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GENERALIZED HANKEL DETERMINANT FOR A GENERAL SUBCLASS OF UNIVALENT FUNCTIONS

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ABSTRACT. Making use of the generalized Hankel determinant, in this work, we consider a general subclass of univalent functions. Moreover, upper bounds are obtained for $\left|a_3 - \mu a_2^2\right|$, where $\mu \in \mathbb{R}$.

Keywords: Analytic and univalent functions, Fekete-Szegö inequality, Hankel determinant.

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1. Introduction

Let A represent the class of functions f which are analytic in the open unit disk

$$U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$$

with in the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Let S be the subclass of A consisting of the form (1) which are also univalent in U.

Let P_β denote the class of functions consisting of p, such that

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots = 1 + \sum_{n=1}^{\infty} p_n z^n,$$

which are regular in the open unit disc U and satisfy $\Re(p(z)) > \beta$ for some β $(0 \leq \beta < 1)$ and for any $z \in U$.

The Fekete-Szegö functional $\left|a_3 - \mu a_2^2\right|$ for normalized univalent functions of the form given by (1) is well known for its rich history in Geometric Function Theory. Its origin was in the disproof by Fekete and Szegö of the 1933 conjecture of Littlewood and Paley that the coefficients of odd univalent functions are bounded by unity (see [3]). The functional has since received great attention, particularly in many subclasses of the family of univalent functions. Nowadays, it seems that this topic had become an interest among the researchers (see, for example, [1], [4], [5], [6]).

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The q^{th} Hankel determinant for $n \geq 0$ and $q \geq 1$ is stated by Noonan and Thomas ([7]) as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix} \qquad (a_1 = 1).$$

This determinant has also been investigated by several authors. For example, Noor [8] determined the rate of growth of $H_q(n)$ as $n \to \infty$ for functions f given by (1) with bounded boundary. In particular, sharp upper bounds on $H_2(2)$ were obtained by the authors of articles ([8], [10]) for different classes of functions.

It is interesting to note that

$$H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2$$

and

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2.$$

The Hankel determinant $H_2(1) = a_3 - a_2^2$ is well-known as Fekete-Szegö functional.

Definition 1.1. [2] *A function $f \in A$ is said to be in the class $Q_\lambda(\beta)$, if the following condition is satisfied:*

$$\Re\left((1-\lambda)\frac{f(z)}{z} + \lambda f'(z)\right) > \beta; \quad 0 \leq \beta < 1, \quad \lambda \geq 0, \quad z \in U.$$

In order to derive our main results, we require the following lemmas.

Lemma 1.1. [9] *If the function $p \in P_\beta$, then*

$$2(1-\beta)p_2 = p_1^2 + x(4(1-\beta)^2 - p_1^2)$$

$$4(1-\beta)^2 p_3 = p_1^3 + 2(4(1-\beta)^2 - p_1^2)p_1 x - p_1(4(1-\beta)^2 - p_1^2)x^2$$

$$+2(1-\beta)(4(1-\beta)^2 - p_1^2)(1-|x|^2)z$$

for some x, z with $|x| \leq 1$ and $|z| \leq 1$.

Lemma 1.2. [9] *If the function $p \in P_\beta$, then*

$$|p_n| \leq 2(1-\beta) \qquad (n \in \mathbb{N} = \{1, 2, \ldots\}).$$

Lemma 1.3. [9] *If the function $p \in P_\beta$, then, for all n and s $(1 \leq s < n)$,*

$$|(1-\alpha)\mu p_n - p_{n-s}p_s| \leq \begin{cases} 2(2-\mu)(1-\beta)^2; & \mu \leq 1 \\ 2(1-\beta)^2\mu; & \mu \geq 1 \end{cases}.$$

The purpose of this paper is to find the upper bounds of generalized Hankel determinant $\left|a_n a_{n+2} - \mu a_{n+1}^2\right|$ for functions in the class $Q_\lambda(\beta)$.

2. Main results

Theorem 2.1. *Let f given by (1) be in the class $Q_\lambda(\beta)$, $0 \le \beta < 1$ and $n = 2, 3, \ldots$. Then*

$$\left|a_n a_{n+2} - \mu a_{n+1}^2\right| \le \begin{cases} \frac{4(1-\beta)^2}{1+2n\lambda+(n^2-1)\lambda^2}\left[1 - \frac{1+2n\lambda+(n^2-1)\lambda^2}{(1+n\lambda)^2}\mu\right]; & \mu \le 0 \\ \frac{4(1-\beta)^2}{1+2n\lambda+(n^2-1)\lambda^2}; & 0 \le \mu \le \frac{(1+n\lambda)^2}{1+2n\lambda+(n^2-1)\lambda^2} \\ \frac{4(1-\beta)^2}{1+2n\lambda+(n^2-1)\lambda^2}\left[\frac{1+2n\lambda+(n^2-1)\lambda^2}{(1+n\lambda)^2}\mu - 1\right]; & \mu \ge \frac{(1+n\lambda)^2}{1+2n\lambda+(n^2-1)\lambda^2} \end{cases}. \tag{2}$$

The equality is satisfied for the function

$$f(z) = (1-\lambda)\frac{z + (1-2\beta)z^2}{1-z} - \lambda\left[(1-2\beta)z + 2(1-\beta)\log(1-z)\right] \quad (\mu \le 0).$$

Proof. Let $f \in Q_\lambda(\beta)$. Then

$$(1-\lambda)\frac{f(z)}{z} + \lambda f'(z) = p(z)$$

or equivalently,

$$\begin{aligned} &1 + (1+\lambda)a_2 z + \cdots + [1+(n-1)\lambda]a_n z^{n-1} + [1+n\lambda]a_{n+1}z^n + [1+(n+1)\lambda]a_{n+2}z^{n+1} + \cdots \\ &= 1 + p_1 z + \cdots + p_{n-1}z^{n-1} + p_n z^n + p_{n+1}z^{n+1} + \cdots . \end{aligned} \tag{3}$$

It follows that

$$a_n = \frac{p_{n-1}}{1+(n-1)\lambda}, \quad a_{n+1} = \frac{p_n}{1+n\lambda} \quad \text{and} \quad a_{n+2} = \frac{p_{n+1}}{1+(n+1)\lambda}.$$

This gives us that

$$\left|a_n a_{n+2} - \mu a_{n+1}^2\right| = \frac{1}{1+2n\lambda+(n^2-1)\lambda^2}\left|p_{n-1}p_{n+1} - \frac{1+2n\lambda+(n^2-1)\lambda^2}{(1+n\lambda)^2}\mu p_n^2\right|.$$

Applying Lemma 1.3, we get

$$\left|a_n a_{n+2} - \mu a_{n+1}^2\right| \le \begin{cases} \frac{4(1-\beta)^2}{1+2n\lambda+(n^2-1)\lambda^2}\left[1 - \frac{1+2n\lambda+(n^2-1)\lambda^2}{(1+n\lambda)^2}\mu\right]; & \mu \le 0 \\ \frac{4(1-\beta)^2}{1+2n\lambda+(n^2-1)\lambda^2}; & 0 \le \frac{1+2n\lambda+(n^2-1)\lambda^2}{(1+n\lambda)^2}\mu \le 1 \\ \frac{4(1-\beta)^2}{1+2n\lambda+(n^2-1)\lambda^2}\left[\frac{1+2n\lambda+(n^2-1)\lambda^2}{(1+n\lambda)^2}\mu - 1\right]; & \frac{1+2n\lambda+(n^2-1)\lambda^2}{(1+n\lambda)^2}\mu \ge 1 \end{cases}$$

This gives the bound on $\left|a_n a_{n+2} - \mu a_{n+1}^2\right|$ as asserted in (2). □

Theorem 2.2. *Let f given by (1) be in the class $Q_\lambda(\beta)$ and $\mu \in \mathbb{R}$. Then*

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} \frac{2(1-\beta)}{1+2\lambda}\left[1 - \frac{2(1-\beta)(1+2\lambda)}{(1+\lambda)^2}\mu\right]; & \mu \le 0 \\ \frac{2(1-\beta)}{1+2\lambda}; & 0 \le \mu \le \frac{(1+\lambda)^2}{(1-\beta)(1+2\lambda)} \\ \frac{2(1-\beta)}{1+2\lambda}\left[\frac{2(1-\beta)(1+2\lambda)}{(1+\lambda)^2}\mu - 1\right]; & \mu \ge \frac{(1+\lambda)^2}{(1-\beta)(1+2\lambda)} \end{cases}.$$

The equality is satisfied for the function

$$f(z) = (1-\lambda)\,\frac{z+(1-2\beta)z^2}{1-z} - \lambda\left[(1-2\beta)z + 2(1-\beta)\log(1-z)\right] \quad (\mu \leq 0).$$

Proof. From (3)

$$\left|(1+2\lambda)a_3 - \mu(1+\lambda)^2 a_2^2\right| = \left|p_2 - \mu p_1^2\right|, \tag{4}$$

and from this equation (4), we obtain

$$(1+2\lambda)\left|a_3 - \mu\frac{(1+\lambda)^2}{1+2\lambda}a_2^2\right| = \left|p_2 - \mu p_1^2\right|.$$

Our result now follows by an application of Lemma 4:

$$\left|a_3 - \mu a_2^2\right| \leq \frac{1}{1+2\lambda}\begin{cases} 2(1-\beta)\left[1 - \frac{2(1-\beta)(1+2\lambda)}{(1+\lambda)^2}\mu\right]; & \mu \leq 0 \\ 2(1-\beta); & 0 \leq \frac{(1+2\lambda)}{(1+\lambda)^2}\mu \leq \frac{1}{1-\beta} \\ 2(1-\beta)\left[\frac{2(1-\beta)(1+2\lambda)}{(1+\lambda)^2}\mu - 1\right]; & \frac{(1+2\lambda)}{(1+\lambda)^2}\mu \geq \frac{1}{1-\beta} \end{cases}.$$

□

Remark 2.1. *Putting* $\lambda = 0$ *in Theorem 2.2 we have the generalized Hankel determinant for the well-known class* $Q_\lambda(\beta) = Q(\beta)$ *as in* [10].

Corollary 2.1. *Let* f *given by (1) be in the class* $Q(\beta)$ *and* $0 \leq \beta < 1$. *Then*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} 2(1-\beta)\left[1 - 2(1-\beta)\mu\right]; & \mu \leq 0 \\ 2(1-\beta); & 0 \leq \mu \leq \frac{1}{1-\beta} \\ 2(1-\beta)\left[2(1-\beta)\mu - 1\right]; & \mu \geq \frac{1}{1-\beta} \end{cases}.$$

Remark 2.2. *Putting* $\lambda = 1$ *in Theorem 2.2 we have the generalized Hankel determinant for the well-known class* $Q_\lambda(\beta) = R(\beta)$ *as in* [10].

Corollary 2.2. *Let* f *given by (1) be in the class* $R(\beta)$ *and* $0 \leq \beta < 1$. *Then*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{2(1-\beta)}{3}\left[1 - \frac{3}{2}(1-\beta)\mu\right]; & \mu \leq 0 \\ \frac{2}{3}(1-\beta); & 0 \leq \mu \leq \frac{4}{3(1-\beta)} \\ \frac{2(1-\beta)}{3}\left[\frac{3}{2}(1-\beta)\mu - 1\right]; & \mu \geq \frac{4}{3(1-\beta)} \end{cases}.$$

Theorem 2.3. *Let* f *given by (1) be in the class* $Q_\lambda(\beta)$ *and* $0 \leq \beta < 1$. *Then*

$$\left|a_2a_4 - \mu a_3^2\right| \leq \frac{3B^2-4B+9}{8(1+\lambda)(1+3\lambda)(1-B)}; \quad for \quad \frac{(1+2\lambda)^2}{2(1+\lambda)(1+3\lambda)} \leq \mu \leq \frac{(1+2\lambda)^2}{(1+\lambda)(1+3\lambda)},$$

where $B = \frac{(1+\lambda)(1+3\lambda)}{(1+2\lambda)^2}\mu$.

Proof. Using (3), one can see easily that

$$\left|a_2a_4 - \mu a_3^2\right| = \frac{1}{(1+\lambda)(1+3\lambda)}\left|p_1p_3 - \frac{(1+\lambda)(1+3\lambda)}{(1+2\lambda)^2}\mu p_2^2\right|.$$

Then, Lemma 2.1 gives us that

$$\begin{aligned}\left|p_1p_3-\tfrac{(1+\lambda)(1+3\lambda)}{(1+2\lambda)^2}\mu p_2^2\right| = & \tfrac{1}{4(1-\beta)^2}\left|p_1^4+2\left\{4(1-\beta)^2-p_1^2\right\}p_1^2x-\left\{4(1-\beta)^2-p_1^2\right\}p_1^2x^2\right.\\ & +2(1-\beta)\left\{4(1-\beta)^2-p_1^2\right\}p_1\left(1-|x|^2\right)z\\ & \left.-\tfrac{(1+\lambda)(1+3\lambda)}{(1+2\lambda)^2}\mu\left[p_1^4+\left[4(1-\beta)^2-p_1^2\right]^2x^2+2\left\{4(1-\beta)^2-p_1^2\right\}p_1^2x\right]\right|.\end{aligned}$$

Letting $|p_1|=p$, for $\eta=|x|\le 1$, $|z|\le 1$ we get

$$\begin{aligned}\left|p_1p_3-\tfrac{(1+\lambda)(1+3\lambda)}{(1+2\lambda)^2}\mu p_2^2\right| \le & \tfrac{1}{4(1-\beta)^2}\left\{\left[1-\tfrac{(1+\lambda)(1+3\lambda)}{(1+2\lambda)^2}\mu\right]p^4\right.\\ & +2\left[1-\tfrac{(1+\lambda)(1+3\lambda)}{(1+2\lambda)^2}\mu\right]\left[4(1-\beta)^2-p^2\right]p^2\eta\\ & +\left[4(1-\beta)^2-p^2\right]\left[p^2+\tfrac{(1+\lambda)(1+3\lambda)}{(1+2\lambda)^2}\mu\left[4(1-\beta)^2-p^2\right]-2(1-\beta)p\right]\eta^2\\ & \left.+2(1-\beta)\left[4(1-\beta)^2-p^2\right]p\right\}\\ & =\tfrac{F(\eta)}{4(1-\beta)^2}.\end{aligned}$$

For $F(\eta)$, we see that

$$\begin{aligned}F'(\eta) &= 2\left[4(1-\beta)^2-p^2\right]\\ &\times\left\{\left(1-\frac{(1+\lambda)(1+3\lambda)}{(1+2\lambda)^2}\mu\right)p^2+\left(p^2+\tfrac{(1+\lambda)(1+3\lambda)}{(1+2\lambda)^2}\mu\left[4(1-\beta)^2-p^2\right]-2(1-\beta)p\right)\eta\right\}.\end{aligned}$$

Therefore, if

$$\frac{(1+2\lambda)^2}{2(1+\lambda)(1+3\lambda)}\le\mu\le\frac{(1+2\lambda)^2}{(1+\lambda)(1+3\lambda)},$$

then $F'(\eta)$ satisfies

$$\begin{aligned}F'(\eta) &= 2\left[4(1-\beta)^2-p^2\right]\left[p^2+\tfrac{1}{2}\left(4(1-\beta)^2-p^2\right)-2(1-\beta)p\right]\eta\\ &= \left[4(1-\beta)^2-p^2\right][p-2(1-\beta)]^2\eta\\ &\ge 0,\end{aligned}$$

because $0\le\eta\le 1$ and $0\le p\le 2(1-\beta)$.

Writing that

$$\begin{aligned}F(1) &\equiv G(p)\\ &= -\frac{1}{A}\left\{2(1-B)p^4-A(3-4B)p^2+A^2B\right\},\end{aligned}$$

where $A=4(1-\beta)^2$ and $B=\frac{(1+\lambda)(1+3\lambda)}{(1+2\lambda)^2}\mu$, we obtain that

$$\begin{aligned} G'(p) &= -\frac{1}{A}\left\{8(1-B)p^3 - 2A(3-4B)p\right\} \\ &= -\frac{8(1-B)p}{A}\left\{p^2 - \frac{A(3-4B)}{4(1-B)}\right\} \\ &= 0 \end{aligned}$$

for $p = 0$ and $p = \frac{(1-\beta)\sqrt{3-4B}}{\sqrt{1-B}}$.

(i) If $0 \le \frac{3-4B}{1-B} \le 4$, then $\frac{(1+\lambda)(1+3\lambda)}{(1+2\lambda)^2}\mu \le \frac{3}{4}$.

In this case, $G(p)$ has the maximum value

$$\begin{aligned} G(p) &= \frac{-1}{4(1-\beta)^2}\left\{\frac{2(1-\beta)^4(3-4B)^2}{1-B} - \frac{4(1-\beta)^4(3-4B)^2}{1-B} + 16(1-\beta)^4 B\right\} \\ &= (1-\beta)^2\left\{4(1-3B) + \frac{1}{2(1-B)}\right\}. \end{aligned}$$

(ii) If $\frac{3-4B}{1-B} > 4$, then we have the contradiction for $G(p)$.

(iii) If $p = 0$, then $G(p)$ takes its minimal value.

Consequently, we say that

$$\left|p_1 p_3 - \frac{(1+\lambda)(1+3\lambda)}{(1+2\lambda)^2}\mu p_2^2\right| \le \frac{G(p)}{4(1-\beta)^2} = \frac{3B^2-4B+9}{8(1-B)}.$$

This completes the proof of the theorem. □

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