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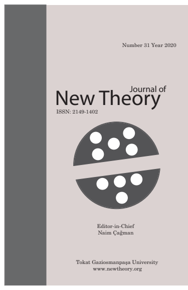


# Hesitant Fuzzy $h$-ideals of $\Gamma$-hemirings

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**Abstract** − The purpose of this paper is to introduce and study hesitant fuzzy h-ideals ( h-bi-ideals, h-quasi-ideals) of a Γ-hemiring. We investigate several properties these ideals. We show that hesitant fuzzy ideals are closed under intersection, cartesian product and composition. We also obtain some inter-relations between these ideals and characterizations of h-regular Γ-hemiring.

***Keywords*** − *Fuzzy, hesitant, ideal, cartesian product, intersection, regular, Γ-hemiring.*

## 1. Introduction

Semiring, introduced by Vandiver [1] in 1934 with two associative binary operations where one distributes over the other. In structure, semirings lie between semigroups and rings. The results which hold in rings but not in semigroups may hold in semirings, since semiring is a generalization of ring. Also, semirings has some applications to the theory of automata, formal languages, optimization theory and other branches of applied mathematics. Ideals of semiring play a central role in the structure theory and useful for many purposes. However they do not in general coincide with the usual ring ideals and for this reason, their use is somewhat limited in trying to obtain analogues of ring theorems for semiring. To ammend this gap Henriksen [2] defined a more restricted class of ideals, which are called $k$-ideals. A still more restricted class of ideals in hemirings are given by Iizuka [3], which are called $h$-ideals. Torre [4], investigated $h$-ideals and $k$-ideals in hemirings in an effort to obtain analogues of ring theorems for hemiring and to amend the gap between ring ideals and semiring ideals. The notion of Γ-semiring was introduced by Rao [5] as a generalization of Γ-ring as well as of semiring. Γ-semirings also includes ternary semirings and provide algebraic home to nonpositives cones of totally ordered rings.

The theory of fuzzy sets, proposed by Zadeh [6], has provided a useful mathematical tool for describing the behavior of the systems that are too complex or illdefined to admit precise mathematical analysis by classical methods and tools. Since then several extensions and generalizations of fuzzy sets have been introduced in the literature, for example, intuitionistic fuzzy sets [7], interval valued fuzzy sets [8], fuzzy multisets [9] etc.. As an important generalization of these notions, in 2010, Torra [10] introduced the hesitant fuzzy set which permits the membership degree of an element to a set to be represented by a set of possible values between 0 and 1. The hesitant fuzzy set therefore provides a more accurate representation of peoples hesitancy in stating their preferences over objects than the fuzzy set or its classical extensions. Hesitant fuzzy set theory has been applied to several practical problems, see [11–18]. Jun et al. [19] applied notion of hesitant fuzzy sets to semigroups and

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investigated several properties. Since then many researchers developed this ideas.
The main aim of this paper is to study some properties of ideals of Γ-hemiring using hesitant fuzzy set. We also obtain some characterizations.

## 2. Preliminaries

We recall the following preliminaries for subsequent use.

**Definition 2.1.** Let S and Γ be two additive commutative semigroups with zero. Then S is called a Γ-hemiring if there exists a mapping
$S \times \Gamma \times S \to S$ ( (a,α,b) $\mapsto$ aαb) satisfying the following conditions:
(i) $(a+b)\alpha c = a\alpha c + b\alpha c$,
(ii) $a\alpha(b+c) = a\alpha b + a\alpha c$,
(iii) $a(\alpha+\beta)b = a\alpha b + a\beta b$,
(iv) $a\alpha(b\beta c) = (a\alpha b)\beta c$.
(v) $0_S\alpha a = 0_S = a\alpha 0_S$,
(vi)$a0_\Gamma b = 0_S = b0_\Gamma a$
for all $a, b, c \in S$ and for all $\alpha, \beta \in \Gamma$.
For simplification we write 0 instead of $0_S$ and $0_\Gamma$.

A subset $A$ of a Γ-hemiring $S$ is called a left(resp. right) ideal of $S$ if $A$ is closed under addition and $S\Gamma A \subseteq A$ (resp. $A\Gamma S \subseteq A$). A subset $A$ of a Γ-hemiring $S$ is called an ideal if it is both left and right ideal of $S$.
A subset $A$ of a Γ-hemiring $S$ is called a quasi-ideal of S if $A$ is closed under addition and $S\Gamma A \cap A\Gamma S \subseteq A$.
A subset $A$ of a Γ-hemiring $S$ is called a bi-ideal if $A$ is closed under addition and $A\Gamma S\Gamma A \subseteq A$.
A left ideal $A$ of $S$ is called a left $h$-ideal if $x, z \in S$, $a, b \in A$ and $x + a + z = b + z$ implies $x \in A$. A right $h$-ideal is defined analogously.

**Definition 2.2.** A fuzzy subset of a non-empty set $S$ is defined as a function $\mu : S \to [0, 1]$.

**Definition 2.3.** Hesitant fuzzy set on S in terms of a function $H$ that when applied to S returns a subset of [0, 1].

Throughout this paper unless otherwise mentioned $S$ denotes the Γ-hemiring and for any two set $P$ and $Q$, we use the following notation:

$$\cap(P, Q) = P \cap Q \text{ and } \cup(P, Q) = P \cup Q.$$

## 3. Hesitant fuzzy h-ideals

In this section, the notions of hesitant fuzzy ideals in Γ-hemiring are introduced and some of their basic properties are investigated.

**Definition 3.1.** Let $H$ be a non empty hesitant fuzzy subset of a Γ-hemiring $S$. Then $H$ is called a hesitant fuzzy left ideal [ hesitant fuzzy right ideal] of S if

(i) $H(x+y) \supseteq \cap\{H(x), H(y)\}$

(ii) $H(x\gamma y) \supseteq H(y)$ [respectively $H(x\gamma y) \supseteq H(x)$].

for all $x, y \in S$ and $\gamma \in \Gamma$.
A hesitant fuzzy ideal of a Γ-hemiring $S$ is a non empty hesitant fuzzy subset of $S$ which is a hesitant fuzzy left ideal as well as a hesitant fuzzy right ideal of $S$.
Note that if $H$ is a hesitant fuzzy left or right ideal of a Γ-hemiring $S$, then $H(0) \supseteq H(x)$ for all $x \in S$.

**Definition 3.2.** A hesitant fuzzy left ideal $H$ of a Γ-hemiring $S$ is called a hesitant fuzzy left $h$-ideal if for all $a, b, x, z \in S$, $x + a + z = b + z \Rightarrow H(x) \supseteq \cap\{H(a), H(b)\}$.

A hesitant fuzzy right $h$-ideal is defined similarly.

**Example 3.3.** Let $S = \Gamma$ =the set of non-positive integers. Then $S$ forms a $\Gamma$-hemiring with usual addition and multiplication of integers. Define $H$ be a hesitant fuzzy subset of $S$ as follows

$$\begin{aligned} H(x) &= [0,1] \quad \text{if } \ x = 0 \\ &= 0.2 \cup (0.3, 0.8] \quad \text{if x is even} \\ &= [0.5, 0.7) \quad \text{if x is odd} \end{aligned}$$

The hesitant fuzzy subset $H$ of $S$ is a hesitant fuzzy ideal $S$.

Throughout this section, we prove results only for hesitant fuzzy left ideals. Similar results can be obtained for hesitant fuzzy right ideals and hesitant fuzzy ideals.

**Definition 3.4.** The characteristic hesitant fuzzy set of $H$ of a set $A$ is defined as

$$H_{\chi_A}(x) = \begin{cases} [0,1], & if \ x \in A; \\ \phi, & if \ x \notin A. \end{cases}$$

**Definition 3.5.** Let $H_1$ and $H_2$ be any two hesitant fuzzy sets of a $\Gamma$-hemiring $S$. Define intersection of $H_1$ and $H_2$ by

$$(H_1 \cap H_2)(x) = \cap(H_1(x), H_2(x))$$

for all $x \in S$.

**Proposition 3.6.** Intersection of a non-empty collection of hesitant fuzzy left $h$-ideals is a hesitant fuzzy left $h$-ideal of $S$.

PROOF. Let $\{H_i : i \in I\}$ be a non-empty family of ideals of $S$. Let $x, y \in S$ and $\gamma \in \Gamma$.
Then

$$\begin{aligned} (\underset{i\in I}{\cap} H_i)(x+y) &= \underset{i\in I}{\cap}\{H_i(x+y)\} \supseteq \underset{i\in I}{\cap}\{\cap\{H_i(x), H_i(y)\}\} \\ &= \cap\{\underset{i\in I}{\cap} H_i(x), \underset{i\in I}{\cap} H_i(y)\} = \cap\{(\underset{i\in I}{\cap} H_i)(x), (\underset{i\in I}{\cap} H_i)(y)\}. \end{aligned}$$

Again

$$(\underset{i\in I}{\cap} H_i)(x\gamma y) = \underset{i\in I}{\cap}\{H_i(x\gamma y)\} \supseteq \underset{i\in I}{\cap}\{H_i(y)\} = (\underset{i\in I}{\cap} H_i)(y).$$

Hence $\underset{i\in I}{\cap} H_i$ is a hesitant fuzzy left ideal of $S$.

Suppose $x \in S$ be such that $x + a + z = b + z$, for $z, a, b \in S$. Then

$$\begin{aligned} (\underset{i\in I}{\cap} H_i)(x) &= \underset{x\in I}{\cap}\{\mu_i(x)\} \supseteq \underset{i\in I}{\cap}\{\cap\{H_i(a), H_i(b)\}\} \\ &= \cap\{\underset{i\in I}{\cap} H_i(a), \underset{i\in I}{\cap} H_i(b)\} = \cap\{(\underset{i\in I}{\cap} H_i)(a), (\underset{i\in I}{\cap} H_i)(b)\}. \end{aligned}$$

Therefore $\underset{i\in I}{\cap} H_i$ is a hesitant fuzzy left $h$-ideal of $S$. □

**Proposition 3.7.** Let $f : R \to S$ be a morphism of $\Gamma$-hemirings ( see, [20])and $H$ be a hesitant fuzzy left h-ideal of S, then $f^{-1}(H)$ is a hesitant fuzzy left h-ideal of R where $f^{-1}(H)(x) = H(f(x))$ for $x \in S$.

PROOF. Let $f : R \to S$ be a morphism of $\Gamma$-hemirings.
Let $H$ be a hesitant fuzzy left ideal of S and $r, s \in R$ and $\gamma \in \Gamma$. Then

$$\begin{aligned} f^{-1}(H)(r+s) &= H(f(r+s)) = H(f(r) + f(s)) \\ &\supseteq \cap\{H(f(r)), H(f(s))\} = \cap\{f^{-1}(H)(r), f^{-1}(H)(s)\} \end{aligned}$$

Again $(f^{-1}(H))(r\gamma s) = H(f(r\gamma s)) = H(f(r)\gamma f(s)) \supseteq H(f(s)) = (f^{-1}(H))(s)$.

Thus $f^{-1}(H)$ is a hesitant fuzzy left ideal of $R$.

Suppose $x, a, b, z \in R$ be such that $x + a + z = b + z$. Then $f(x) + f(a) + f(z) = f(b) + f(z)$.
$(f^{-1}(H))(x) = H(f(x)) \supseteq \cap\{H(f(a)), H(f(b))\} = \cap\{f^{-1}(H)(a), f^{-1}(H)(b)\}$.
Therefore $f^{-1}(H)(x)$ is a hesitant fuzzy left $h$-ideal of $R$.

□

**Definition 3.8.** Let $H_1$ and $H_2$ be hesitant fuzzy subsets of X. The cartesian product of $H_1$ and $H_2$ is defined by

$$(H_1 \times H_2)(x,y) = \cap(H_1(x), H_2(y))$$

for all $x, y \in X$.

**Theorem 3.9.** Let $H_1$ and $H_2$ be two hesitant fuzzy left h-ideals of a Γ-hemiring S. Then $H_1 \times H_2$ is a hesitant fuzzy left h-ideal of the Γ-hemiring $S \times S$.

PROOF. Let $(x_1, x_2), (y_1, y_2) \in S \times S$ and $\gamma \in \Gamma$. Then

$$\begin{aligned} (H_1 \times H_2)((x_1,x_2)+(y_1,y_2)) &= (H_1 \times H_2)(x_1+y_1, x_2+y_2) \\ &= \cap\{H_1(x_1+y_1), H_2(x_2+y_2)\} \\ &\supseteq \cap\{\cap\{H_1(x_1), H_1(y_1)\}, \cap\{H_2(x_2), H_2(y_2)\}\} \\ &= \cap\{\cap\{H_1(x_1), H_2(x_2)\}, \cap\{H_1(y_1), H_2(y_2)\}\} \\ &= \cap\{(H_1 \times H_2)(x_1,x_2), (H_1 \times H_2)(y_1,y_2)\} \end{aligned}$$

and

$$\begin{aligned} (H_1 \times H_2)((x_1,x_2)\gamma(y_1,y_2)) &= (H_1 \times H_2)(x_1\gamma y_1, x_2\gamma y_2) = \cap\{H_1(x_1\gamma y_1), H_2(x_2\gamma y_2)\} \\ &\supseteq \cap\{H_1(y_1), H_2(y_2)\} = (H_1 \times H_2)(y_1,y_2). \end{aligned}$$

Hence $H_1 \times H_2$ is a hesitant fuzzy left ideal of $S \times S$.
Now, let $(a_1, a_2), (b_1, b_2), (x_1, x_2), (z_1, z_2) \in S \times S$ be such that $(x_1, x_2) + (a_1, a_2) + (z_1, z_2) = (b_1, b_2) + (z_1, z_2)$ i.e., $(x_1 + a_1 + z_1, x_2 + a_2 + z_2) = (b_1 + z_1, b_2 + z_2)$. Then $x_1 + a_1 + z_1 = b_1 + z_1$ and $x_2 + a_2 + z_2 = b_2 + z_2$ so that

$$\begin{aligned} (H_1 \times H_2)(x_1,x_2) &= \cap\{H_1(x_1), H_2(x_2)\} \\ &\supseteq \cap\{\cap\{H_1(a_1), H_1(b_1)\}, \cap\{H_2(a_2), H_2(b_2)\}\} \\ &= \cap\{\cap\{H_1(a_1), H_2(a_2)\}, \cap\{H_1(b_1), H_2(b_2)\}\} \\ &= \cap\{(H_1 \times H_2)(a_1,a_2), (H_1 \times H_2)(b_1,b_2)\}. \end{aligned}$$

Therefore $H_1 \times H_2$ is a hesitant fuzzy left $h$-ideal of $S \times S$. □

## 4. Hesitant fuzzy $h$-bi-ideals and $h$-quasi-ideals

**Definition 4.1.** Let $H_1$ and $H_2$ be two hesitant fuzzy sets of a Γ-hemiring $S$. Define composition of $H_1$ and $H_2$ by

$$\begin{aligned} H_1 o H_2(x) &= \underset{x+\sum_{i=1}^{n} a_i\gamma_i b_i + z = \sum_{i=1}^{n} c_i\delta_i d_i + z}{\cup}[\cap_i\{\cap\{H_1(a_i), H_1(c_i), H_2(b_i),\ H_2(d_i)\}\}] \\ &= \phi, \text{if x cannot be expressed as above} \end{aligned}$$

where $x, z, a_i, b_i, c_i, d_i \in S$, $\gamma_i, \delta_i \in \Gamma$ and i=1,...,n.

**Lemma 4.2.** Let $H_1$ and $H_2$ be two hesitant fuzzy $h$-ideal of a Γ-hemiring $S$. Then $H_1 o H_2 \subseteq H_1 \cap H_2 \subseteq H_1, H_2$.

PROOF. Suppose $H_1$ and $H_2$ be two hesitant fuzzy $h$-ideal of a Γ-hemiring $S$. Then

$$\begin{aligned} (H_1 o H_2)(x) &= \underset{x+\sum_{i=1}^{n} a_i\gamma_i b_i + z = \sum_{i=1}^{n} c_i\delta_i d_i + z}{\cup}\{\cap_i\{\cap\{H_1(a_i), H_1(c_i), H_2(b_i),\ H_2(d_i)\}\}\} \\ &\quad \text{where } x, a_i, b_i, c_i, d_i \in S, \gamma_i, \delta_i \in \Gamma \text{ and } i = 1, ..., n. \\ &\subseteq \cup\{\cap_i\{H_1(a_i), H_1(c_i)\}\} \\ &\subseteq \underset{x+\sum_{i=1}^{n} a_i\gamma_i b_i + z = \sum_{i=1}^{n} c_i\delta_i d_i + z}{\cup}\{\cap\{H_1(\sum_{i=1}^{n} a_i\gamma_i b_i), H_1(\sum_{i=1}^{n} c_i\delta_i d_i)\}\} = H_1(x) \end{aligned}$$

Since this is true for every representation of $x$, $H_1oH_2 \subseteq H_1$.
Similarly we can prove that $H_1oH_2 \subseteq H_2$.
Therefore $H_1oH_2 \subseteq H_1 \cap H_2 \subseteq H_1, H_2$.
Hence the lemma. □

**Definition 4.3.** A hesitant fuzzy subset $H$ of a $\Gamma$-hemiring $S$ is called hesitant fuzzy $h$-bi-ideal if for all $a, b, x, y, z \in S$ and $\alpha, \beta \in \Gamma$ we have
(i) $H(x+y) \supseteq \cap\{H(x), H(y)\}$
(ii) $H(x\alpha y) \supseteq \cap\{H(x), H(y)\}$
(iii) $H(x\alpha y\beta z) \supseteq \cap\{H(x), H(z)\}$
(iv) $x+a+z = b+z \Rightarrow H(x) \supseteq \cap\{H(a), H(b)\}$

**Definition 4.4.** A hesitant fuzzy subset $H$ of a $\Gamma$-hemiring $S$ is called hesitant fuzzy $h$-quasi-ideal if for all $a, b, x, y, z \in S$ we have
(i) $H(x+y) \supseteq \cap\{H(x), H(y)\}$
(ii) $(HoH_{\chi_S}) \cap (H_{\chi_S}oH) \subseteq H$
(iii) $x+a+z = b+z \Rightarrow H(x) \supseteq \cap\{H(a), H(b)\}$

**Theorem 4.5.** A hesitant fuzzy subset $H$ of a $\Gamma$-hemiring $S$ is a hesitant fuzzy left $h$-ideal of $S$ if and only if for all $a, b, x, y, z \in S$, we have
(i) $H(x+y) \supseteq \cap\{H(x), H(y)\}$
(ii) $H_{\chi_S}oH \subseteq H$.
(iii) $x+a+z = b+z \Rightarrow H(x) \supseteq \cap\{H(a), H(b)\}$.

Proof. Assume that $H$ is a hesitant fuzzy left $h$-ideal of $S$. Then it is sufficient to show that the condition (ii) is satisfied. Let $x \in S$. If $x$ can be expressed as $x + \sum_{i=1}^{n} a_i\gamma_i b_i + z = \sum_{i=1}^{n} c_i\delta_i d_i + z$, for $a_i, b_i, c_i, d_i \in S$, $\gamma_i, \delta_i \in \Gamma$ and i=1,...,n, then we have

$$\begin{aligned}
(H_{\chi_S}oH)(x) &= \bigcup_{x+\sum_{i=1}^{n} a_i\gamma_i b_i + z = \sum_{i=1}^{n} c_i\delta_i d_i + z}[\bigcap_i\{\cap\{H_{\chi_S}(a_i), H_{\chi_S}(c_i), H(b_i), H(d_i)\}\}] \\
&\subseteq \bigcup_{x+\sum_{i=1}^{n} a_i\gamma_i b_i + z = \sum_{i=1}^{n} c_i\delta_i d_i + z}[\bigcap_i\{\cap\{H(a_i\gamma_i b_i), H(c_i\delta_i d_i)\}\}] \\
&\subseteq \bigcup_{x+\sum_{i=1}^{n} a_i\gamma_i b_i + z = \sum_{i=1}^{n} c_i\delta_i d_i + z}[\cap\{H(\sum_{i=1}^{n} a_i\gamma_i b_i), H(\sum_{i=1}^{n} c_i\delta_i d_i)\}] = H(x).
\end{aligned}$$

This implies that $H_{\chi_S}oH \subseteq H$.
Conversely, assume that the given conditions hold. Then it is sufficient to show the second condition of the definition of hesitant fuzzy left $h$-ideal. Let $x, y \in S$ and $\gamma \in \Gamma$. Then we have

$$\begin{aligned}
H(x\gamma y) &\supseteq (H_{\chi_S}oH)(x\gamma y) = \bigcup_{x\gamma y+\sum_{i=1}^{n} a_i\gamma_i b_i + z = \sum_{i=1}^{n} c_i\delta_i d_i + z}[\bigcap_i\{\cap\{H_{\chi_S}(a_i), H_{\chi_S}(c_i), H(b_i), H(d_i)\}\}] \\
&\supseteq H(y)(\text{since } x\gamma y + 0 + 0 = x\gamma y + 0).
\end{aligned}$$

Hence $H$ is a hesitant fuzzy left $h$-ideal of $S$. □

**Theorem 4.6.** Let $H_1$ and $H_2$ be a hesitant fuzzy right $h$-ideal and a hesitant fuzzy left $h$-ideal of a $\Gamma$-hemiring $S$, respectively. Then $H_1 \cap H_2$ is a hesitant fuzzy $h$-quasi-ideal of $S$.

PROOF. Let $x, y$ be any two elements of $S$. Then

$$\begin{aligned}(H_1 \cap H_2)(x+y) &= \cap\{H_1(x+y), H_2(x+y)\}\\ &\supseteq \cap\{\cap\{H_1(x), H_1(y)\}, \cap\{H_2(x), H_2(y)\}\}\\ &= \cap\{\cap\{H_1(x), H_2(x)\}, \cap\{H_1(y), H_2(y)\}\}\\ &= \cap\{(H_1 \cap H_2)(x), (H_1 \cap H_2)(y)\}.\end{aligned}$$

On the other hand, we have

$$((H_1 \cap H_2)oH_{\chi_S}) \cap (H_{\chi_S}o(H_1 \cap H_2)) \subseteq (H_1oH_{\chi_S}) \cap (H_{\chi_S}oH_2) \subseteq (H_1 \cap H_2).$$

Now let $a, b, x, z \in S$ such that $x + a + z = b + z$. Then

$$\begin{aligned}(H_1 \cap H_2)(x) &= \cap(H_1(x), H_2(x))\\ &\supseteq \cap(\cap(H_1(a), H_1(b)), \cap(H_2(a), H_2(b)))\\ &= \cap(\cap(H_1(a), H_2(a)), \cap(H_1(b), H_2(b)))\\ &= \cap((H_1 \cap H_2)(a), (H_1 \cap H_2)(b))\end{aligned}$$

This completes the proof. □

**Lemma 4.7.** Any hesitant fuzzy $h$-quasi-ideal of $S$ is a hesitant fuzzy $h$-bi-ideal of $S$.

PROOF. Let $H$ be any hesitant fuzzy $h$-quasi-ideal of $S$. It is sufficient to show that $H(x\alpha y\beta z) \supseteq \cap\{H(x), H(z)\}$ and $H(x\alpha y) \supseteq \cap\{H(x), H(y)\}$ for all $x, y, z \in S$ and $\alpha, \beta \in \Gamma$.
In fact, by the assumption, we have

$$\begin{aligned}H(x\alpha y\beta z) &\supseteq ((HoH_{\chi_S}) \cap (H_{\chi_S}oH))(x\alpha y\beta z)\\ &= \cap\{(HoH_{\chi_S})(x\alpha y\beta z), (H_{\chi_S}oH)(x\alpha y\beta z)\}\\ &= \cap\{\underset{x\alpha y\beta z+\sum_{i=1}^{n} a_i\gamma_i b_i + p = \sum_{i=1}^{n} c_i\delta_i d_i + p}{\cup(\cap(H(a_i), H(c_i)), \cup(\cap(H(b_i), H(d_i))}\}\\ &\supseteq \cap\{H(x), H(z)\} \text{ since } x\alpha y\beta z + 0 + 0 = x\alpha y\beta z + 0.\end{aligned}$$

Similarly, we can show that $H(x\alpha y) \supseteq \cap\{H(x), H(y)\}$ for all $x, y \in S$ and $\alpha \in \Gamma$.

□

**Definition 4.8.** A Γ-hemiring $S$ is said to be $h$-hemiregular if for each $x \in S$, there exist $a, b \in S$ and $\alpha, \beta, \gamma, \delta \in \Gamma$ such that $x + x\alpha a\beta x + z = x\gamma b\delta x + z$.

**Theorem 4.9.** Let $S$ be a $h$-hemiregular Γ-hemiring. Then for any hesitant fuzzy right $h$-ideal $H_1$ and any hesitant fuzzy left $h$-ideal $H_2$ of S we have $H_1oH_2 = H_1 \cap H_2$.

PROOF. Let $S$ be a $h$-hemiregular Γ-hemiring. By Lemma 4.2, we have $H_1oH_2 \subseteq H_1 \cap H_2$.
For any $a \in S$, there exist $x, y, z \in S$ and $\alpha, \beta, \gamma, \delta \in \Gamma$ such that $a + a\alpha x\beta a + z = a\gamma y\delta a + z$.
Then

$$\begin{aligned}(H_1oH_2)(a) &= \underset{a+\sum_{i=1}^{n} a_i\gamma_i b_i + z = \sum_{i=1}^{n} c_i\delta_i d_i + z}{\cup\{\cap\{H_1(a_i), H_1(c_i), H_2(b_i), H_2(d_i)\}\}}\\ &\supseteq \cap\{H_1(a\alpha x), H_1(a\gamma y), H_2(a)\}\\ &\supseteq \cap\{H_1(a), H_2(a)\} = (H_1 \cap H_2)(a).\end{aligned}$$

Therefore $(H_1 \cap H_2) \subseteq (H_1oH_2)$.
Hence $H_1oH_2 = H_1 \cap H_2$. □

**Theorem 4.10.** Let $S$ be a $h$-hemiregular Γ-hemiring. Then

(i) $H \subseteq HoH_{\chi_S}oH$ for every hesitant fuzzy $h$-bi-ideal $H$ of $S$.

(ii) $H \subseteq HoH_{\chi_S}oH$ for every hesitant fuzzy $h$-quasi-ideal $H$ of $S$.

PROOF. (i) Suppose that $H$ be any hesitant fuzzy $h$-bi-ideal of $S$ and $x$ be any element of $S$. Since $S$ is $h$-hemiregular there exist $a, b, z \in S$ and $\alpha, \beta, \gamma, \delta \in \Gamma$ such that $x + x\alpha a\beta x + z = x\gamma b\delta x + z$. Now

$$\begin{aligned}(HoH_{\chi_S}oH)(x) &= \cup(\cap\{(HoH_{\chi_S})(a_i), (HoH_{\chi_S})(c_i), H(b_i), H(d_i)\}) \\ &\qquad x+\sum_{i=1}^{n} a_i\gamma_i b_i + z = \sum_{i=1}^{n} c_i\delta_i d_i + z \\ &\supseteq \cap\{(HoH_{\chi_S})(x\alpha a), (HoH_{\chi_S})(x\gamma b)H(x)\} \\ &\supseteq \cap\{H(x), H(x)\} \\ &\qquad (since\ x\alpha a + x\alpha a\beta x\alpha a + z\alpha a = x\gamma b\delta x\alpha a + z\alpha a, \\ &\qquad x\gamma b + x\alpha a\beta x\gamma b + z\gamma b = x\gamma b\delta x\gamma b + z\gamma b). \\ &= H(x)\end{aligned}$$

This implies that $H \subseteq HoH_{\chi_S}oH$.
(ii) This is straight forward from Lemma 4.7 □

**Theorem 4.11.** Let $S$ be a $h$-hemiregular Γ-hemiring. Then

(i) $H_1 \cap H_2 \subseteq H_1oH_2oH_1$ for every hesitant fuzzy $h$-bi-ideal $H_1$ and every hesitant fuzzy $h$-ideal $H_2$ of S.

(ii) $H_1 \cap H_2 \subseteq H_1oH_2oH_1$ for every hesitant fuzzy $h$-quasi-ideal $H_1$ and every hesitant fuzzy $h$-ideal $H_2$ of S.

PROOF. (i) Suppose $S$ is a $h$-hemiregular Γ-hemiring and $H_1$, $H_2$ be any hesitant fuzzy $h$-bi-ideal and hesitant fuzzy $h$-ideal of $S$, respectively and $x$ be any element of $S$. Since $S$ is $h$-hemiregular, there exist $a, b, z \in S$ and $\alpha, \beta, \gamma, \delta \in \Gamma$ such that $x + x\alpha a\beta x + z = x\gamma b\delta x + z$.

$$\begin{aligned}(H_1oH_2oH_1)(x) &= \cup(\cap\{(H_1oH_2)(a_i), (H_1oH_2)(c_i), H_1(b_i), H_1(d_i)\}) \\ &\qquad x+\sum_{i=1}^{n} a_i\gamma_i b_i + z = \sum_{i=1}^{n} c_i\delta_i d_i + z \\ &\supseteq \cap\{(H_1oH_2)(xa), H_1oH_2)(xb), H_1(x)\} \\ &\supseteq \cap\{\cap\{H_1(x), H_2(a\beta x\alpha a), H_2(b\delta x\alpha a), H_2(a\beta x\gamma b), H_2(b\delta x\gamma b), H_1(x)\} \\ &\qquad (since\ x\alpha a + x\alpha a\beta x\alpha a + z\alpha a = x\gamma b\delta x\alpha a + z\alpha a, \\ &\qquad x\gamma b + x\alpha a\beta x\gamma b + z\gamma b = x\gamma b\delta x\gamma b + z\gamma b) \\ &\supseteq \cap\{H_1(x), H_2(x)\} = (H_1 \cap H_2)(x).\end{aligned}$$

(ii) follows from Lemma 4.7. □

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