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ON (p,q)-ANALOG OF STANCU-BETA OPERATORS AND THEIR APPROXIMATION PROPERTIES

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Abstract. In this paper we introduce the (p,q) -analogue of the Stancu-Beta operators and call them as the (p,q)-Stancu-Beta operators. We study approximation properties of these operators based on the Korovkin's approximation theorem and also study some direct theorems. Also, we study the Voronovskaja type estimate for these operators.

Keywords: (p,q)-calculus; Stancu-Beta operators; modulus of continuity; positive linear operators; Korovkin type approximation theorem; Lipschitz class of functions.

AMS Subject Classification: 40A30, 41A10, 41A25, 41A36

1. Introduction

During the last two decades, there has been intensive research on the approximation of functions by positive linear operators introduced by making use of q-calculus. Lupas [10] was the first who used q-calculus to define q-Bernstein polynomials and later Phillips [20] considered another q-analogue of the classical Bernstein polynomials. Most recently, Mursaleen *et al* applied (p,q)-calculus in approximation theory and introduced first (p,q)-analogue of Bernstein operators [13]. They investigated uniform convergence of the operators and rate of convergence, obtained Voronovskaya type theorem as well. Also, (p,q)-analogue of Bernstein-Stancu operators [14], Bleimann- Butzer-Hahn operators [16], Bernstein-Schurer operarors [17] were introduced and their approximation properties were investigated. Most recently, the (p,q)-analogue of some more operators have been defined and studied their approximation properties in [1], [2], [3], [5], [8], [11], [12], [15], [18], [19] and [23]. So motivated by this, we introduce the (p,q)-analogue of the Stancu-Beta operators and study their approximation properties. We also study the Voronovskaja type estimate for these operators. We recall certain notations and definitions of (p,q)-calculus. The (p,q)-integer is a generalization of q-integer which is defined by

$$[n]_{p,q}=\frac{p^n-q^n}{p-q},\ n=0,1,2,...,\ 0<q<p\leq 1;\ n\in\mathbb{N}.$$

The (p,q)-Binomial expansion and (p,q)-binomial coefficients are defined by

$$(ax+by)^n_{p,q}=\sum_{k=0}^{n}\binom{n}{k}_{p,q}q^{\frac{k(k-1)}{2}}p^{\frac{(n-k)(n-k-1)}{2}}a^{n-k}b^kx^{n-k}y^k$$

$$(x+y)^n_{p,q}=(x+y)(px+qy)(p^2x+q^2y)...(p^{n-1}x+q^{n-1}y).$$

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$$\binom{n}{k}_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}.$$

The definite integrals of a function f are defined by

$$\int_0^a f(x)d_{p,q}x = (q-p)a\sum_{k=0}^{\infty}\frac{p^k}{q^{k+1}}f\left(\frac{p^k}{q^{k+1}}a\right),\ when\left|\frac{p}{q}\right|<1$$

and

$$\int_0^a f(x)d_{p,q}x = (p-q)a\sum_{k=0}^{\infty}\frac{q^k}{p^{k+1}}f\left(\frac{q^k}{p^{k+1}}a\right),\ when\left|\frac{q}{p}\right|<1.$$

There are two (p,q)-analogues of the classical exponential function defined as follows

$$e_{p,q}(x) = \sum_{k=0}^{\infty}\frac{p^{\frac{n(n-1)}{2}}}{[n]_{p,q}!}x^n$$

and

$$E_{p,q}(x) = \sum_{k=0}^{\infty}\frac{q^{\frac{n(n-1)}{2}}}{[n]_{p,q}!}x^n.$$

It is easily seen that

$$e_{p,q}(x)E_{p,q}(-x) = 1.$$

For $m,n\in N$, the (p,q)-Beta and the (p,q)-Gamma functions are defined by

$$B_{p,q}(m,n) = \int_0^{\infty}\frac{x^{m-1}}{(1+x)^{m+n}}d_{p,q}x$$

and

$$\Gamma_{p,q}(n) = \int_0^{\infty}p^{\frac{n(n-1)}{2}}E_{p,q}(-qx)d_{p,q}x,\ \Gamma_{p,q}(n+1) = [n]_{p,q}!$$

respectively. The two functions are connected through

$$B_{p,q}(m,n) = q^{\frac{2-m(m-1)}{2}}p^{\frac{-m(m-1)}{2}}\frac{\Gamma_{p,q}(n)\Gamma_{p,q}(m)}{\Gamma_{p,q}(m+n)}. \tag{1}$$

If $p=1$, then the above notions of (p,q)-calculus (see [9], [21], [22].) reduce to the corresponding notations of q-calculus.

2. Discussion

Stancu introduced in [24] the Beta operator as follows

$$L_n(f,x) = \frac{1}{B(nx,n+1)}\int_0^{\infty}\frac{t^{nx}}{(1+t)^{nx+n+1}}f(t)dt.$$

Aral and Gupta [4] gave the q-analogue of the Stancu-Beta operators as follows

$$L_n(f,x) = \frac{K(A,[n]_qx)}{B([n]_qx,[n]_q+1)}\int_0^{\infty/A}\frac{u^{[n]_qx-1}}{(1+u)^{[n]_qx+[n]_q+1}}f(q^{[n]_qx}u)d_qu.$$

Here, we introduce the (p,q)-analogue of the Stancu-Beta operators and study their approximation properties. We also study the Voronovskaja type estimate for these operators.

Let $0<q<p\le 1$ and $x\in[0,\infty)$. We introduce the (p,q)-Stancu-Beta operators as follows

$$L_n^{p,q}(f,x) = \frac{1}{B_{p,q}([n]_{p,q}x,[n]_{p,q}+1)}\int_0^{\infty}\frac{u^{[n]_{p,q}x-1}}{(1+u)^{[n]_{p,q}x+[n]_{p,q}+1}}f(p^{[n]_{p,q}x}q^{[n]_{p,q}x}u)d_{p,q}u. \tag{2}$$

We have the following auxilary result.

Lemma 2.1. *Let $L_n^{p,q}(f,x)$ be given by (2). Then for the polynomials t^m, $m = 0, 1, 2, ...$ we have*

$$L_n^{p,q}(t^m,x) = \frac{p^{\frac{-m(m-1)}{2}} q^{\frac{-m(m-1)}{2}}([n]_{p,q}x+m-1)!([n]_{p,q}-m)!}{([n]_{p,q}x-1)!([n]_{p,q})!} \tag{3}$$

Proof. By (1), we have

$$\begin{aligned} L_n^{p,q}(t^m,x) &= \frac{B_{p,q}([n]_{p,q}x+m,[n]_{p,q}-m+1)}{B_{p,q}([n]_{p,q}x,[n]_{p,q}+1)} \\ &= \frac{p^{\frac{-m(m-1)}{2}} q^{\frac{-m(m-1)}{2}}([n]_{p,q}x+m-1)!([n]_{p,q}-m)!}{([n]_{p,q}x-1)!([n]_{p,q})!} \end{aligned}$$

which establishes (3). □

To examine the approximation results for the operators in (2), we need the following lemmas.

Lemma 2.2. *Let $L_n^{p,q}(f,x)$ be given by (2). Then the followings hold*

(i) $L_n^{p,q}(1,x) = 1$,

(ii) $L_n^{p,q}(t,x) = x$,

(iii) $L_n^{p,q}(t^2,x) = \frac{[n]_{p,q}}{pq([n]_{p,q}-1)}x^2 + \frac{1}{pq([n]_{p,q}-1)}x$.

Proof. Putting $m = 0$ in (3), we have

$$L_n^{p,q}(1,x) = \frac{([n]_{p,q}x-1)!([n]_{p,q})!}{([n]_{p,q}x-1)!([n]_{p,q})!} = 1,$$

which proves (i). Putting $m = 1$ in (3),

$$\begin{aligned} L_n^{p,q}(t,x) &= \frac{([n]_{p,q}x)!([n]_{p,q}-1)!}{([n]_{p,q}x-1)!([n]_{p,q})!} \\ &= \frac{[n]_{p,q}x([n]_{p,q}x-1)!([n]_{p,q}-1)!}{([n]_{p,q}x-1)![n]_{p,q}([n]_{p,q}-1)!} \\ &= x. \end{aligned}$$

Finally, putting $m = 2$ in (3),

$$\begin{aligned} L_n^{p,q}(t,x) &= \frac{p^{-1}q^{-1}([n]_{p,q}x+1)!([n]_{p,q}-2)!}{([n]_{p,q}x-1)!([n]_{p,q})!} \\ &= \frac{1}{pq}\frac{([n]_{p,q}x+1)[n]_{p,q}x([n]_{p,q}x-1)}{([n]_{p,q}x-1)[n]_{p,q}([n]_{p,q}-1)} \\ &= \frac{1}{pq}\frac{([n]_{p,q}x+1)}{([n]_{p,q}-1)} \\ &= \frac{[n]_{p,q}}{pq([n]_{p,q}-1)}x^2 + \frac{1}{pq([n]_{p,q}-1)}x, \end{aligned}$$

and this proves (iii).

This completes the proof of the lemma. □

Lemma 2.3. *Let $q \in (0,1)$ and $p \in (q,1]$. Then for $x \in [0,\infty)$, we have*

(a) $L_n^{p,q}((t-x),x) = 0$,

(b) $L_n^{p,q}(t-x)^2,x) = \frac{([n]_{p,q}-pq[n]_{p,q}+pq)}{pq([n]_{p,q}-1)}x^2 + \frac{1}{pq([n]_{p,q}-1)}x$.

Proof. Using the Lemma (3), (i)-(ii), (a) is obvious.
Also,

$$\begin{aligned} L_n^{p,q}(t-x)^2, x) &= L_n^{p,q}t^2, x) - 2xL_n^{p,q}t, x) + x^2 L_n^{p,q}(1, x) \\ &= \frac{[n]_{p,q}x + 1}{pq([n]_{p,q} - 1)}x - x^2 \\ &= \frac{([n]_{p,q} - pq[n]_{p,q} + pq)}{pq([n]_{p,q} - 1)}x^2 + \frac{1}{pq([n]_{p,q} - 1)}x, \end{aligned}$$

which proves (b). □

3. Main results

This section is devoted to prove some direct theorems for the operators $L_n^{p,q}(f, x)$.
By $C_B[0, \infty)$, we denote the space of all real valued continuous bounded functions f on the interval $[0, \infty)$ equipped with the norm

$$\|f\| = \sup_{0 \le x < \infty} |f(x)|.$$

If $f \in C(I)$, $\delta > 0$ and $W^2 = \{h : h', h'' \in C(I)\}$, where $I = [0, \infty)$, then the Peetre's K-functional is defined by

$$K_2(f, \delta) = \inf_{h \in W^2} \{\|f - h\| + \delta\|h''\|\}.$$

Then there exists a constant $C > 0$ such that (see [6])

$$K_2(f, \delta) \le C\omega_2(f, \sqrt{\delta}), \tag{4}$$

where $\omega_2(f, \sqrt{\delta})$ is the second order modulus of continuity of $f \in C(I)$ defined by

$$\omega_2(f, \sqrt{\delta}) = \sup_{0 < p < \delta^{\frac{1}{2}}} \sup_{x \in I} |f(x + 2p) - 2f(x + p) + f(x)|.$$

The first order modulus of continuity of $f \in C(I)$ is given by

$$\omega(f, \delta) = \sup_{0 < p < \delta} \sup_{x \in I} |f(x + p) - f(x)|.$$

We prove the following theorem.

Theorem 3.1. *Let $f \in C_B[0, \infty)$, $x \in [0, \infty)$ and $n \in N$. Then there exists a constant C such that*

$$|L_n^{p,q}(f; x) - f(x)| \le C\omega_2(f, \delta_n(x)),$$

where

$$\delta_n^2(x) = \frac{([n]_{p,q} - pq[n]_{p,q} + pq)}{pq([n]_{p,q} - 1)}x^2 + \frac{1}{pq([n]_{p,q} - 1)}x.$$

and $0 < p, q < 1$.

Proof. Let $g \in W^2$. By the Taylor's expansion we can write

$$g(t) = g(x) + g'(x)(t - x) + \int_x^t (t - u)g''(u)du, \ t \in [0, \infty).$$

Operating $L_n^{p,q}(., x)$ on both sides,

$$L_n^{p,q}(g; x) = g(x) + L_n^{p,q}\left(\int_x^t (t - u)g''(u)du; x\right).$$

So,

$$
\begin{aligned}
|L_n^{p,q}(g;x)-g(x)| &\leq \left| L_n^{p,q}\left(\int_x^t (t-u)g''(u)du;x\right)\right| \\
&\leq L_n^{p,q}\left(\left|\int_x^t |t-u||g''(u)|du\right|;x\right) \\
&\leq L_n^{p,q}((t-x)^2;x)\|g''\|.
\end{aligned}
$$

By Lemma 2.3, we get

$$
|L_n^{p,q}(g;x)-g(x)| \leq \left(\frac{([n]_{p,q}-pq[n]_{p,q}+pq)}{pq([n]_{p,q}-1)}x^2+\frac{1}{pq([n]_{p,q}-1)}x\right)\|g''\|.
$$

By the definition of $L_n^{p,q}(f,x)$,

$$
|L_n^{p,q}(f;x)| \leq \|f\|.
$$

Then,

$$
\begin{aligned}
|L_n^{p,q}(f;x)-f(x)| &\leq |L_n^{p,q}(f-g;x)-(f-g)(x)|+|L_n^{p,q}(g;x)-g(x)| \\
&\leq \|f-g\|+\left(\frac{([n]_{p,q}-pq[n]_{p,q}+pq)}{pq([n]_{p,q}-1)}x^2+\frac{1}{pq([n]_{p,q}-1)}x\right)\|g''\|.
\end{aligned}
$$

Taking infimum over $g \in W^2$,

$$
|L_n^{p,q}(f;x)-f(x)| \leq K_2(f,\delta_n^2(x)).
$$

Therefore

$$
|L_n^{p,q}(f;x)-f(x)| \leq C\omega_2(f,\delta_n(x)).
$$

for every $q \in (0,1)$ and hence the proof is completed. $\square$

Let $B_{x^2}[0,\infty)$ denote the set of all functions f which are bounded by $M_f(1+x^2)$, where M_f is a constant depending on f. By $C_{x^2}[0,\infty)$, we denote the subspace of all continuous functions in the space $B_{x^2}[0,\infty)$. Also we denote by $C^*_{x^2}[0,\infty)$, the subspace of all functions $f \in C_{x^2}[0,\infty)$ for which $\lim_{x\to\infty}\frac{f(x)}{1+x^2}$ is finite.

We prove the following result.

Theorem 3.2. *Let $f \in C^*_{x^2}[0,\infty)$ be such that $f', f'' \in C^*_{x^2}[0,\infty)$ and $p=(p_n), q=(q_n)$ with $p_n,\ q_n \in (0,1)$ such that $p_n \to 1, q_n \to 1$ as $n \to \infty$. Then*

$$
\lim_{n\to\infty}[n]_{p_n,q_n}(L_n^{p_n,q_n}(f;x)-f(x)) = \frac{x(1+x)}{2}f''(x)
$$

uniformly on $[0,A], A>0$.

Proof. Using the Taylor's formula, we can get

$$
f(t)=f(x)+f'(x)(t-x)+\frac{1}{2}f''^2+r(t,x)(t-x)^2, \tag{5}
$$

where $r(t,x)$ is the remainder such that

$$
\lim_{t\to x} r(t,x)=0.
$$

Operating by $L_n^{p_n,q_n}(.;x)$ on both sides of (5),

$$
\begin{aligned}
[n]_{p_n,q_n}(L_n^{p_n,q_n}(f;x)-f(x)) &= [n]_{p_n,q_n}L_n^{p_n,q_n}(t-x;x)f'(x) \\
&+ [n]_{p_n,q_n}L_n^{p_n,q_n}((t-x)^2;x)\frac{f''(x)}{2} \\
&+ [n]_{p_n,q_n}L_n^{p_n,q_n}(r(t,x)(t-x)^2;x).
\end{aligned}
$$

By the Cauchy-Schwarz inequality, we have

$$L_n^{p_n,q_n}(r(t,x)(t-x)^2;x) \leq \sqrt{L_n^{p_n,q_n}(r^2(t,x);x)}\sqrt{L_n^{p_n,q_n}((t-x)^4;x)}. \tag{6}$$

Noting that $r^2(t,x)=0$ and $r^2(.,x) \in C^*_{x^2}[0,\infty)$, it follows from the Theorem 3.1 that

$$\lim_{n\to\infty} L_n^{p_n,q_n}(r^2(t,x);x) = r^2(x,x) = 0 \tag{7}$$

uniformly with respect to $x \in [0,A]$. By (3), (6) and (7) we immediately get

$$\lim_{n\to\infty}[n]_{p_n,q_n}L_n^{p_n,q_n}(r(t,x)(t-x)^2;x) = 0.$$

Using the Lemma 2.3, we get the following

$$\begin{aligned}\lim_{n\to\infty}[n]_{p_n,q_n}(L_n^{p_n,q_n}(f;x)-f(x)) &= \lim_{n\to\infty}[n]_{p_n,q_n}\Bigg(f'(x)L_n^{p_n,q_n}((t-x);x)\\ +\frac{1}{2}f''(x)L_n^{p_n,q_n}((t-x)^2;x) &+ L_n^{p_n,q_n}(r(t,x)(t-x)^2;x)\Bigg)\\ &= \frac{x(1+x)}{2}f''(x).\end{aligned}$$

This completes the proof of the theorem. □

Next we present the weighted approximation theorem for operators (2).

Theorem 3.3. *Let $p=p_n,\ q=q_n$ be two sequences such that $0<p_n,q_n<1$ and $p_n\to 1,\ q_n\to 1$ $(n\to\infty)$. Then*

$$\lim_{n\to\infty}\|L_n^{p_n,q_n}(f)-f\|_{x^2}=0,$$

for $f\in C^*_{x^2}[0,\infty)$.

Proof. We show that

$$\lim_{n\to\infty}\|L_n^{p_n,q_n}(t^i)-x^i\|_{x^2}=0,\ \ for\ i=0,1 \text{ and } 2.$$

By using (i)–(ii) of Lemma 2.2, the conditions are easily fulfilled for $i=0$ and 1. For $i=2$, we can write

$$\begin{aligned}\|L_n^{p_n,q_n}(t^2)-x^2\|_{x^2} &\leq \sup_{x\in[0,\infty)}\frac{|[n]_{p_n,q_n}-p_nq_n[n]_{p_n,q_n}+p_nq_n|}{p_nq_n([n]_{p_n,q_n}-1)}\frac{x^2}{1+x^2}\\ &+ \sup_{x\in[0,\infty)}\frac{1}{p_nq_n([n]_{p_n,q_n}-1)}\frac{x}{1+x^2}.\end{aligned}$$

By the Korovkin's theorem [7], we get

$$\lim_{n\to\infty}\|L_n^{p_n,q_n}(t^2,x)-x^2\|_{x^2}=0.$$

This completes the proof of the theorem. □

4. Conclusion

Recently, (p,q)-calculus has been used in constructing (p,q)-analogues of several classical operators and investigated their approximation properties. In this paper, we have introduced the (p,q)-analogue of the Stancu-Beta operators and established some results on their approximation properties by using Korovkin's approximation theorem as well as direct theorems. We have also studied the Voronovskaja type estimate for our operators.

Competing interests
The authors declare that they have no competing interests.

Authors contributions
Both authors of the manuscript have read and agreed to its content and are accountable for all aspects of the accuracy and integrity of the manuscript.

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