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ON THE NUMERICAL SOLUTION OF FRACTIONAL ORDER DIFFERENTIAL EQUATIONS USING TRANSFORMS AND QUADRATURE

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Abstract. In this work, we extended the work of [12] to approximate the solution of fractional order differential equations by an integral representation in the complex plane. The resultant integral is approximated to high order accuracy using quadrature. The accuracy of the method depends on the selection of optimal contour of integration. Several contour have been proposed in the literature for solving fractional differential equations. In the present work, we will investigate the applicability of the recently developed optimal contour in [16] for solving fractional differential equations. Various fractional order differential equations are approximated and the results are compared with other methods to demonstrate the efficiency and accuracy of the method for various optimal contour of integrations.

Keywords: Fractional order ODE, Laplace Transform, Quadrature.

AMS Subject Classification: 44A, 30C30, 26A33.

1. Introduction

In this work, we want to investigate the applicability of the proposed numerical algorithm for the solution of fractional order differential equations of the form,

$$D^{\alpha}w(\tau) = f(\tau, w(\tau)), \tag{1}$$

where p is the integer defined by $p-1 < \alpha \leq p$, and α is not necessarily an inter, then one can specify suitable initial conditions,

$$w^{(j)}(0) = w_0^{(j)},\ j =, 0, 1, 3, ..., p-1, \tag{2}$$

where the Reimann-Liouville differential operators of fractional order $\alpha > 0$, is defined as

$$D^{\alpha}w(\tau) = \frac{1}{\Gamma(p-\alpha)}\frac{d^p}{dt^p}\int_0^{\tau}\frac{w(u)}{(\tau-u)^{\alpha-p+1}}du, \tag{3}$$

where p is the integer defined by $p-1 < \alpha \leq p$ (see [3, 9]). Equations of the form (1) have many applications and valuable tools in the modelling of various phenomena in physical and engineering sciences [8, 6, 1, 14, 15]. For the particular case $0 < \alpha < 1$, many important applications occurs, but for $\alpha > 1$ there are also some important applications. In the work of [3], it is shown that the problem (1)-(2) has a unique solution under some strong condition like (linearity of the differential equations). In the present work we

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extended the work of [12] to investigate the applicability of the recently developed contour of integration [16] to solve the fractional order differential equations of the form (1)-(2). In the last section, it is shown how the theoretical results may be incorporated for practical cases. Particularly we investigated the performance of the proposed numerical method for solving fractional differential equations. The problem is approximated by choosing optimal contour of integration subject to validation of errors bounds. We considered, the optimal choices of three different types of contours of integration for our method to gain maximum possible accuracy in approximating the solution with minimum cost of computations.

Lemma 1.1. *For $p-1<\alpha\leq p$, $p\in N$, the expression for the Laplace transformation of Caputo fractional order differential operator can be written as:*

$$\pounds\left[\frac{d^{\alpha}}{dy^{\alpha}}f(\tau)\right]=\frac{s^{p}F(s)-s^{p-1}f(0)-s^{p-2}f'(0)-...-f^{p-1}(0)}{s^{p-\alpha}}. \tag{4}$$

2. Analysis of the Method

In this section, we describe numerical method. We select quadrature step as $k>0$ and the equal weight quadrature rule is applied. The procedure is follows as applying Laplace Transformation to the problem (1)-(2), we get

$$\widehat{w}(s)=s^{-\alpha}g(s), \tag{5}$$

where

$$g(s)=\left[w_0s^{\alpha-1}+\widehat{f}(s)\ \right] \tag{6}$$

and

$$\widehat{f}(s)=\pounds(f(\tau)). \tag{7}$$

Now applying the formal inverse Laplace transform, we get

$$w(\tau)=\pounds^{-1}(\widehat{w}(s))=\frac{1}{2\pi\iota}\int_{\Gamma}e^{s\tau}\widehat{w}(s)ds,\ R(s)>0. \tag{8}$$

where Γ is the contour of integration such that $\Gamma\subset\Omega_r$ and

$$s=s(u), \tag{9}$$

are the points on Γ. Now using equation (8) in equation (9), we find the expression for $w(\tau)$ as an integral in u,

$$w(\tau)=\frac{1}{2\pi\iota}\int_{\Gamma}e^{s(u)\tau}\widehat{w}(s(u))s'(u)du. \tag{10}$$

Now we choose the step $k>0$ for the quadrature rule to be use on equation (10). For simplicity we set $s_j=s(u_j)$, $s_j'=s'(u_j)$ where $u_j=jk$ for $-n\leq j\leq n$, we get

$$w_n(\tau)=\frac{k}{2\pi\iota}\sum_{j=-n}^{n}e^{s_j\tau}\widehat{w}(s_j)s_j'. \tag{11}$$

Now to find the approximate solution w_n, we solve the $2n+1$ equations given in (5) for $-n\leq j\leq n$.

3. Error Analysis of the method

As the numerical solution w_n determined the approximate solution (11) for all $\tau > 0$. In practice, however, the accuracy of approximate solution depends on the choice of the contour Γ. A number of such contour available one such path is due to [7] given as

$$s = \lambda + \omega(1 - \sin(\delta - iu)),\ (C_1) \tag{12}$$

In fact, when $Im\ u = \eta$, (12) reduces to the left branch of the hyperbola,

$$\left(\frac{x - \lambda - \omega}{\omega \sin(\delta + \eta)}\right)^2 - \left(\frac{y}{\omega \cos(\delta + \eta)}\right)^2 = 1, \tag{13}$$

here the strip $Z_r = \{u : Im\ u \leq r\}$ with $r > 0$ transformed into the hyperbola $\Omega_r = \{s : u \in Z_r\} \supset \Gamma$. let $\Sigma_\phi = \{s \neq 0 : |args| \leq \phi\} \cup 0$, with $0 < \phi < (1 - \alpha)\pi/2$, and let $\Sigma_\beta^\lambda := \lambda + \Sigma_\beta$, $\Gamma \subset \Omega_r \subset \Sigma_\beta^\lambda$.

The error bound of the proposed method for hyperbolic path is based on the following theorem.

Theorem 3.1. [7] *Let $w(\tau)$ be the solution of (1), with $\widehat{f}$ analytic in Σ_β^λ. Let $0 < \theta < 1$, $0 < \tau_0 < T$, , and let $b > 0$ be defined by $\cosh b = 1/\theta T \sin \delta$, where $T = t_0/t$. Let $\Gamma \subset \Omega_r \subset \Sigma_\beta^\lambda$, and let the scaling factor be $\omega = \theta \overline{r} n/(bt)$. Then we have, for the approximate solution w_n defined by (11), with $k = b/n \leq \overline{r}/\log 2$, $|w_n(\tau) - w(\tau)| \leq CMe^{\lambda\tau} l(\rho_r n) e^{-\mu n}(\| w_0 \| + \| \hat{f} \|_{\Sigma_\beta^\lambda})$, for $\tau_0 \leq \tau \leq t$, where $\mu = \overline{r}(1 - \theta)/b$, $\rho_r = \theta \overline{r} T \sin(\delta - r)/b$, and $C = C_{\delta,r,\beta}$, $\overline{r} = 2\pi r$, $l(\tau) = max(\log(1/\tau), 1)$.*

Since Talbot's original contour has a cotangent shape, and is much complicated to analyze see [10]. Very recently the authors [16] proposed a parabolic contour, which circumvented the slow decay due to Talbot [13]. The parabola is parameterized as

$$s = \mu(1 + iz)^2,\ (C_2), \tag{14}$$

while the hyperbolic path is re-defined as

$$s = \mu(1 + \sin(\iota w - q),\ (C_3), \tag{15}$$

for the strip $z = u + \iota c$, where $c > 0$, $-\infty < u < \infty$ the parabolic contour reduces to

$$s = \mu\left((1 - c)^2 - u^2\right) + 2\mu\iota u(1 - c), \tag{16}$$

Since we are using the trapezoidal rule on the real line, the error bound for the parabolic path is based on the following theorem

Theorem 3.2. [11] *Let $z = u + \iota y$, where u and y are real. Suppose $F(z)$ is analytic in the strip $-c < y < c$, for some $c > 0$, and $F(z) \to 0$ as $|z| \to \infty$ in the strip. Let for some $M > 0$ the function $F(z)$ satisfy*

$$\int_{\infty}^{\infty} |F(u + \iota r)| du \leq M,\ 0 < r < c,$$

then the error estimate is

$$|w_n(\tau) - w(\tau)| = \frac{2M}{e^{2\pi c/k} - 1}$$

.

The numerical accuracy mainly depends on the optimal contour of integration. The parabolic path contains parameters μ and c. In the work of [16] these parameters are optimized as

$$\mu = \frac{\pi}{4\sqrt{8\rho + 1}} \frac{n}{t},\ k = \frac{\sqrt{8\rho + 1}}{n},\ t_0 < \tau < t,\ \rho = \frac{t}{t_0}.$$

In such a case the error estimate becomes as

$$|w_n(\tau) - w(\tau)| = O(e^{-(2\pi/\sqrt{8\rho+1})n}), \ n \to \infty \tag{17}$$

In special case when $\rho = 1$, the optimal parameters and its corresponding error estimate becomes as

$$\mu = \frac{\pi}{12}\frac{n}{t}, \ k = \frac{3}{n}, \tag{18}$$

$$|w_n(\tau) - w(\tau)| = O(e^{-(2\pi n/3}), \ n \to \infty \tag{19}$$

It should be noted that the optimal parameter μ contains the ratio n/t, so the accuracy will effect with increasing nodes n for a given fixed t. To fix this problem t may be re-adjusted for better accuracy. This fact is evident from results when using the contours C_2 and C_3 shown in all figures and tables respectively.

4. Application of the method

In many cases the analytical method is unavailable for solving fractional order differential equations. So in practical problems of relevance, we have to use some numerical methods to approximate the solution of fractional order differential equations. In this section we validated our algorithms for three different problems, and to show the efficiency of our method, we compared our results with other methods available in the literature.

4.1. **Problem 1.** Here we apply the present numerical method to the equation,

$$D^\alpha w(\tau) = \beta w(\tau) + f(\tau), \tag{20}$$

For example if we choose $f(\tau) = \tau^2 + 2/\Gamma(3-\alpha)\ \tau^{2-\alpha}$, and $\alpha = 0.5$, $w(0) = 0$, $\beta = -1$. In such a case the exact solution is given as $w(\tau) = \tau^2$. Here we apply our method to approximate the problem (1)-(2), using three different types of contours C_1, C_2 and C_3 respectively. We used various number of points n along the contour of integration and the results are shown in table 1 and figure 1 respectively. It is observed that the convergence is achieved for very large number of nodes n using path C_1, while a small number of nodes n are used to get better accuracy using paths C_2 and C_3 respectively. The reason for this that the optimal parameters μ depends on number of nodes n and time τ see for example (18). As it is evident from the results that the accuracy drops with the increase in number of nodes n along C_2 and C_3 respectively. These results are calculated for fixed value of τ. This problem may be avoided if we increase the time τ with increasing nodes. It is demonstrated that the present method achieved better accuracy than [3, 2]. In this computations the parameters used for C_1 ($\alpha = 0.5$, $\tau = 5$, $\theta = 0.1$, $\lambda = 0$, $\delta = 0.1541$, $r = 0.1387$, $[\tau_o, t] = [0.5, 5]$), for C_2 ($k = 3/n$, $\mu = \pi n/12\tau$), those for C_3 ($q = 1.1721$, $A_q = cosh^{-1}(2q/(4q-\pi)\sin q)$, $k = A_q/n$, $\mu = (4\pi q - \pi^2)n/A_q\tau$) respectively.

4.2. **Problem 2.** Next we consider the test problem,

$$D^\alpha w(\tau) = \Gamma(2+\alpha)\tau + \frac{1}{4}(w(t) - \omega - \tau^{1+\alpha}), \tag{21}$$

with exact solution $w(\tau) = \tau^{1+\alpha} + \omega$. The present method is applied to approximate the problem (21) for the three contours C_1, C_2 and C_3 respectively. We used various number of nodes n along the contour of integration and the results are shown in table 2 and figure 2 respectively. We observed similar convergence behavior as achieved in case problem 1. The corresponding parameters for the three paths are C_1 ($\alpha = 0.5$, $\tau = 2$, $\theta = 0.1$, $w = 0$,

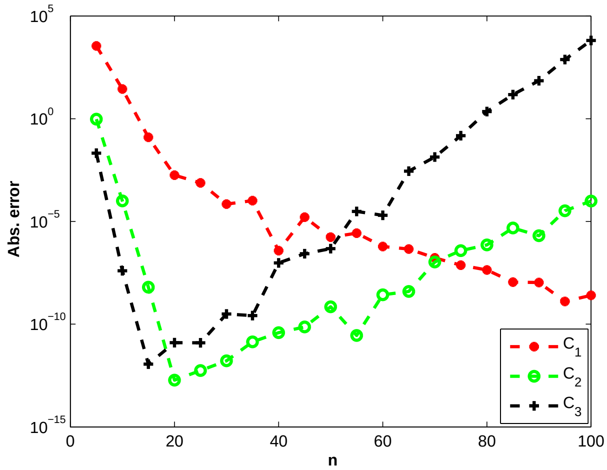

FIGURE 1. Numerical solution: Plots of absolute error versus n, for the given three types of contours corresponding to problem (20).

n	Error (C_1)	Error (C_2)	Error (C_3)
15	1.24e-001	6.26e-009	1.13e-012
20	1.80e-003	1.88e-013	1.24e-011
30	7.00e-005	1.63e-012	3.12e-010
40	3.81e-007	3.85e-011	9.51e-008
80	4.33e-008	7.25e-009	2.22e-000
100	2.54e-009	9.85e-005	6.44e+003
	[3]	[2]	
	3.56e-004	1.02e-003	

TABLE 1. Numerical solution corresponding to (20).

$\delta = 0.1541$, $r = 0.1387$, $[\tau_o, t] = [0.5, 5]$), C_2 ($k = 3/n$, $\mu = \pi n/12\tau$) and C_3 ($q = 1.1721$, $A_q = \cosh^{-1}(2q/(4q-\pi)\sin q)$, $k = A_q/n$, $\mu = (4\pi q - \pi^2)n/A_q t$) respectively.

n	Error (C_1)	Error (C_2)	Error (C_3)
15	0.0026	1.1613e-010	2.2765e-013
20	5.5944e-004	3.3468e-014	2.6197e-012
30	7.8852e-005	3.0465e-013	1.0969e-010
40	3.4510e-007	8.5219e-012	3.3494e-008
80	1.8357e-008	2.3197e-007	0.9869
100	5.6914e-010	3.7423e-005	3.0538e+003
	[4]	[5]	
	2.6300e-006	7.0900e-007	

TABLE 2. Numerical solution corresponding to (21).

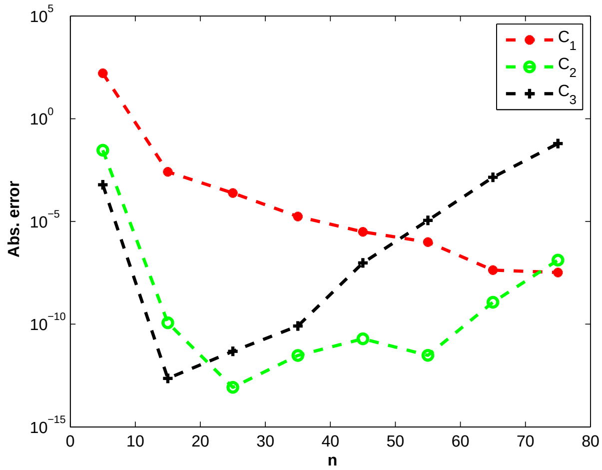

FIGURE 2. Numerical solution: Plots of absolute error versus n, for the given three types of contours corresponding to problem (21).

4.3. **Problem 3.** In the last test problem

$$Dw(\tau) + D^{\alpha}w(\tau) = f(t), \quad w(0) = 1. \tag{22}$$

In the present problem, we select a function $f(t) = e^{-t}\cos(\pi t)$, whose transformed function $\hat{f}(s) = (s+1)/((s+1)^2 + \pi^2)$ has a singularity at $s = -1 + \iota\pi$ and $s = -1 - \iota\pi$. But the present method successfully and easily approximated the problem by choosing optimal contour subject to the condition $\beta < \frac{\pi}{2} + \tan^{-1}(\frac{1+\lambda}{\pi})$. For $\alpha = \frac{1}{2}$ and using Mittage-Leffler function and substituting $s = ty^2$ the exact solution is given by

$$w(t) = E_{\frac{1}{2}}(-\sqrt{t}) + \int_0^1 E_{\frac{1}{2}}(-\sqrt{t}y)f(t - ty^2)2tydy. \tag{23}$$

The present method is applied to approximate the solution of the problem (22) for the hyperbolic and parabolic contours respectively. We used various number of points n along the contour of integration and the results in terms of actual error and corresponding error estimates $l(n\rho)e^{-\mu N}$ and $e^{-2\pi n/3}$ of the method for the contours C_1 and C_2 respectively. These results are shown in table 3 and figure 3 respectively. The actual errors agreed with theoretical error bounds of the method for both hyperbolic and parabolic contours.

5. Conclusion

In this work, we used the Laplace transform inversion numerical method to approximate the fractional order differential equations. The accuracy of proposed numerical method mainly depends on the optimal selection of contour of integration. In this work three different types of contour have been used to approximate fractional order differential equations. Comparison of the present method with other methods validated the efficiency over other methods for approximating the solution of fractional order differential equations. This procedure is an excellent alternative for approximating same types of fractional order partial differential equations. Solving fractional order differential with time-stepping methods may faces the problem of instability. On the other hand the present method avoids the problem of instability.

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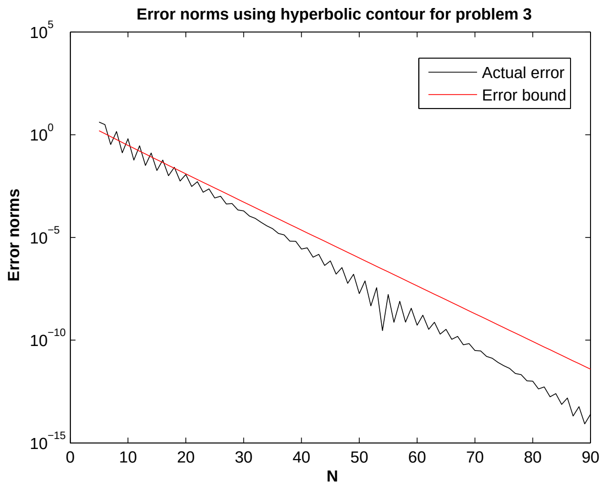

FIGURE 3. Numerical solution: Plots of absolute error versus n, for the given three types of contours corresponding to problem (22).

n	Error (C_1)	Error bound (C_1)	Error (C_2)	Error bound (C_2)
5	0.4490	1.5675	0.0216	2.8328e-005
10	0.0343	0.3034	9.3804e-006	8.0249e-010
15	7.9301e-005	0.0609	4.1278e-008	2.2733e-014
30	1.0560e-005	5.2844e-004	1.0112e-012	5.1680e-028
40	1.4798e-007	2.2889e-005	4.1793e-011	4.1472e-037
50	1.2408e-009	1.0013e-006	2.1124e-009	3.3281e-046
60	2.1145e-011	4.4085e-008	5.6829e-009	2.6708e-055
70	1.4735e-012	1.9500e-009	5.3427e-007	2.1433e-064
80	4.9337e-014	8.6551e-011	3.3696e-006	1.7200e-073
90	1.9424e-015	3.8520e-012	6.2873e-005	1.3802e-082

TABLE 3. Numerical solution using hyperbolic contour C_1 ($\alpha = -0.5$, $\tau = 1$, $\theta = 0.1$, $w = 02$, $\delta = 0.3812$, $r = 0.3431$, $[\tau_o, t] = [0.5, 5]$), and parabolic contour C_2 ($k = 3/n$, $\mu = \pi n/12\tau$), corresponding to (22).

REFERENCES

[1] Daul, L., Klein, P., and Kempfle, S., (1991) Damping description involving fractional operators, Mech. systems Signal processing, 5, 81–88.

[2] Diethelm, K., (1997), An algorithm for the numerical solution of differential equations of fractional order, Electronic Transactions on Numerical Analysis, 5, 1–6.

[3] Diethelm, K. and Ford, N. J., (2002), Analysis of fractional differential equations, Journal of Mathematical Analysis and Applications, 265, 229–248.

[4] Diethelm, K., (2015), Increasing the efficiency of shooting methods for terminal value problems of fractional order, Journal of Computational Physics, 293, 135–141.

[5] Ford, N. J., Morgado, M. L., and Rebelo, M., High order numerical methods for fractional terminal value problems, Computational Methods in Applied Mathematics, 14, 55–70.

[6] Glockle, W. G., and Nonnenmacher, T. F., (1995), A fractional calculas approach to self-similar protein dynamics, Biophys J., 68, 46–53.

[7] McLean, M., Thomee, V., (2007) Numerical solution via laplace transforms of a fractional order evolution equation, Journal of Integral Equations and Applications, 22, 57–94.

[8] Metzler, R., Schick, W., Kilian, H. G., and Nonnenmacher, T. F., (1995) Relaxation in filled ploymers: A fractional calculus approach, J. Chem. Phys. 103, 7180–7186.

[9] Oldham, K. B., and Spanier, J., (1974), The fractional calculas, Mathematics in Science and Engineering, Vol, 111, Academic Press, New York/London.
[10] Rizzardi, M., (1995), A modification of Talbot's method for the simultaneous approximation of several values of the inverse Laplace transform, ACM Trans. Math. Software, 21, 347–371.
[11] Schwarz, H. R., (1989), Numerical Analysis, John Wiley and Sons Ltd., Chichester.
[12] Sheen, D., Sloan, I. H., and Thomee, V., (2003), A parallel method for time discretization of parabolic equations based on Laplace transformations and quadrature, IMA J. Numer. Anal., 23, 269–299.
[13] Talbot, A., (1079), The accurate numerical inversion of Laplace transform, J. Inst. Math. Appl., 23, 97–120.
[14] Uddin, M., and Ahmad, Suleman., (2017), On the numerical solution of Bagley-Torvik equation via the Laplace transform, Tbilisi Mathematical Journal 10(1), 279-284.
[15] Uddin, M., and Haq, S. (2011), RBFs approximation method for time fractional partial differential equations, Communications in Nonlinear Science and Numerical Simulation, 16(11), 4208–4214.
[16] Weideman, J. A. C., and Trefethen, L. N., (2007) Parabolic and hyperbolic contours for computing the Bromwich integral, Mathematics of Computation, 76, 1341–1356.

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