{B_1} \cup \mu^N_{B_2})(I_1I_2)$ *if* $I_1I_2 \in E_1 \cup E_2$.

(iii) $(\mu^P_{B_1} + \mu^P_{B_2})(I_1I_2) = max\{h(I_1), h(I_2)\}$
$(\mu^N_{B_1} + \mu^N_{B_2})(I_1I_2) = min\{d(I_1), d(I_2)\}$ *if* $I_1I_2 \in E'$ *where* E' *is the set of all edges joining the edges of* V_1 *and* V_2.

Proposition 6.4. *If* G_1 *and* G_2 *are two bipolar fuzzy interval graphs, then* $G_1 + G_2$ *is a bipolar fuzzy interval graph.*

Proof. Similar to the other cases. □

7. Conclusions

The use of intersection graphs, fuzzy intersection graphs and in particular, interval graphs, fuzzy interval graphs are very important to understand a number of real world problems. So, in this paper, a natural extension to fuzzy intervals, fuzzy interval graphs are introduced and studied several results of it. Finally, bipolar fuzzy interval graph is introduced. The developed theoretical results of this paper will certainly help to find more important results and algorithms on the mentioned field. Our next plan is to extend our research work on applications of bipolar fuzzy interval graphs, m-polar fuzzy graphs, m-polar fuzzy intersection graphs, m-polar fuzzy interval graphs, m-polar fuzzy hypergraphs, etc.

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