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A CRITICAL STUDY OF MEROMORPHIC STARLIKE FUNCTIONS

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Abstract. An attempt has been made to introduce a new criterion to make it possible to change meromorphic analytic function into a meromorphic starlike function of particular order. This criterion is based on a differential operator which is defined in a punctured unit disk $\mathbb{U}^*$. By using this criterion, one can find easily different types of meromorphic starlike functions of specific order.

Keywords: Meromorphic functions, meromorphic starlike functions, differential operators.

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1. Introduction

Let $\sum_p$ denote the class of meromorphic functions [cf.[1]] of the form

$$f(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} a_k z^k, p \in \mathbf{N} = \{1, 2, 3...\}, \tag{1}$$

which are analytic in $\mathbb{U}^* = \{z : 0 < |z| < 1\}$. For $f \in \sum_p$, we define

$$\Theta^n_{p,\lambda}(\alpha, \beta, \mu)f(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} \left(\frac{\alpha + (\mu + \lambda)(k + p) + \beta}{\alpha + \beta}\right)^n a_k z^k, \tag{2}$$

where $\alpha \geq 0$, $\beta > 0$, $\mu \geq 0$, $\lambda \geq 0$ and $n \in \mathbf{N} \cup \{0\}$.
Also by specializing the parameters α, β, p, μ and λ, we obtain the following operators studied by various authors:

$\Theta^m_{p,\lambda}(0, l, 0)f(z) = I^m_p(\lambda, l)f(z)$ (see R.M. El-Ashwah [2]);

$\Theta^m_{1,1}(0, l, 0)f(z) = I(m, l)f(z)$ (Cho et al. [3, 4]);

$\Theta^m_{p,1}(0, l, 0)f(z) = D^m_p f(z)$ (see Aouf and Hossen [5], Liu and Owa [6], Liu and Srivastava [7], Srivastava and Patel [8]);

$\Theta^m_{1,1}(0, 1, 0)f(z) = I^m f(z)$ (see Uralegaddi and Somanatha [9] and Ashwah and Aouf [10]), respectively.

Note that:

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For $\beta = 0$ we get $\Theta^n_{p,\lambda}(\alpha, 0, \mu) = \Upsilon^n_{p,\lambda}(\alpha, \mu)$ where

$$\Upsilon^n_{p,\lambda}(\alpha, \mu)f(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} \left(\frac{\alpha + (\mu + \lambda)(k + p)}{\alpha}\right)^n a_k z^k.$$

If $\beta = 0$ and $\alpha = 1$ then $\Theta^n_{p,\lambda}(1, 0, \mu) = \Phi^n_{p,\lambda}(\mu)$ where

$$\Phi^n_{p,\lambda}(\mu)f(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} (1 + (\mu + \lambda)(k + p))^n a_k z^k.$$

When $\beta = 0$, $\alpha = \lambda = 1$ we get $\Theta^n_{p,1}(1, 0, \mu) = \Theta^n_p(\mu)$ where

$$\Theta^n_p(\mu)f(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} (1 + (\mu + 1)(k + p))^n a_k z^k.$$

A function $f \in \sum_p$ is said to be meromorphic starlike functions of order ξ i.e. $f \in S^*_p(\xi)$ if

$$\Re\left(-\frac{zf'(z)}{f(z)}\right) > \xi, 0 \leq \xi < p. \tag{3}$$

For more details about meromorphic functions, we suggest the readers to study [[11]-[19]].

2. Criterion for Meromorphic Starlike Functions

Theorem 2.1. *Let the meromorphic function $f \in \sum_p$ be regular in $\mathbb{U}^*$ and $\Theta^n_{p,\lambda}(\alpha, \beta, \mu)f$ be a differential operator defined in* (2)*. For $p = 1$ if*

$$g(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} \left(\frac{\alpha + \beta}{\alpha + (\mu + \lambda)(k + p) + \beta}\right)^n \left(\frac{(1 - \xi)^{k+1}}{(k + 1)!}\right) z^k,$$

then $\Theta^n_{p,\lambda}(\alpha, \beta, \mu)g$ belongs to the class $S^*_p(\xi)$, i.e. $\Theta^n_{p,\lambda}(\alpha, \beta, \mu)g$ is a meromorphic starlike functions of order ξ, where α, β, λ and μ have the same constraints as given in (1), (2) *and* (3) *respectively.*

Proof. First we suppose the function

$$g(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} \left(\frac{\alpha + \beta}{\alpha + (\mu + \lambda)(k + p) + \beta}\right)^n \left(\frac{(1 - \xi)^{k+1}}{(k + 1)!}\right) z^k,$$

where ξ, p, α, β, λ and μ have the same constraints as given in (1), (2) and (3).
By using (2), for $f(z) = \frac{1}{z^p} + \sum\limits_{k=0}^{\infty} a_k z^k,\ z \in \mathbb{U} = \mathbb{U}^* \cup \{0\}$, we have

$$\Theta^1_{p,\lambda}(\alpha, \beta, \mu)f(z) = \left(1 + \frac{p(\mu + \lambda)}{\alpha + \beta}\right) f(z) + \left(\frac{\mu + \lambda}{\alpha + \beta}\right) zf'(z),$$

therefore for the function $g \in \sum_p$ we define

$$\Theta^1_{p,\lambda}(\alpha, \beta, \mu)g(z) == \left(1 + \frac{p(\mu + \lambda)}{\alpha + \beta}\right) g(z) + \frac{\mu + \lambda}{\alpha + \beta} \left(zg'(z)\right), \tag{4}$$

where

$$g(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} \left(\frac{\alpha + \beta}{\alpha + (\mu + \lambda)(k + p) + \beta}\right)^n \left(\frac{(1 - \xi)^{k+1}}{(k + 1)!}\right) z^k, \tag{5}$$

implies

$$zg'(z) = \frac{-p}{z^p} + \sum_{k=0}^{\infty} k \left(\frac{\alpha+\beta}{\alpha+(\mu+\lambda)(k+p)+\beta} \right)^n \left(\frac{(1-\xi)^{k+1}}{(k+1)!} \right) z^k. \quad (6)$$

By using (4), (5) and (6) and doing some calculation, we get

$$\Theta^1_{p,\lambda}(\alpha,\beta,\mu)g(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} \left(\frac{\alpha+\beta}{\alpha+(\mu+\lambda)(k+p)+\beta} \right)^{n-1} \left(\frac{(1-\xi)^{k+1}}{(k+1)!} \right) z^k.$$

Suppose $h(z)$ on temporary

$$h(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} \left(\frac{\alpha+\beta}{\alpha+(\mu+\lambda)(k+p)+\beta} \right)^{n-1} \left(\frac{(1-\xi)^{k+1}}{(k+1)!} \right) z^k, \quad (7)$$

and define

$$\Theta^2_{p,\lambda}(\alpha,\beta,\mu)f(z) = \left(1 + \frac{p(\mu+\lambda)}{\alpha+\beta}\right) h(z) + \left(\frac{\mu+\lambda}{\alpha+\beta}\right) zh'(z), \quad (8)$$

then by using (7) and (8), and after simplification

$$\Theta^2_{p,\lambda}(\alpha,\beta,\mu)g(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} \left(\frac{\alpha+\beta}{\alpha+(\mu+\lambda)(k+p)+\beta} \right)^{n-2} \left(\frac{(1-\xi)^{k+1}}{(k+1)!} \right) z^k,$$

continuing the same process, finally we obtain

$$\Theta^n_{p,\lambda}(\alpha,\beta,\mu)g(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} \left(\frac{\alpha+\beta}{\alpha+(\mu+\lambda)(k+p)+\beta} \right)^{n-n} \left(\frac{(1-\xi)^{k+1}}{(k+1)!} \right) z^k,$$

hence for $p = 1$ and $0 \le \xi < 1$

$$\Theta^n_{1,\lambda}(\alpha,\beta,\mu)g(z) = \frac{1}{z} + \sum_{k=0}^{\infty} \left(\frac{(1-\xi)^{k+1}}{(k+1)!} \right) z^k,$$

where

$$\frac{1}{z} + \sum_{k=0}^{\infty} \frac{(1-\xi)^{k+1}}{(k+1)!} z^k = \frac{e^{(1-\xi)z}}{z}.$$

Let us define the function $F(z)$ by

$$F(z) = \frac{e^{(1-\xi)z}}{z},$$

this give us that

$$\Re\left(\frac{zF'(z)}{F(z)}\right) = \Re(-1 + (1-\xi)z) = -\xi,$$

Therefore we see that $\frac{e^{(1-\xi)z}}{z} \in S^*_p(\xi)$, implies $\Theta^n_{p,\lambda}(\alpha,\beta,\mu)g(z) \in S^*_p(\xi)$, for $p = 1$ as required. □

Corollary 2.1. *Let the meromorphic function $f \in \sum_p$ be regular in $\mathbb{U}^*$ and $\Theta^n_{p,\lambda}(\alpha,\beta,\mu)f$ be a differential operator defined in* (2). *For $p = 1$ if*

$$g(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} \left(\frac{\alpha+\beta}{\alpha+(\mu+\lambda)(k+p)+\beta} \right)^n \left(\frac{1}{(k+1)!} \right) z^k,$$

then $\Theta^n_{p,\lambda}(\alpha,\beta,\mu)g$ belongs to the class $S^*_p(0)$, i.e. $\Theta^n_{p,\lambda}(\alpha,\beta,\mu)g$ is a meromorphic starlike functions of order 0*, where α, β, λ and μ have the same constraints as given in* (1), (2) *and* (3) *respectively.*

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