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ON FUNCTIONAL GENERALIZATION OF OSTROWSKI INEQUALITY FOR CONFORMABLE FRACTIONAL INTEGRALS

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Abstract. In this study, we establish a generalized Ostrowski type integral inequality for conformable fractional integrals. We also give some applications for p-norms and exponential.

Keywords: Ostrowski inequality, conformable fractional integral, Hölder inequality, p-norm, exponential.

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1. Introduction

In 1938, Ostrowski established the following interesting integral inequality for differentiable mappings with bounded derivatives [11]. This inequality is well known in the literature as the *Ostrowski inequality.*

Theorem 1.1 (Ostrowski inequality)**.** *Let $f : [a,b] \to R$ be a differentiable mapping on (a,b) whose derivative $f' : (a,b) \to R$ is bounded on (a,b), i.e. $\|f'\|_\infty := \sup\limits_{t\in(a,b)} |f'(t)| < \infty$. Then, we have the inequality*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2} \right](b-a)\left\|f'\right\|_\infty, \tag{1}$$

for all $x \in [a,b]$. The constant $\frac{1}{4}$ is the best possible.

Ostrowski inequality has applications in numerical integration, probability and optimization theory, stochastic, statistics, information and integral operator theory. Until now, a large number of research papers and books have been written on generalizations of Ostrowski inequalities and their numerous applications. One of these generalizations

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is given by S.S. Dragomir in [5]. In this paper, Dragomir show that if $f : [a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$ and $\Phi : \mathbb{R} \to \mathbb{R}$ is convex on $\mathbb{R}$, then

$$\Phi\left(f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right) \tag{2}$$

$$\leq \frac{1}{b-a}\left[\int_a^x \Phi\left((t-a)\, f'(t)\right) dt + \int_x^b \Phi\left((t-b)\, f'(t)\right) dt\right]$$

for any $x \in [a,b]$.

The main aim of our study is to establish the conformable fractional version of the inequality (2). The remainder of this work is organized as follows: In Section 2, we give the definitions and properties of the conformable fractional derivatives and integrals. Then, in Section 3, we present a functional generalization of Ostrowski type integral inequality for conformable fractional integrals and give some special cases of this inequality. Using these results, we obtain some inequalities for p-norms and exponential in Section 3 and Section 4, respectively.

Now, we will introduce the conformable integral and derivative:

2. Definitions and properties of conformable fractional derivative and integral

The following definitions and theorems with respect to conformable fractional derivative and integral were referred in (see, [1]-[4], [6]-[10], [12]-[17]).

Definition 2.1. *(Conformable fractional derivative) Given a function $f : [0,\infty) \to \mathbb{R}$. Then the "conformable fractional derivative" of f of order α is defined by*

$$D_\alpha (f)(t) = \lim_{\varepsilon \to 0} \frac{f\left(t + \varepsilon t^{1-\alpha}\right) - f(t)}{\varepsilon} \tag{3}$$

for all $t > 0$, $\alpha \in (0,1)$. If f is α-differentiable in some $(0,a)$, $\alpha > 0$, $\lim_{t \to 0^+} f^{(\alpha)}(t)$ exist, then define

$$f^{(\alpha)}(0) = \lim_{t \to 0^+} f^{(\alpha)}(t). \tag{4}$$

We can write $f^{(\alpha)}(t)$ for $D_\alpha (f)(t)$ to denote the conformable fractional derivatives of f of order α. In addition, if the conformable fractional derivative of f of order α exists, then we simply say f is α-differentiable.

Theorem 2.1. *Let $\alpha \in (0,1]$ and f, g be α-differentiable at a point $t > 0$. Then*

i. $D_\alpha (af + bg) = aD_\alpha (f) + bD_\alpha (g)$, for all $a, b \in \mathbb{R}$,
ii. $D_\alpha (\lambda) = 0$, for all constant functions $f(t) = \lambda$,
iii. $D_\alpha (fg) = fD_\alpha (g) + gD_\alpha (f)$,
iv. $D_\alpha \left(\frac{f}{g}\right) = \frac{fD_\alpha (g) - gD_\alpha (f)}{g^2}$.

If f is differentiable, then

$$D_\alpha (f)(t) = t^{1-\alpha} \frac{df}{dt}(t). \tag{5}$$

Definition 2.2 (Conformable fractional integral)**.** *Let $\alpha \in (0,1]$ and $0 \leq a < b$. A function $f : [a,b] \to \mathbb{R}$ is α-fractional integrable on $[a,b]$ if the integral*

$$\int_a^b f(x)\, d_\alpha x := \int_a^b f(x)\, x^{\alpha-1} dx \tag{6}$$

exists and is finite. All α-fractional integrable on $[a,b]$ is indicated by $L_\alpha^1([a,b])$.

Remark 2.1.

$$I_\alpha^a(f)(t) = I_1^a\left(t^{\alpha-1} f\right) = \int_a^t \frac{f(x)}{x^{1-\alpha}} dx,$$

where the integral is the usual Riemann improper integral, and $\alpha \in (0,1]$.

Theorem 2.2. *Let $f : (a,b) \to \mathbb{R}$ be differentiable and $0 < \alpha \leq 1$. Then, for all $t > a$ we have*

$$I_\alpha^a D_\alpha^a f(t) = f(t) - f(a). \tag{7}$$

We will also use the following important results, which can be derived from the results above.

Lemma 2.1. *Let the conformable differential operator D^α be given as in (3), where $\alpha \in (0,1]$ and $t \geq 0$, and assume the functions f and g are α-differentiable as needed. Then*

i. $D^\alpha(\ln t) = t^{-\alpha}$ *for* $t > 0$

ii. $D^\alpha\left[\int_a^t f(t,s)\, d_\alpha s\right] = f(t,t) + \int_a^t D^\alpha[f(t,s)]\, d_\alpha s$

iii. $\int_a^b f(x)\, D^\alpha(g)(x)\, d_\alpha x = fg|_a^b - \int_a^b g(x)\, D^\alpha(f)(x)\, d_\alpha x.$

The following lemma and theorem was given by Anderson in [3].

Lemma 2.2 (Montgomery Identity)**.** *Let $a, b, t, x \in \mathbb{R}$ with $0 \leq a < b$, and let $f : [a,b] \to \mathbb{R}$ be α-fractional differentiable for $\alpha \in (0,1]$. Then*

$$f(x) = \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(t) d_\alpha t + \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b p(x,t) D_\alpha f(t) d_\alpha t \tag{8}$$

where

$$p(x,t) = \begin{cases} \frac{t^\alpha - a^\alpha}{\alpha}, & a \leq t < x \\ \frac{t^\alpha - b^\alpha}{\alpha} & x \leq t \leq b. \end{cases}$$

Theorem 2.3 (Jensen inequality)**.** *Let $\alpha \in (0,1]$ and $a, b, x, y \in [0,\infty)$. If $w : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to (x,y)$ are nonnegative, continuous functions with $\int_a^b w(t) d_\alpha t > 0$, and $F : (x,y) \to \mathbb{R}$ is continuous and convex, then*

$$F\left(\frac{\int_a^b g(t) w(t) d_\alpha t}{\int_a^b w(t) d_\alpha t}\right) \leq \frac{\int_a^b F(g(t)) w(t) d_\alpha t}{\int_a^b w(t) d_\alpha t}. \tag{9}$$

Corollary 2.1. *Under assumptions of Theorem 2.3 with $w(t) = 1$, we have*

$$F\left(\frac{\alpha}{b^\alpha - a^\alpha} \int_a^b g(t) d_\alpha t\right) \leq \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b F(g(t)) d_\alpha t. \tag{10}$$

Throughout the paper we consider the norm $\|f\|_{[a,b],p}$, $p \geq 1$ as

$$\|f\|_{[a,b],p} = \left(\int_a^b |f(t)|^p\, d_\alpha t\right)^{\frac{1}{p}}.$$

Now, we present the main results:

3. Generalized Ostrowski Type Inequality for Conformable fractional integrals

Theorem 3.1. *Let $f:[a,b]\to\mathbb{R}$ be α-fractional differentiable for $\alpha\in(0,1]$. If $F:\mathbb{R}\to\mathbb{R}$ is convex on $\mathbb{R}$, then we have the following inequality*

$$F\left(f(x)-\frac{\alpha}{b^\alpha-a^\alpha}\int_a^b f(t)d_\alpha t\right) \tag{11}$$

$$\leq \frac{\alpha}{b^\alpha-a^\alpha}\int_a^x F\left(\left(\frac{t^\alpha-a^\alpha}{\alpha}\right)D_\alpha f(t)\right)d_\alpha t$$

$$+\frac{\alpha}{b^\alpha-a^\alpha}\int_x^b F\left(\left(\frac{t^\alpha-b^\alpha}{\alpha}\right)D_\alpha f(t)\right)d_\alpha t$$

for any $x\in[a,b]$.

Proof. Using the identity (8), we have

$$f(x)-\frac{\alpha}{b^\alpha-a^\alpha}\int_a^b f(t)d_\alpha t$$

$$= \frac{\alpha}{b^\alpha-a^\alpha}\left[\int_a^x\left(\frac{t^\alpha-a^\alpha}{\alpha}\right)D_\alpha f(t)d_\alpha t+\int_x^b\left(\frac{t^\alpha-b^\alpha}{\alpha}\right)D_\alpha f(t)d_\alpha t\right]$$

$$= \frac{x^\alpha-a^\alpha}{b^\alpha-a^\alpha}\left[\frac{\alpha}{x^\alpha-a^\alpha}\int_a^x\left(\frac{t^\alpha-a^\alpha}{\alpha}\right)D_\alpha f(t)d_\alpha t\right]$$

$$+\frac{b^\alpha-x^\alpha}{b^\alpha-a^\alpha}\left[\frac{\alpha}{b^\alpha-x^\alpha}\int_x^b\left(\frac{t^\alpha-b^\alpha}{\alpha}\right)D_\alpha f(t)d_\alpha t\right]$$

for any $x\in(a,b)$.

Since F is a convex function, we obtain

$$F\left(f(x)-\frac{\alpha}{b^\alpha-a^\alpha}\int_a^b f(t)d_\alpha t\right) \tag{12}$$

$$= F\left(\frac{x^\alpha-a^\alpha}{b^\alpha-a^\alpha}\left[\frac{\alpha}{x^\alpha-a^\alpha}\int_a^x\left(\frac{t^\alpha-a^\alpha}{\alpha}\right)D_\alpha f(t)d_\alpha t\right]\right.$$

$$\left.+\frac{b^\alpha-x^\alpha}{b^\alpha-a^\alpha}\left[\frac{\alpha}{b^\alpha-x^\alpha}\int_x^b\left(\frac{t^\alpha-b^\alpha}{\alpha}\right)D_\alpha f(t)d_\alpha t\right]\right)$$

$$\leq \frac{x^\alpha - a^\alpha}{b^\alpha - a^\alpha} F\left(\frac{\alpha}{x^\alpha - a^\alpha}\int_a^x \left(\frac{t^\alpha - a^\alpha}{\alpha}\right) D_\alpha f(t) d_\alpha t\right) + \frac{b^\alpha - x^\alpha}{b^\alpha - a^\alpha} F\left(\frac{\alpha}{b^\alpha - x^\alpha}\int_x^b \left(\frac{t^\alpha - b^\alpha}{\alpha}\right) D_\alpha f(t) d_\alpha t\right).$$

By using inequality (10), we have

$$F\left(\frac{\alpha}{x^\alpha - a^\alpha}\int_a^x \left(\frac{t^\alpha - a^\alpha}{\alpha}\right) D_\alpha f(t) d_\alpha t\right) \leq \frac{\alpha}{x^\alpha - a^\alpha}\int_a^x F\left(\left(\frac{t^\alpha - a^\alpha}{\alpha}\right) D_\alpha f(t)\right) d_\alpha t \quad (13)$$

and

$$F\left(\frac{\alpha}{b^\alpha - x^\alpha}\int_x^b \left(\frac{t^\alpha - b^\alpha}{\alpha}\right) D_\alpha f(t) d_\alpha t\right) \leq \frac{\alpha}{b^\alpha - x^\alpha}\int_x^b F\left(\left(\frac{t^\alpha - b^\alpha}{\alpha}\right) D_\alpha f(t)\right) d_\alpha t. \quad (14)$$

Substituting the inequalities (13) and (14) in (12), we obtain the desired inequality (11). □

Corollary 3.1. *Under assumptions of Theorem 3.1,*
i) if $x = a$, then

$$F\left(f(a) - \frac{\alpha}{b^\alpha - a^\alpha}\int_a^b f(t) d_\alpha t\right) \leq \frac{\alpha}{b^\alpha - a^\alpha}\int_a^b F\left(\left(\frac{t^\alpha - b^\alpha}{\alpha}\right) D_\alpha f(t)\right) d_\alpha t,$$

ii) if $x = b$, then

$$F\left(f(b) - \frac{\alpha}{b^\alpha - a^\alpha}\int_a^b f(t) d_\alpha t\right) \leq \frac{\alpha}{b^\alpha - a^\alpha}\int_a^b F\left(\left(\frac{t^\alpha - a^\alpha}{\alpha}\right) D_\alpha f(t)\right) d_\alpha t.$$

Corollary 3.2. *With the assumptions of Theorem 3.1, we have*

$$\begin{aligned} F(0) &\leq \frac{\alpha}{b^\alpha - a^\alpha}\int_a^b F\left(f(x) - \frac{\alpha}{b^\alpha - a^\alpha}\int_a^b f(t) d_\alpha t\right) d_\alpha x \qquad (15) \\ &\leq \left(\frac{\alpha}{b^\alpha - a^\alpha}\right)^2 \left[\int_a^b \left(\frac{b^\alpha - x^\alpha}{\alpha}\right) F\left(\left(\frac{x^\alpha - a^\alpha}{\alpha}\right) D_\alpha f(x)\right) d_\alpha x \right. \\ &\quad \left. + \int_a^b \left(\frac{x^\alpha - a^\alpha}{\alpha}\right) F\left(\left(\frac{x^\alpha - b^\alpha}{\alpha}\right) D_\alpha f(x)\right) d_\alpha x\right] \end{aligned}$$

for any $x \in [a, b]$.

Proof. By using the inequality (10), we have

$$\begin{aligned}
&\frac{\alpha}{b^\alpha-a^\alpha}\int_a^b F\left(f(x)-\frac{\alpha}{b^\alpha-a^\alpha}\int_a^b f(t)d_\alpha t\right)d_\alpha x\\
\geq\ & F\left(\frac{\alpha}{b^\alpha-a^\alpha}\int_a^b\left(f(x)-\frac{\alpha}{b^\alpha-a^\alpha}\int_a^b f(t)d_\alpha t\right)d_\alpha x\right)\\
=\ & F(0)
\end{aligned}$$

which completes the proof of left-hand side of the inequality (15).

On the other hand, integrating the inequality (11) with respect to x on $[a,b]$, we get

$$\begin{aligned}
&\frac{\alpha}{b^\alpha-a^\alpha}\int_a^b F\left(f(x)-\frac{\alpha}{b^\alpha-a^\alpha}\int_a^b f(t)d_\alpha t\right)d_\alpha x \qquad (16)\\
\leq\ & \left(\frac{\alpha}{b^\alpha-a^\alpha}\right)^2\left[\int_a^b\left(\int_a^x F\left(\left(\frac{t^\alpha-a^\alpha}{\alpha}\right)D_\alpha f(t)\right)d_\alpha t\right)d_\alpha x\right.\\
&\left.+\int_a^b\left(\int_x^b F\left(\left(\frac{t^\alpha-b^\alpha}{\alpha}\right)D_\alpha f(t)\right)d_\alpha t\right)d_\alpha x\right]\\
=\ & \left(\frac{\alpha}{b^\alpha-a^\alpha}\right)^2[I_1+I_2].
\end{aligned}$$

Using integration by parts for conformable fractional integral, we have

$$\begin{aligned}
I_1 &= \int_a^b\left(\int_a^x F\left(\left(\frac{t^\alpha-a^\alpha}{\alpha}\right)D_\alpha f(t)\right)d_\alpha t\right)d_\alpha x \qquad (17)\\
&= \frac{x^\alpha}{\alpha}\left(\int_a^x F\left(\left(\frac{t^\alpha-a^\alpha}{\alpha}\right)D_\alpha f(t)\right)d_\alpha t\right)\Bigg|_a^b\\
&\quad -\int_a^b\frac{x^\alpha}{\alpha}F\left(\left(\frac{x^\alpha-a^\alpha}{\alpha}\right)D_\alpha f(x)\right)d_\alpha x\\
&= \int_a^b\left(\frac{b^\alpha-x^\alpha}{\alpha}\right)F\left(\left(\frac{x^\alpha-a^\alpha}{\alpha}\right)D_\alpha f(x)\right)d_\alpha x,
\end{aligned}$$

and similarly

$$I_2 = \int_a^b \left(\int_x^b F\left(\left(\frac{t^\alpha - b^\alpha}{\alpha} \right) D_\alpha f(t) \right) d_\alpha t \right) d_\alpha x \tag{18}$$
$$= \int_a^b \left(\frac{x^\alpha - a^\alpha}{\alpha} \right) F\left(\left(\frac{x^\alpha - b^\alpha}{\alpha} \right) D_\alpha f(x) \right) d_\alpha x.$$

Substituting the inequalities (17) and (18) in (16), we obtain the right-hand side of the inequality (15). Therefore, the proof is completed. □

Corollary 3.3. *If we write the inequality (11) for the convex function $F(x) = |x|^p$, $p \geq 1$, then we have the inequality*

$$\left| f(x) - \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(t) d_\alpha t \right|^p \tag{19}$$
$$\leq \frac{\alpha}{b^\alpha - a^\alpha} \left[\int_a^x \left(\frac{t^\alpha - a^\alpha}{\alpha} \right)^p |D_\alpha f(t)|^p d_\alpha t + \int_x^b \left(\frac{t^\alpha - b^\alpha}{\alpha} \right)^p |D_\alpha f(t)|^p d_\alpha t \right]$$

for any $x \in [a,b]$.

Using the Hölder inequality for conformable fractional integrals, we get

$$\int_a^x \left(\frac{t^\alpha - a^\alpha}{\alpha} \right)^p |D_\alpha f(t)|^p d_\alpha t + \int_x^b \left(\frac{t^\alpha - b^\alpha}{\alpha} \right)^p |D_\alpha f(t)|^p d_\alpha t \tag{20}$$
$$\leq \begin{cases} \frac{1}{p+1} \left(\frac{x^\alpha - a^\alpha}{\alpha} \right)^{p+1} \|D_\alpha f\|^p_{[a,x],\infty}, & \text{if } D_\alpha f \in L^\alpha_\infty [a,x] \\ \left(\frac{1}{\gamma p+1} \right)^{\frac{1}{\gamma}} \left(\frac{x^\alpha - a^\alpha}{\alpha} \right)^{p+1/\gamma} \|D_\alpha f\|^p_{[a,x],p\beta} & \text{if } D_\alpha f \in L^\alpha_{p\beta} [a,x],\ \gamma > 1,\ \frac{1}{\gamma} + \frac{1}{\beta} = 1 \\ \left(\frac{x^\alpha - a^\alpha}{\alpha} \right)^p \|D_\alpha f\|^p_{[a,x],p} & \text{if } D_\alpha f \in L^\alpha_p [a,x] \end{cases}$$
$$+ \begin{cases} \frac{1}{p+1} \left(\frac{b^\alpha - x^\alpha}{\alpha} \right)^{p+1} \|D_\alpha f\|^p_{[x,b],\infty}, & \text{if } D_\alpha f \in L^\alpha_\infty [x,b] \\ \left(\frac{1}{\gamma p+1} \right)^{\frac{1}{\gamma}} \left(\frac{b^\alpha - x^\alpha}{\alpha} \right)^{p+1/\gamma} \|D_\alpha f\|^p_{[x,b],p\beta} & \text{if } D_\alpha f \in L^\alpha_{p\beta} [x,b],\ \gamma > 1,\ \frac{1}{\gamma} + \frac{1}{\beta} = 1 \\ \left(\frac{b^\alpha - x^\alpha}{\alpha} \right)^p \|D_\alpha f\|^p_{[x,b],p} & \text{if } D_\alpha f \in L^\alpha_p [x,b]. \end{cases}$$

Using the inequalities (19) and (20) for $x \in [a,b]$, we have

$$\left| f(x) - \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(t) d_\alpha t \right|^p \tag{21}$$
$$\leq \frac{\alpha}{b^\alpha - a^\alpha} \frac{1}{p+1} \left[\left(\frac{x^\alpha - a^\alpha}{\alpha} \right)^{p+1} + \left(\frac{b^\alpha - x^\alpha}{\alpha} \right)^{p+1} \right] \|D_\alpha f\|^p_{[a,b],\infty},$$

if $D_\alpha f \in L_\infty^\alpha [a,b]$,

$$\left| f(x) - \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(t) d_\alpha t \right|^p \tag{22}$$
$$\leq \frac{\alpha}{b^\alpha - a^\alpha} \left(\frac{1}{\gamma p + 1} \right)^{\frac{1}{\gamma}} \left[\left(\frac{x^\alpha - a^\alpha}{\alpha} \right)^{p+\frac{1}{\gamma}} + \left(\frac{b^\alpha - x^\alpha}{\alpha} \right)^{p+\frac{1}{\gamma}} \right] \| D_\alpha f \|_{[a,b],p\beta}^p ,$$

if $D_\alpha f \in L_{p\beta}^\alpha [a,b]$ *and*

$$\left| f(x) - \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(t) d_\alpha t \right|^p \tag{23}$$
$$\leq \frac{\alpha}{b^\alpha - a^\alpha} \max \left\{ \left(\frac{x^\alpha - a^\alpha}{\alpha} \right)^p , \left(\frac{b^\alpha - x^\alpha}{\alpha} \right)^p \right\} \| D_\alpha f \|_{[a,b],p}^p$$
$$= \left(\frac{b^\alpha - a^\alpha}{\alpha} \right)^{p-1} \left[\frac{1}{2} + \left| \frac{x^\alpha - \frac{a^\alpha + b^\alpha}{2}}{b^\alpha - a^\alpha} \right| \right]^p \| D_\alpha f \|_{[a,b],p}^p$$

if $D_\alpha f \in L_p^\alpha [a,b]$.

Remark 3.1. *If we take* $p = 1$ *in the above inequalities, then for* $x \in [a,b]$ *we have*

$$\left| f(x) - \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(t) d_\alpha t \right|$$
$$\leq \frac{1}{2\alpha (b^\alpha - a^\alpha)} \left[(x^\alpha - a^\alpha)^2 + (b^\alpha - x^\alpha)^2 \right] \| D_\alpha f \|_{[a,b],\infty} ,$$

given by Anderson in [3],

$$\left| f(x) - \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(t) d_\alpha t \right|$$
$$\leq \frac{\alpha}{b^\alpha - a^\alpha} \left(\frac{1}{\gamma + 1} \right)^{\frac{1}{\gamma}} \left[\left(\frac{x^\alpha - a^\alpha}{\alpha} \right)^{1+\frac{1}{\gamma}} + \left(\frac{b^\alpha - x^\alpha}{\alpha} \right)^{1+\frac{1}{\gamma}} \right] \| D_\alpha f \|_{[a,b],\beta} ,$$

and

$$\left| f(x) - \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(t) d_\alpha t \right| \leq \left[\frac{1}{2} + \left| \frac{x^\alpha - \frac{a^\alpha + b^\alpha}{2}}{b^\alpha - a^\alpha} \right| \right] \| D_\alpha f \|_{[a,b],1} .$$

4. Applications for p-Norms

Theorem 4.1. *Let* $f : [a,b] \to \mathbb{R}$ *be* α*-fractional differentiable for* $\alpha \in (0,1]$.

i. if $D_\alpha f \in L_\infty^\alpha [a,b]$ *, then*

$$\left\| f - \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(t) d_\alpha t \right\|_{[a,b],p} \tag{24}$$

$$\leq \left(\frac{2}{(p+2)(p+1)} \right)^{\frac{1}{p}} \left(\frac{b^\alpha - a^\alpha}{\alpha} \right)^{1+1/p} \|D_\alpha f\|_{[a,b],\infty},$$

ii. if $D_\alpha f \in L_{p\beta}^\alpha [a,b]$

$$\left\| f - \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(t) d_\alpha t \right\|_{[a,b],p} \tag{25}$$

$$\leq \left(\frac{2}{(\gamma p + 1)^{\frac{1}{\gamma}} (p + 1 + 1/\gamma)} \right)^{\frac{1}{p}} \left(\frac{b^\alpha - a^\alpha}{\alpha} \right)^{1+\frac{1}{p\gamma}} \|D_\alpha f\|_{[a,b],p\beta},$$

and

iii. if $D_\alpha f \in L_p^\alpha [a,b]$

$$\left\| f - \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(t) d_\alpha t \right\|_{[a,b],p} \tag{26}$$

$$\leq \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left(\frac{2^{p+1} - 1}{2^p} \right)^{\frac{1}{p}} \left(\frac{b^\alpha - a^\alpha}{\alpha} \right) \|D_\alpha f\|_{[a,b],p}.$$

Proof. Integrating the inequality (21), we have

$$\int_a^b \left| f(x) - \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(t) d_\alpha t \right|^p d_\alpha x$$

$$\leq \frac{\alpha}{b^\alpha - a^\alpha} \frac{1}{p+1} \|D_\alpha f\|_{[a,b],\infty}^p \int_a^b \left[\left(\frac{x^\alpha - a^\alpha}{\alpha} \right)^{p+1} + \left(\frac{b^\alpha - x^\alpha}{\alpha} \right)^{p+1} \right] d_\alpha x$$

$$= \frac{2}{(p+2)(p+1)} \left(\frac{b^\alpha - a^\alpha}{\alpha} \right)^{p+1} \|D_\alpha f\|_{[a,b],\infty}^p$$

which gives (24).

Integrating the inequality (22), we have

$$\int_a^b \left| f(x) - \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(t) d_\alpha t \right|^p d_\alpha x$$

$$\leq \frac{\alpha}{b^\alpha - a^\alpha} \left(\frac{1}{\gamma p + 1} \right)^{\frac{1}{\gamma}} \|D_\alpha f\|_{[a,b],p\beta}^p \int_a^b \left[\left(\frac{x^\alpha - a^\alpha}{\alpha} \right)^{p+\frac{1}{\gamma}} + \left(\frac{b^\alpha - x^\alpha}{\alpha} \right)^{p+\frac{1}{\gamma}} \right] d_\alpha x$$

$$= \frac{2}{(\gamma p + 1)^{\frac{1}{\gamma}} (p + 1 + 1/\gamma)} \left(\frac{b^\alpha - a^\alpha}{\alpha} \right)^{p+\frac{1}{\gamma}} \|D_\alpha f\|_{[a,b],p\beta}^p$$

which completes the proof of (25).

Integrating the inequality (23), we have

$$\begin{aligned}
&\int_a^b \left| f(x) - \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(t) d_\alpha t \right|^p d_\alpha x \\
\leq\ & \frac{\alpha}{b^\alpha - a^\alpha} \|D_\alpha f\|_{[a,b],p}^p \int_a^b \max\left\{ \left(\frac{x^\alpha - a^\alpha}{\alpha}\right)^p, \left(\frac{b^\alpha - x^\alpha}{\alpha}\right)^p \right\} d_\alpha x \\
=\ & \frac{\alpha}{b^\alpha - a^\alpha} \|D_\alpha f\|_{[a,b],p}^p \left[\int_a^{\left(\frac{a^\alpha + b^\alpha}{2}\right)^{\frac{1}{\alpha}}} \left(\frac{b^\alpha - x^\alpha}{\alpha}\right)^p d_\alpha x + \int_{\left(\frac{a^\alpha + b^\alpha}{2}\right)^{\frac{1}{\alpha}}}^b \left(\frac{x^\alpha - a^\alpha}{\alpha}\right)^p d_\alpha x \right] \\
=\ & \frac{1}{p+1} \left(\frac{2^{p+1} - 1}{2^p}\right) \left(\frac{b^\alpha - a^\alpha}{\alpha}\right)^p \|D_\alpha f\|_{[a,b],p}^p .
\end{aligned}$$

This completes the proof of Theorem. □

5. Applications for the Exponential

If we write the inequality (11) for the convex function $F(x) = \exp(x)$, then we obtain the following inequality

$$\begin{aligned}
&\exp\left(f(x) - \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(t) d_\alpha t \right) \\
\leq\ & \frac{\alpha}{b^\alpha - a^\alpha} \int_a^x \exp\left(\left(\frac{t^\alpha - a^\alpha}{\alpha}\right) D_\alpha f(t) \right) d_\alpha t \\
&+ \frac{\alpha}{b^\alpha - a^\alpha} \int_x^b \exp\left(\left(\frac{t^\alpha - b^\alpha}{\alpha}\right) D_\alpha f(t) \right) d_\alpha t
\end{aligned} \tag{27}$$

for all $x \in [a,b]$.

Theorem 5.1. *Let $f : [a,b] \to (0,\infty)$ be α-fractional differentiable for $\alpha \in (0,1]$. Then we have the following inequalities*

$$\begin{aligned}
&\frac{f(x)}{\exp\left(\frac{\alpha}{b^\alpha - a^\alpha} \int_a^b \ln f(t) d_\alpha t \right)} \\
\leq\ & \frac{\alpha}{b^\alpha - a^\alpha} \int_a^x \exp\left(\left(\frac{t^\alpha - a^\alpha}{\alpha}\right) \frac{D_\alpha f(t)}{f(t)} \right) d_\alpha t
\end{aligned} \tag{28}$$

$$+\frac{\alpha}{b^{\alpha}-a^{\alpha}}\int_{x}^{b}\exp\left(\left(\frac{t^{\alpha}-b^{\alpha}}{\alpha}\right)\frac{D_{\alpha}f(t)}{f(t)}\right)d_{\alpha}t$$

and

$$\frac{\int_{a}^{b}f(x)d_{\alpha}x}{\exp\left(\frac{\alpha}{b^{\alpha}-a^{\alpha}}\int_{a}^{b}\ln f(t)d_{\alpha}t\right)} \tag{29}$$

$$\leq \int_{a}^{b}\left(\frac{b^{\alpha}-x^{\alpha}}{\alpha}\right)\exp\left(\left(\frac{x^{\alpha}-a^{\alpha}}{\alpha}\right)\frac{D_{\alpha}f(x)}{f(x)}\right)d_{\alpha}x$$

$$+\int_{a}^{b}\left(\frac{x^{\alpha}-a^{\alpha}}{\alpha}\right)\exp\left(\left(\frac{x^{\alpha}-b^{\alpha}}{\alpha}\right)\frac{D_{\alpha}f(x)}{f(x)}\right)d_{\alpha}x$$

for all $x\in[a,b]$.

Proof. In (27), if we replace f by $\ln f$, we get

$$\exp\left(\ln f(x)-\frac{\alpha}{b^{\alpha}-a^{\alpha}}\int_{a}^{b}\ln f(t)d_{\alpha}t\right)$$

$$\leq \frac{\alpha}{b^{\alpha}-a^{\alpha}}\int_{a}^{x}\exp\left(\left(\frac{t^{\alpha}-a^{\alpha}}{\alpha}\right)\frac{D_{\alpha}f(t)}{f(t)}\right)d_{\alpha}t$$

$$+\frac{\alpha}{b^{\alpha}-a^{\alpha}}\int_{x}^{b}\exp\left(\left(\frac{t^{\alpha}-b^{\alpha}}{\alpha}\right)\frac{D_{\alpha}f(t)}{f(t)}\right)d_{\alpha}t$$

for all $x\in[a,b]$.

Using the fact that

$$\exp\left(\ln f(x)-\frac{\alpha}{b^{\alpha}-a^{\alpha}}\int_{a}^{b}\ln f(t)d_{\alpha}t\right)$$

$$=\exp\left(\ln f(x)-\ln\left\{\exp\left(\frac{\alpha}{b^{\alpha}-a^{\alpha}}\int_{a}^{b}\ln f(t)d_{\alpha}t\right)\right\}\right)$$

$$=\exp\left(\ln\left(\frac{f(x)}{\exp\left(\frac{\alpha}{b^{\alpha}-a^{\alpha}}\int_{a}^{b}\ln f(t)d_{\alpha}t\right)}\right)\right)=\frac{f(x)}{\exp\left(\frac{\alpha}{b^{\alpha}-a^{\alpha}}\int_{a}^{b}\ln f(t)d_{\alpha}t\right)}$$

for any $x\in[a,b]$, we can obtain the inequality (28).

Integrating the both sides of the inequality (28) with respect to x over $[a,b]$, we have

$$\begin{aligned}
&\frac{\int_a^b f(x)d_\alpha x}{\exp\left(\frac{\alpha}{b^\alpha-a^\alpha}\int_a^b \ln f(t)d_\alpha t\right)} \\
\leq\ & \frac{\alpha}{b^\alpha-a^\alpha}\int_a^b\left(\int_a^x \exp\left(\left(\frac{t^\alpha-a^\alpha}{\alpha}\right)\frac{D_\alpha f(t)}{f(t)}\right)d_\alpha t\right)d_\alpha x \\
&+\frac{\alpha}{b^\alpha-a^\alpha}\int_a^b\left(\int_x^b \exp\left(\left(\frac{t^\alpha-b^\alpha}{\alpha}\right)\frac{D_\alpha f(t)}{f(t)}\right)d_\alpha t\right)d_\alpha x.
\end{aligned}$$

Using integration by parts for conformable fractional integral, we have

$$\begin{aligned}
&\int_a^b\left(\int_a^x \exp\left(\left(\frac{t^\alpha-a^\alpha}{\alpha}\right)\frac{D_\alpha f(t)}{f(t)}\right)d_\alpha t\right)d_\alpha x \\
=\ & \frac{x^\alpha}{\alpha}\int_a^x \exp\left(\left(\frac{t^\alpha-a^\alpha}{\alpha}\right)\frac{D_\alpha f(t)}{f(t)}\right)d_\alpha t\Bigg|_a^b \\
&-\int_a^b \frac{x^\alpha}{\alpha}\exp\left(\left(\frac{x^\alpha-a^\alpha}{\alpha}\right)\frac{D_\alpha f(x)}{f(x)}\right)d_\alpha x \\
=\ & \frac{b^\alpha}{\alpha}\int_a^b \exp\left(\left(\frac{t^\alpha-a^\alpha}{\alpha}\right)\frac{D_\alpha f(t)}{f(t)}\right)d_\alpha t \\
&-\int_a^b \frac{x^\alpha}{\alpha}\exp\left(\left(\frac{x^\alpha-a^\alpha}{\alpha}\right)\frac{D_\alpha f(x)}{f(x)}\right)d_\alpha x \\
=\ & \int_a^b \left(\frac{b^\alpha-x^\alpha}{\alpha}\right)\exp\left(\left(\frac{x^\alpha-a^\alpha}{\alpha}\right)\frac{D_\alpha f(x)}{f(x)}\right)d_\alpha x
\end{aligned}$$

and similarly,

$$\begin{aligned}
&\int_a^b\left(\int_x^b \exp\left(\left(\frac{t^\alpha-b^\alpha}{\alpha}\right)\frac{D_\alpha f(t)}{f(t)}\right)d_\alpha t\right)d_\alpha x \\
=\ & \frac{x^\alpha}{\alpha}\int_x^b \exp\left(\left(\frac{t^\alpha-b^\alpha}{\alpha}\right)\frac{D_\alpha f(t)}{f(t)}\right)d_\alpha t\Bigg|_a^b
\end{aligned}$$

$$
\begin{aligned}
&+\int_a^b \frac{x^\alpha}{\alpha}\exp\left(\left(\frac{x^\alpha-b^\alpha}{\alpha}\right)\frac{D_\alpha f(x)}{f(x)}\right)d_\alpha x \\
= &-\frac{a^\alpha}{\alpha}\int_a^b \exp\left(\left(\frac{t^\alpha-b^\alpha}{\alpha}\right)\frac{D_\alpha f(t)}{f(t)}\right)d_\alpha t \\
&+\int_a^b \frac{x^\alpha}{\alpha}\exp\left(\left(\frac{x^\alpha-b^\alpha}{\alpha}\right)\frac{D_\alpha f(x)}{f(x)}\right)d_\alpha x \\
= &\int_a^b \left(\frac{x^\alpha-a^\alpha}{\alpha}\right)\exp\left(\left(\frac{x^\alpha-b^\alpha}{\alpha}\right)\frac{D_\alpha f(x)}{f(x)}\right)d_\alpha x.
\end{aligned}
$$

Hence, the proof of Theorem is completed. □

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