TWMS J. App. Eng. Math. V.8, N.1, 2018, pp. 284-294

COMMON FIXED POINT THEOREMS FOR WEAKLY SUBSEQUENTIALLY CONTINUOUS MAPPINGS IN FUZZY METRIC SPACES VIA IMPLICIT RELATION

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Abstract. The aim of this paper is to prove some common fixed point theorems for two weakly subsequentially continuous and compatible of type (E) pairs of self mappings satisfying an implicit relation in fuzzy metric spaces. Two examples are given to illustrate our results.

Keywords: Common fixed point, weakly subsequentially continuous, compatible of type (E), implicit relation, fuzzy metric space.

AMS Subject Classification: 47H10, 54H25

1. Introduction and preliminaries

Zadeh [24] introduced the concept of fuzzy sets. Later George and Veeramanti [6] modified the concept of the fuzzy metric spaces introduced by Kramosil and Michalek [13]. Many authors have proved fixed point and common fixed point theorems involving fuzzy metric spaces, Jungck [10] introduced the commuting mappings in order to establish a common fixed point in metric spaces and lateron Jungck [11] generalized it to compatible mappings concept. The same author Jungck and Rhoades [12] introduced the weakly compatible mappings concept which is weaker than the concept of compatible mappings. In 2009 Bouhadjera and Godet Tobie [4] introduced the concept of subcompatibility and subsequential continuity. Later Imdad et al. [9] improved the results of Bouhadjera et al.[4] and mentioned that these results can easily recovered by replacing subcompatibility with compatibility or subsequential continuity with reciprocal continuity. Gopal and Imdad [7] proved some results in (GV)-fuzzy metric spaces by using these concepts. More recently, the present author [3] gave the notion of weakly subsequential continuity and used it with compatibility of type (E) to establish some common fixed point results in metric spaces. Singh et.al [22] defined the concept of compatible mappings in fuzzy metric spaces. The concept of implicit relation has been introduced by Popa [15]. There are some interesting references concerning fixed point and common fixed results involving the notion of implicit relation in fuzzy metric spaces for instance [1, 7, 14, 16]. In this paper we will improve and generalize results in paper [2] by combining the concept of weakly subsequential continuity with compatibility of type (E) due to Singh et.al [19].

Definition 1.1. [24] *Let X be a non empty set. A function M with domain X and values in $[0, 1]$ is called to be a fuzzy set.*

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§ Manuscript received: September 21, 2016; accepted: August 8, 2017.
TWMS Journal of Applied and Engineering Mathematics, Vol.8, No.1a

Definition 1.2. [18] *A binary operation* $* : [0,1] \times [0,1] \to [0,1]$ *is a continuous t-norm if it satisfies the following conditions:*

(1) $*$ *is associative and commutative.*
(2) $*$ *is continuous function.*
(3) $a * 1 = a$, *for all* $a \in [0,1]$
(4) $a * b \leq c * d$, *if* $a \leq c$ *and* $b \leq d$, *for all* $a, b, c, d \in [0,1]$.

Definition 1.3. [8] *A triplet* $(X, M, *)$ *is a fuzzy metric space if* X *is an arbitrary set,* $*$ *is a continuous t-norm and* M *is a fuzzy set in* $X^2 \times (0, \infty)$ *satisfying the following conditions for every* $x, y, z \in X$ *and* $s, t > 0$*:*

(1) $M(x, y, t) > 0$,
(2) $M(x, y, t) = 1$ *for all* $t > 0$ *if and only if* $x = y$,
(3) $M(x, y, t) = M(y, x, t)$,
(4) $M(x, y, t) * M(y, z, s) = M(x, z, t + s)$,
(5) $M(x, y, .) : [0, \infty) \to [0, 1]$ *is a continuous function, for all* $x, y, z \in X$ *and* $t, s > 0$.

Note that $M(x, y, t)$ *can be considered as the degree of nearness between* x *and* y *with respect to* t*. We identify* $x = y$ *with* $M(x, y, t) = 1$, *for all* $t > 0$*. The following example shows that every metric space induces a fuzzy metric space.*

Example 1.1. [8] *Let* (X, d) *be a metric space. Define* $a * b = \min\{a, b\}$ *(or* $a * b = ab$*) for all* $a, b \in [0, 1]$ *and for each* $t > 0$*. Let* M *be a fuzzy set on* $X^2 \times [0, \infty)$ *defined as follows:*

$$M(x, y, t) = \frac{t}{t + d(x, y)},$$

then $(X, M, *)$ *is a fuzzy metric space.*

Definition 1.4. [8] *Let* $(X, M, *)$ *be a fuzzy metric space.*

(1) *A sequence* $\{x_n\}$ *in* X *is said to be convergent to a point* $x \in X$, *if* $\lim_{n\to\infty} M(x_n, x, t) = 1$, *for all* $t > 0$.
(2) *A sequence* $\{x_n\}$ *in* X *is said to be a Cauchy sequence if for each* $0 < \varepsilon < 1$ *and* $t > 0$, *there exists* $n_0 \in \mathbb{N}$ *such that* $M(x_n, x_m, t) > 1 - \varepsilon$, *for all* $n, m \geq 0$.
(3) *A fuzzy metric space in which every Cauchy sequence convergent to* $x \in X$ *is to be complete.*

Singh et al. [22] extended the concept of compatible mappings due to Jungck [10] in the context of fuzzy metric spaces as follows:

Definition 1.5. *Two self mappings* A *and* S *on fuzzy metric space are said to be compatible if* $\lim_{n\to\infty} M(ASx_n, SAx_n) = 1$, *for all* $t > 0$, *whenever* $\{x_n\}$ *is a sequence in* X *such* $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} SAx_n = z$, *for some* $z \in X$.

Definition 1.6. [23] *Mappings* A *and* S *on a fuzzy metric space* $(X, M, *)$ *are said to be reciprocally continuous if* $\lim_{n\to\infty} ASx_n = Az$ *and* $\lim_{n\to\infty} SAx_n = Sz$, *whenever* $\{x_n\}$ *is a sequence in* X *such* $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} SAx_n = z$, *for some* $z \in X$.

Definition 1.7. [3] *Let* (X, d) *be a metric space. Let* A *and* S *be self mappings on* X*. The pair* (A, S) *is said to be weakly subsequentially continuous, if there exists a sequence* $\{x_n\}$ *such that* $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$, *for some* $z \in X$, *and* $\lim_{n\to\infty} ASx_n = Az$ *or* $\lim_{n\to\infty} SAx_n = Sz$.

Motivated by definition 1.7, we define:

Definition 1.8. *Two self mappings A and S on a fuzzy metric space $(X, M, *)$ are said to be weakly subsequentially continuous (wsc), if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$, for some $z \in X$ and $\lim_{n\to\infty} ASx_n = Az$ or $\lim_{n\to\infty} SAx_n = Sz$*

Definition 1.9. *The pair (A, S) is said to be A-subsequentially continuous (S-subsequentially continuous), if there exists a sequence $\{x_n\}$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ and($\lim_{n\to\infty} ASx_n = Az$($\lim_{n\to\infty} SAx_n = Sz$)*

Example 1.2. *Let $X = [0, 2]$ and $M(x, y, t) = \frac{t}{t+|x-y|}$ with a continuous t-norm: $* : (a, b) \mapsto ab$ for all $t > 0$. Define A, S as follows:*

$$Ax = \begin{cases} 1 + x, & 0 \le x \le 1 \\ \frac{x+1}{2}, & 1 < x \le 2 \end{cases}, \quad Sx = \begin{cases} 1 - x, & 0 \le x \le 1 \\ 2 - x, & 1 < x \le 2 \end{cases}$$

Clearly A and S are discontinuous at 1.
We consider a sequence $\{x_n\}$ defined for each $n \ge 1$ by: $x_n = \dfrac{1}{n}$.
Now $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = 1$. Also we have:

$$\lim_{n\to\infty} ASx_n = \lim_{n\to\infty} A(1 - \frac{1}{n}) =$$
$$\lim_{n\to\infty} (2 - \frac{1}{n}) = 2 = A(1).$$

Then the pair (A, S) is A-subsequentially continuous as well as weakly subsequentially continuous.

Singh et al. [19, 20] introduced the notion of compatibility of type (E) in metric space. As an extension to the setting of the fuzzy metric spaces. Define:

Definition 1.10. *Mappings A and S on a fuzzy metric space $(X, M, *)$ are said to be compatible of type (E), if $\lim_{n\to\infty} S^2x_n = \lim_{n\to\infty} SAx_n = Az$ and $\lim_{n\to\infty} A^2x_n = \lim_{n\to+\infty} ASx_n = Sz$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$, for some $t \in X$.*

Definition 1.11. *Two self mappings A and S of a fuzzy metric space $(X, M, *)$ into itself are said to be A-compatible of type (E), if $\lim_{n\to\infty} A^2x_n = \lim_{n\to\infty} ASx_n = Sz$, for some $z \in X$ and said to be S-compatible of type (E), if $\lim_{n\to\infty} S^2x_n = \lim_{n\to\infty} SAx_n = Az$, for some $z \in X$.*

Notice that if A and S are compatible of type (E), then they are A-compatible and S-compatible of type (E), but the converse is not true.

Example 1.3. *Let $X = [0, \infty)$ and the t-norm $a * b = ab$ with $M(x, y, t) = \frac{t}{t+|x-y|}$. We define A, S as follows:*

$$Ax = \begin{cases} 2, & 0 \le x \le 2 \\ x + 1, & x > 2 \end{cases} \quad Sx = \begin{cases} x, & 0 \le x \le 2 \\ 0, & x > 2 \end{cases}$$

Consider a sequence $\{x_n\}$ defined by: $x_n = 2 - \dfrac{1}{n}$, for all $n \ge 1$. we have

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = 2,$$

and

$$\lim_{n\to\infty} A^2x_n = \lim_{n\to\infty} ASx_n = 2 = S(2)$$

$$\lim_{n\to\infty} S^2x_n = \lim_{n\to\infty} SAx_n = 2 = A(2)$$

then (A,S) is compatible of type (E).

2. Implicit relation

Let $\mathcal{F}$ be the set of all continuous functions $F : [0,1]^6 \to [0,1]$ satisfying:

(F_1) : F is non increasing in u_5 and u_6.

(F_2) : If, for some constant $k \in (0,1)$, we have

$$(F_a) : F(u(kt), v(t), v(t), u(t), 1, u(\frac{t}{2}) * v(\frac{t}{2})) \geq 1,$$

or

$$(F_b) : F(u(kt), v(t), u(t), v(t), u(\frac{t}{2}) * v(\frac{t}{2}), 1) \geq 1,$$

for any fixed $t > 0$ and any nondecreasing functions $u, v : [0,1) \to [0,1]$, then there exists $h \in (0,1)$ such $u(ht) \geq v(t) * u(t)$.

(F_3) : If, for some constant $k \in (0,1)$, we have $F(u(kt), u(t), 1, 1, u(t), u(t)) \geq 1$, for any fixed $t > 0$ and any nondecreasing function $u : [0,1) \to [0,1]$, then $u(kt) \geq u(t)$.

Example 2.1. *Let*

$$F(u_1, u_2, u_3, u_4, u_5, u_6) = \frac{u_1}{\min\{u_2, u_3, u_4, u_5, u_6\}}$$

and $a * b = \min\{a, b\}$.

(F_1) : *Obviously satisfied.*

(F_2) : *If we have*

$$\frac{u(kt)}{\min\{v(t), v(t), u(t), 1, u(\frac{t}{2}) * v(\frac{t}{2})\}} \geq 1,$$

for some $k \in (0,1)$, then $u(kt) \geq \min\{u(\frac{t}{2}), v(\frac{t}{2})\}$, putting $h = 2k$, we obtain $u(ht) \geq u(t) * v(t)$. For F_b, it is similar as F_a.

(F_3) : *If there exists $k \in (0,1)$ such that*

$$u(kt) \geq \min\{u(t), 1, 1, u(t), u(t)\},$$

then $u(kt) \geq u(t)$.

Example 2.2.

$$F(u_1, u_2, u_3, u_4, u_5, u_6) = \frac{u_1 \max\{u_2, u_3, u_4\}}{\min\{u_5, u_6\}}.$$

(F_1) *Obviously satisfied.*

(F_2) *Suppose there is $k \in (0,1)$ such:*

$$u(kt) \max\{u(t), v(t)\} \geq u(\frac{t}{2}) * v(\frac{t}{2}),$$

choosing $h = 2k$, we obtain $u(ht) \geq u(t) * v(t)$. For F_b, it is similar as F_a.

(F_3) *Clearly*

$$\frac{u(kt)}{u(t)} \geq 1,$$

implies that $u(kt) \geq u(t)$.

Example 2.3. *Let*

$$F(u_1,u_2,u_3,u_4,u_5,u_6) = \frac{u_1^3}{[u_2 * u_3 * u_4]\max\{u_5,u_6\}}$$

and $a*b=ab$.

(F_1) *Obviously satisfied.*

(F_2) *Suppose there is* $k\in(0,1)$ *such:*

$$u^3(kt) \geq (v(t)*v(t)*u(t))\max\{1,u(\frac{t}{2})*v(\frac{t}{2})\} = v^2(t)u(t),$$

so

$$u(kt) \geq v^{\frac{2}{3}}(t)u^{\frac{1}{3}}(t) \geq v(t)u(t) = u(t)*v(t),$$

then it suffices to taking $k=h$. *For* F_b *it is similar.*

(F_3) *If there exists* $k\in(0,1)$ *such*

$$u^3(kt) \geq u^2(t),$$

since $u(t)\in[0,1]$ *for all* $s>0$, *then* $u(kt)\geq u(t)$.

We need in a sequel of our main results the following lemma:

Lemma 2.1. [14] *Let* $(X,M,*)$ *be a fuzzy metric space. If there exists a constant* $k\in(0,1)$ *such that* $M(x,y,kt)\geq M(x,y,t)$, *for all* $t>0$ *and fixed* $x,y\in X$, *then* $x=y$.

3. Main results

Theorem 3.1. *Let* $(X,M,*)$ *be a fuzzy metric space, with* $a*b=\min\{a,b\}$, *for all* $a,b\in[0,1]$. A,B,S *and* T *are four self mappings on* X *such that the two pairs* (A,S) *and* (B,T) *are weakly subsequentially continuous (wsc) and compatible of type (E), then the pairs* (A,S) *and* (B,T) *has a coincidence point.*
Moreover, A,B,S *and* T *have a unique common fixed point in* X *provided for all* $x,y\in X$ *and each* $t>0$, *there exists* $k\in(0,1)$ *and* $F\in\mathcal{F}$ *such that:*

$$F\left(\begin{array}{c} M(Sx,Ty,kt),M(Ax,By,t),M(Ax,Sx,t), \\ M(By,Ty,t)M(Sx,By,t),M(Ty,Ax,t)\end{array}\right) \geq 1 \qquad (1)$$

Proof. Since (A,S) is weakly subsequentially continuous, there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$, for some $z\in X$ and $\lim_{n\to\infty} ASx_n = Az($ or $\lim_{n\to\infty} SAx_n = Sz)$. Also (A,S) is compatible of type (E) implies that

$$\lim_{n\to\infty} ASx_n = \lim_{n\to\infty} A^2x_n = Sz$$

and

$$\lim_{n\to\infty} SAx_n = \lim_{n\to\infty} S^2x_n = Az.$$

Consequently we obtain $Az=Sz$ and z is a coincidence point of A and S. Similarly for B and T, since (B,T) is wsc (suppose that it is B-subsequentially continuous) there exists a sequence $\{y_n\}$ such

$$\lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = w,$$

for some $w\in X$ and

$$\lim_{n\to\infty} BTy_n = Bw.$$

Also (B,T) is compatible of type (E), we get

$$\lim_{n\to\infty} BTy_n = \lim_{n\to\infty} B^2y_n = Tw$$

$$\lim_{n\to\infty} TBy_n = \lim_{n\to\infty} T^2 y_n = Bw.$$

Hence $Bw = Tw$ and w is a coincidence point of B and T.
We claim $Az = Bw$, if not by using (1) we get:

$$F\left(\begin{array}{c} M(Sz,Tw,kt), M(Az,Bw,t), M(Az,Sz,t), \\ M(Bw,Tw,t)M(Sz,Bw,t), M(Tw,Az,t) \end{array}\right) =$$

$$F\left(\begin{array}{c} M(Az,Bw,kt), M(Az,Bw,t), 1, \\ 1, M(Az,Bw,t), M(Bw,Az,t) \end{array}\right) \geq 1.$$

From (F_2), we get $M(Az,Bw,kt) \geq M(Az,Bw,t)$.
Hence by Lemma 2.1 we obtain $Az = Bw$.
Now we show $z = Az$, if not by using (1) we get:

$$F\left(\begin{array}{c} M(Sx_n,Tw,kt), M(Ax_n,Bw,t), M(Ax_n,Sz,t), \\ M(Bw,Tw,t), M(Sx_n,Bw,t), M(Tw,Ax_n,t) \end{array}\right) \geq 1.$$

Letting $n \to \infty$ we get:

$$F\left(\begin{array}{c} M(z,Tw,kt), M(z,Bw,t), M(z,Sz,t), \\ 1, M(z,Tw,t), M(Bw,z,t) \end{array}\right) =$$

$$F\left(\begin{array}{c} M(z,Az,kt), M(z,Az,t), M(z,Az,t), \\ 1, M(z,Az,t), M(Az,z,t) \end{array}\right) \geq 1.$$

From (F_1) we have $M(z,Az,kt) \geq M(z,Az,t$. Then $z = Az = Sz$ (by Lemma 2.1).
Nextly we prove $z = w$, if not by using (1) we get:

$$F\left(\begin{array}{c} M(Sx_n,Ty_n,kt), M(Ax_n,By_n,t), M(Ax_n,Sx_n,t), \\ M(By_n,Ty_n,t), M(Ax_n,Ty_n,t), M(By_n,Ax_n,t) \end{array}\right) \geq 1.$$

Letting $n \to \infty$ we get:

$$F\left(\begin{array}{c} M(z,w,kt), M(z,w,t), M(z,z,t), \\ M(w,w,t), M(z,w,t), M(w,z,t) \end{array}\right) \geq 1.$$

From (F_2) we get $M(z,w,kt) \geq M(z,w,t)$. Hence, by Lemma 2.1 we obtain $z = Bz$,i.e., z is a fixed point of A, B, S and T.
For the uniqueness suppose that there is another fixed point q and using (1), we get:

$$F\left(\begin{array}{c} M(Sz,Tq,kt), M(Az,Bq,t), M(Az,Sz,t), \\ M(Bq,Tq,t), M(Az,Tq,t), M(Sz,Bq,t) \end{array}\right) =$$

$$F\left(\begin{array}{c} M(z,q,kt), M(z,q,t), 1, \\ 1, M(z,q,t), M(z,q,t) \end{array}\right) \geq 1.$$

From (F_2) and Lemma 2.1, we get $z = q$. Consequently z is unique. □

Theorem 3.1 improves Theorem 1 of Altun and Türkoğlu [2].
If $A = B$ and $S = T$, we obtain the following corollary:

Corollary 3.1. *Let $(X, M, *)$ be a fuzzy metric space, with $a * b = \min\{a, b\}$, for all $a, bin[0, 1]$. A and S are two self mappings on X such that the pair (A, S) is weakly subsequentially continuous (wsc) and compatible of type (E), then pair (A, S) has a coincidence point each. Moreover, A, B, S and T have a unique common fixed point in X provided for all x, y in X and each $t > 0$, there exists $k \in (0, 1)$ and $F \in \mathcal{F}$ such that:*

$$F\left(\begin{array}{c} M(Sx,Sy,kt), M(Ax,Ay,t), M(Ax,Sx,t), \\ M(Ay,Sy,t), M(Ax,Sy,t), M(Sx,Ay,t) \end{array}\right) \geq 1.$$

If we combine Theorem 3.1 with Example 2.1 we obtain the following corollary:

Corollary 3.2. *Let $(X, M, *)$ be a fuzzy metric space, with $a * b = \min\{a, b\}$, for all $a, bin[0, 1]$. A and S are two self mappings on X such that the pair (A, S) is weakly subsequentially continuous (wsc) and compatible of type (E), then pair (A, S) has a coincidence point each. Moreover, A, B, S and T have a unique common fixed point in X provided for all x, y in X and each $t > 0$, there exists $k \in (0, 1)$ and $F \in \mathcal{F}$ such that:*

$$M(Sx, Ty, kt) \geq \min\{M(Ax, By, t), M(Ax, Sx, t), M(By, Ty, t), M(Ax, Ty, t), M(Sx, By, t)\},$$

Theorem 3.2. *Let $(X, M, *)$ be a fuzzy metric space with $a * b = \min\{a, b\}$, for all $a, b \in [0, 1]$ and let A, B, S be self mappings on X satisfying (1), suppose*

(1) *the pair (A, S) (or (B, T)) is weakly subsequentially continuous and compatible of type (E),*
(2) *$\overline{S(X)} \subset B(X)$ (or $\overline{T(X)} \subset A(X)$),*
(3) *(B, T) is weakly compatible (or (A, S) is weakly compatible).*

Then A, B, S and T have unique common fixed point in X.

Proof. Since the pair (A, S) is wsc and compatible of type (E), there exists a sequence $\{x_n\}$ in X such $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$ and $Az = Sz$.
$\overline{S(X)} \subset B(X)$ implies that there is $v \in X$ such $z = Bv$. We check $z = Tv$, if it is not, by using (1) we get:

$$F\begin{pmatrix} M(Sx_n, Tv, kt), M(Ax_n, Bv, t), M(Ax_n, Sx_n, t), \\ M(Bv, Tv, t), M(Sx_n, Bv, t), M(Tv, Ax_n, t) \end{pmatrix} \geq 1.$$

Letting $n \to \infty$, we get:

$$F\begin{pmatrix} M(z, Tv, kt), M(z, Tv, t), 1, \\ M(z, Tv, t), 1, M(z, Tv, t) \end{pmatrix} \geq 1.$$

Since

$$M(z, Tv, t) \geq M(z, Tv, \frac{t}{2}) = M(z, Tv, \frac{t}{2}) * 1,$$

and F is non increasing in sixth variable we get:

$$F\begin{pmatrix} M(z, Tv, kt), 1, 1, \\ M(z, Tv, t), 1, M(z, Tv, \frac{t}{2}) * 1 \end{pmatrix} \geq 1.$$

From (F_2), we get $M(z, Tv, kt) \geq M(z, Tv, t)$. Hence by Lemma 2.1 we obtain $z = Tv = Bv$.

The point v is a coincidence point for B and T, and since the pair (B, T) is weakly compatible we get $Bz = Tz$.
Now, we claim $z = Az$, if is not by using (1), we get:

$$F\begin{pmatrix} M(Az, z, kt), M(Az, z, t), 1, 1, \\ M(Az, z, t), M(Az, z, t), M(z, Tv, t) \end{pmatrix} =$$

$$F\begin{pmatrix} M(Sz, Tv, kt), M(Az, Bv, t), 1, 1, \\ M(Sz, Bv, t), M(Az, Tv, t), M(z, Tv, t) \end{pmatrix} \geq 1.$$

By using (F_3) and Lemma 2.1, we get $z = Az$.
Now, if $Az \neq Bz$, by using (1) we get;

$$F\begin{pmatrix} M(Az, Bz, kt), M(Az, Bz, t), 1, 1, \\ M(Az, Bz, t), M(Az, Bz, t), M(z, Tv, t) \end{pmatrix} =$$

$$F\begin{pmatrix} M(Sz, Tz, kt), M(Az, Bz, t), 1, 1, \\ M(Sz, Bz, t), M(Az, Tz, t), M(z, Tv, t) \end{pmatrix} \geq 1.$$

Then from (F_3) and Lemma 2.1, we obtain $Az = Bz$. Hence z is a common fixed point of A, B, S and T.
The uniqueness is similar as in proof of Theorem 3.1. □

Theorem 3.3. *Let $(X, M, *)$ be a fuzzy metric space, with $a*b = \min\{a, b\}$ and let A, B, S be self mappings on X satisfying (1) and the following conditions:*

(1) *the pair (A, S) (or (B, T)) is wsc and compatible of type (E),*
(2) *$S(X) \subset B(X)$ (or $T(X) \subset A(X)$),*
(3) *$\{Ty_n\}$ converges for every sequence $\{y_n\}$ in X, whenever $\{By_n\}$ converges(or $\{Sx_n\}$ converges for every sequence $\{x_n\}$ in X, whenever $\{Ax_n\}$ converges,*
(4) *(B, T) is weakly compatible (or (A, S) is weakly compatible).*

Then A, B, S and T have unique common fixed point in X.

Proof. As in proof of Theorem 3.1, if the pair (A, S) is wsc and compatible of type (E), then there exists a sequence $\{x_n\}$ in X satisfying $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$ and $Az = Sz$.
The inclusion $A(X) \subset T(X)$ implies that there is a sequence $\{y_n\}$ in X such $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Ty_n = z$.
Suppose $\lim_{n\to\infty} Ty_n = l$, by using 1 we get:

$$F\begin{pmatrix} M(Sx_n, Ty_n, kt), M(Ax_n, By_n, t), M(Ax_n, Sx_n, t), \\ M(By_n, Ty_n, t), M(Sx_n, By_n, t), M(Ax_n, Ty_n, t) \end{pmatrix} \geq 1.$$

Letting $n \to \infty$, we get:

$$F\begin{pmatrix} M(z, l, kt), 1, 1, \\ M(z, l, t), 1, M(z, l, t) \end{pmatrix} \geq 1.$$

From (F_2) and Lemma 2.1, we obtain $l = z$.
We prove $z = Az$, if it is not, by using (1) we get:

$$F\begin{pmatrix} M(Sz, Ty_n, kt), M(Az, By_n, t), M(Az, Sz, t), \\ M(By_n, Ty_n, t), M(Sz, By_n, t), M(Az, Ty_n, t) \end{pmatrix} \geq 1.$$

Letting $n \to \infty$, we get:

$$F\begin{pmatrix} M(Az, z, kt), M(Az, z, t), 1, \\ 1, M(Az, z, t), M(Az, z, t) \end{pmatrix} \geq 1.$$

From (F_3) and Lemma 2.1, we obtain $z = Az = Sz$.
Since $S(X) \subset B(X)$, there exists $w \in X$ satisfying $z = Sz = Bw$, if $Bw \neq w$, by using (1) we get:

$$F\begin{pmatrix} M(Bw, Tw, kt), 1, 1, \\ M(Bw, Tw, t), 1, M(Bw, Tw, t) \end{pmatrix} =$$

$$F\begin{pmatrix} M(Sz, Tw, kt), M(Az, Bw, t), M(Az, Sx_n, t), \\ M(Bw, Tw, t), M(Sz, Bw, t), M(Az, Tw, t) \end{pmatrix} \geq 1.$$

From (F_2) and Lemma 2.1, we obtain $Bw = Tw$ and w is a coincidence point for B and T, since (B, T) is weakly compatible, then $z = Bz = Tz$.
The uniqueness is similar as in proof of Theorem 3.1. □

Theorem 3.3 improves theorem 2 of Altun and Türkoğlu [2].

Example 3.1. *Let* $X = [0, \infty)$ *and* $M = \frac{t}{t+|x-y|}$ *with t-norm defined by* $a * b = \min\{a, b\}$, *for all* $x, y \in X$ *and* $t > 0$. *Define* A *and* S *by:*

$$Ax = \begin{cases} 2x, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases} \qquad Sx = \begin{cases} \frac{x}{2}, & 0 \leq x \leq 1 \\ \frac{1}{4}, & x > 1 \end{cases}$$

Consider a sequence $\{x_n\}$ *such for each* $n \geq 1$ *we have:*
$x_n = \frac{1}{n}$, *clearly that* $\lim\limits_{n\to\infty} Ax_n = \lim\limits_{n\to\infty} Sx_n = 0$, *also we have:*

$$\lim_{n\to\infty} ASx_n = A(0) = 0$$

$$\lim_{n\to\infty} A^2x_n = S(0) = 0,$$

then (A, S) *is A-subsequentially continuous and A-compatible of type (E).*
For the inequality (1), taking

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = \frac{t_1}{\min\{t_i\}}, i = 2, 3, 4, 5, 6,$$

we have the following cases:

(1) *For* $x, y \in [0, 1]$, *we have*

$$M(Sx, Sy, t) = \frac{t}{t + \frac{1}{2}|x - y|}.$$

For $k = \frac{1}{2}$, *we get:*

$$M(Sx, Sy, kt)) = \frac{t}{t + |x - y|} \geq \frac{t}{t + 2|x - y|} = M(Ax, Ay, t).$$

(2) *For* $x \in [0, 1]$ *and* $y > 1$, *we have*

$$M(Sx, Sy, t) = \frac{t}{t + \frac{1}{2}|x - \frac{1}{2}|}.$$

For $k = \frac{1}{2}$, *we get:*

$$M(Sx, Ty, \frac{1}{2}t) = \frac{t}{t + |x - \frac{1}{2}|} \geq \frac{t}{t + \frac{3}{4}} = M(Ay, Sy, t).$$

(3) *For* $x \in (1, \infty)$ *and* $y \in [0, 1]$, *we have*

$$M(Sx, Sy) = \frac{t}{t + \frac{1}{2}|y - \frac{1}{2}|}.$$

For $k = \frac{1}{2}$, *we get:*

$$M(Sx, Ty, kt) = \frac{t}{t + |y - \frac{1}{2}|} \geq \frac{t}{t + \frac{3}{4}} = M(Ax, Sx, t).$$

(4) *For* $x, y \in (1, \infty)$, *we have* $M(Sx, Sy, t) = 1$, *for any* $k \in (0, 1)$, *so obviously (1) is satisfied.*

Consequently all the hypotheses of Corollary 3.2 (with $A = B$ *and* $S = T$*) are satisfied, therefore the point* 0 *is the unique common fixed of* A *and* S.

Example 3.2. *Let* $X = [0, 2]$ *and* $M = \frac{t}{t+|x-y|}$ *with a t-norm defined by* $a * b = \min\{a, b\}$, *for all* $x, y \in X$ *and* $t > 0$,*define* A, B, S *and* T *as follows:*

$$Ax = \begin{cases} x, & 0 \leq x \leq 1 \\ \frac{1}{2}, & 1 < x \leq 2 \end{cases} \qquad Bx = \begin{cases} 2 - x, & 0 \leq x \leq 2 \\ 2, & 1 < x \leq 2 \end{cases}$$

$$Sx = \begin{cases} 1, & 0 \le x \le 1 \\ \frac{1}{4}, & 1 < x \le 2 \end{cases} \qquad Tx = \begin{cases} \frac{x+1}{2}, & 0 \le x \le 1 \\ \frac{3}{4}, & 1 < x \le 2 \end{cases}$$

We consider a sequence $\{x_n\}$ *defined for each* $n \ge 1$ *by:*
$x_n = 1 - \frac{1}{n}$, *clearly that* $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = 1$, *also we have:*

$$\lim_{n\to\infty} ASx_n = A(1) = S(1) = 1$$

$$\lim_{n\to\infty} A^2x_n = S(1) = 1,$$

then (A, S) *is A-subsequentially continuous and A-compatible of type (E).*

$S(X) = \overline{S(X)} = \{\frac{1}{4}, 1\} \subset [0, 1] \cup \{2\} = B(X)$. *The mappings* B *and* T *have a coincidence point* $x = 1$ *and* $BT(1) = TB(1) = 1$, *so the pair* (B, T) *is weakly compatible.*
To check the inequality (1) is satisfied, we take F *as in Example3.1, and so we have the following cases:*

(1) *For* $x, y \in [0, 1]$, *there is* $k = \frac{1}{4}$ *such that:*

$$M(Sx, Ty, kt)) = \frac{t}{t + |x - 1|} \ge \frac{t}{t + |x - 1|} = M(By, Sx, t)$$

(2) *For* $x \in [0, 1]$ *and* $y \in (1, 2]$, *there is* $k = \frac{1}{2}$ *such that:*

$$M(Sx, Ty, kt) = \frac{t}{t + \frac{1}{2}} \ge \frac{t}{t + 1} = M(By, Sx, t).$$

(3) *For* $x \in (1, 2]$, $y \in [0, 1]$, *there is* $k = \frac{1}{2}$ *such that:*

$$M(Sx, Ty, kt) = \frac{t}{t + |\frac{1}{2} - y|} \ge \frac{t}{t + \frac{1}{2}} = M(Ax, Sx, t),$$

because, for all $y \in [0, 1]$, *we have:* $|\frac{1}{2} - y| \le \frac{1}{2}$.

(4) *For* $x, y \in (1, 2]$,*there is* $k = \frac{1}{2}$ *such that:*

$$M(Sx, Ty, kt) = \frac{t}{t + 1} \ge \frac{t}{t + 1} = M(Ax, By, t)$$

Consequently all hypotheses of Theorem3.2 are satisfied, therefore 1 *is the unique common fixed of* A, B, S *and* T.

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