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NEW CLASSES OF HARMONIC FUNCTIONS DEFINED BY FRACTIONAL OPERATOR

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Abstract. In the present study, we introduce an investigation of new subclasses of harmonic functions which are defined by fractional operator. Firstly, using by fractional operator, we define new subclasses of harmonic functions. Later, we obtain main theorems of our study which contain sufficient and necessary coefficient bounds for functions related to the classes newly defined. Also, several particular characterization properties of these classes are given. Some of these properties involve extreme points, convex combination, distortion bounds. Finally, several corollaries of the main theorems are presented.

Keywords: Harmonic functions, fractional operator, coefficient estimates, univalent.

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1. Introduction

Let u, v be real harmonic functions in the simply connected domain Ω, then the continuous function $f = u + iv$ defined in Ω is said to be harmonic in Ω. In any simply connected domain $\Omega \subset \mathbb{C}$ we can write $f = h + \overline{g}$, where h and g are analytic in Ω. A necessary and sufficient condition for f to be locally univalent and sense-preserving in Ω is that $|h'(z)| > |g'(z)|$ in Ω (see [5]).Let H be the family of functions $f = h + \overline{g}$ which are harmonic univalent and sense-preserving in the open unit disk $U = \{z : |z| < 1\}$ that $f = h + \overline{g}$ normalized by $f(0) = h(0) = f'(0) - 1 = 0$. Let HW be the set harmonic univalent and sense-preserving functions in U [1] of the form $f(z) = h(z) + \overline{g(z)}$, where

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad g(z) = \sum_{n=1}^{\infty} b_n z^n \qquad |b_1| < 1 \tag{1}$$

are analytic in U.

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Let $\overline{HW}$ be the subclass of HW consisting of functions of the form $f(z) = h(z) + \overline{g(z)}$, where

$$h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \qquad g(z) = \sum_{n=1}^{\infty} |b_n| z^n \qquad |b_1| < 1. \tag{2}$$

In 1984 Clunie and Sheil-Small [5] investigated the class SH as well as its geometric subclasses and obtained some coefficient bounds. Since then, there has been several related papers on SH and its subclasses studied by Avcı and Zlotkiewicz [3], Silverman [11], Silverman and Silvia [13], Jahangiri [6] and Jahangiri et al. [7]. Furthermore, some other authors (e.g see [2] and [14]) have recently studied the harmonic univalent functions using by certain operators. In current study, a new subclasses of harmonic functions has been investigated by using fractional operator especially.

A fairly complate treatment, with applications of the fractional calculus, is given in the books [10] by Oldham and Spanier, and [9] by Miller and Ross. We refer to [12] for more insight into the concept of the fractional calculus. For further details on the materials in this paper see [8].

For convenience, we shall remind some definitions of the fractional calculus (i.e., fractional integral and fractional derivative).

The fractional integral of order λ for analytical function$f(z)$ in D is defined by

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}} d\zeta, (\lambda > 0) \tag{3}$$

where the multiplicity of $(z-\zeta)^{(\lambda-1)}$ is removed by requiring $log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

The fractional derivative of order λ for an analytic function $f(z)$ in D is defined by

$$D_z^{\lambda} f(z) = \frac{d}{dz}(D_z^{-\lambda} f(z)) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}} d\zeta, (0 \leq \lambda < 1), \tag{4}$$

where the multiplicity of $(z-\zeta)^{-\lambda}$ is removed by requiring $log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

Under the hypothesis of the fractional derivative, the fractional derivative of order $(n+\lambda)$ for an analytic function $f(z)$ in D is defined by

$$D_z^{n+\lambda} f(z) = \frac{d^n}{dz^n}(D_z^{\lambda} f(z)), \qquad (0 \leq \lambda < 1, n \in N_0 = \{0, 1, 2, ...\}). \tag{5}$$

From the definitions of the fractional calculus, we see that

$$D_z^{-\lambda} z^k = \frac{\Gamma(k+1)}{\Gamma(k+1+\lambda)} z^{k+\lambda}, \qquad (\lambda > 0, k > 0) \tag{6}$$

$$D_z^{\lambda} z^k = \frac{\Gamma(k+1)}{\Gamma(k+1+\lambda)} z^{k-\lambda}, \qquad (0 \leq \lambda < 1, k > 0) \tag{7}$$

and

$$D_z^{n+\lambda} z^k = \frac{\Gamma(k+1)}{\Gamma(k+1-\lambda-n)} z^{k-n-\lambda}, \qquad (0 \le \lambda < 1, k > 0, n \in N_0, k-n \neq -1, -2, ...). \tag{8}$$

Therefore, we see that for any real λ,

$$D_z^{\lambda} z^k = \frac{\Gamma(k+1)}{\Gamma(k+1-\lambda)} z^{k-\lambda}, \qquad (k > 0, k-\lambda \neq -1, -2, ...). \tag{9}$$

Using the rule of the fractional derivative, which is mentioned in the preceding, Aydoğan et al. [4] defined the $\lambda - fractional\ operator$ as follows,

$$f(z) = z + a_2 z^2 + ... + a_n z^n + ... \Rightarrow D_z^{\lambda} f(z) = D_z^{\lambda}(z + a_2 z^2 + ... + a_n z^n + ...)$$

$$D_z^{\lambda} f(z) = \Gamma(2-\lambda) z^{\lambda} D_z^{\lambda} f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)} a_n z^n. \tag{10}$$

From the definition of $D^{\lambda} f(z)$ we have the following properties:

$i. D' f(z) = Df(z) = lim_{\lambda \to 1} D^{\lambda} f(z) = z f'(z);$

$ii. D^{\lambda}(D^{\delta} f(z)) = D^{\delta}(D^{\lambda} f(z)) = z + \sum_{n=2}^{\infty} \frac{\Gamma(2-\lambda)\Gamma(2-\delta)(\Gamma(n+1))^2}{\Gamma(n+1-\lambda)\Gamma(n+1-\delta)} a_n z^n;$

$iii. D(D^{\delta} f(z)) = z + \sum_{n=2}^{\infty} n \frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)} a_n z^n = z(D^{\delta} f(z))' = \Gamma(2-\lambda) z^{\lambda}(\lambda D_z^{\lambda} + z D_z^{\lambda+1} f(z));$

$$iv. \frac{D\left(D^{\lambda} f(z)\right)}{D^{\lambda} f(z)} = z\frac{f'(z)}{f(z)}, for\ \lambda = 0$$

$$= 1 + z\frac{f''(z)}{f'(z)}, \quad for\ \ \lambda = 1.$$

Thus, we define the following class of functions.

Definition 1.1. *Let* $f(z) = h(z) + \overline{g(z)}$*, be the harmonic univalent function given by Eq.(1), then* $f \in HW(\lambda, \beta, t, \alpha)$ *if and only if*

$$\Re\left\{(1-\beta)(1-t)\frac{D^{\lambda} f(z)}{z} + (\beta + t)\frac{(D^{\lambda} f(z))'}{z'} + \beta t\frac{(D^{\lambda} f(z))''}{z''} - 2\beta t\right\} \ge \alpha \tag{11}$$

where $0 \le \alpha < 1$*,* $\beta \ge 0$*,* $0 \le t \le 1$ *and* $z = re^{i\theta} \in U$*.*

We also let $\overline{HW}(\lambda, \beta, t, \alpha) = HW(\lambda, \beta, t, \alpha) \cap \overline{HW}$*.*

When we take $\beta = 0$, the class of $HW(\lambda, \beta, t, \alpha)$ reduces to the class of harmonic functions $HG(\lambda, t, \alpha) \equiv HW(\lambda, 0, t, \alpha)$ as follows

$$\Re\left\{(1-t)\frac{D^{\lambda} f(z)}{z} + t\frac{(D^{\lambda} f(z))'}{z'}\right\} \ge \alpha. \tag{12}$$

When we take $\beta = 1$, the class of $HW(\lambda, \beta, t, \alpha)$ reduces to the $HA(\lambda, t, \alpha) \equiv HW(\lambda, 1, t, \alpha)$ class of harmonic functions $HA(\lambda, t, \alpha)$ as follows:

$$\Re\left\{(1+t)\frac{(D^{\lambda}f(z))'}{z'}+t\frac{(D^{\lambda}f(z))''}{z''}-2t\right\}\geq\alpha. \tag{13}$$

2. Coefficient Bounds

Now, firstly we presented sufficient coefficient conditions for functions to be in the class of $HW(\lambda,\beta,t,\alpha)$. Furthermore, it is shown that these conditions are also necessary for the functions of $\overline{HW}(\lambda,\beta,t,\alpha)$.

Theorem 2.1. *Let $f=h+\overline{g}$, h and g be given by (1) and*

$$\sum_{n=2}^{\infty}\left|(\beta+t)(n-1)+\beta t(1+n^2)+1\right|\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)}|a_n|$$
$$+\sum_{n=1}^{\infty}\left|(\beta+t)(n+1)-\beta t(1+n^2)-1\right|\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)}|b_n|\leq 1-\alpha. \tag{14}$$

Then $f(z)\in HW(\lambda,\beta,t,\alpha)$.

Proof. Using the fact that $Rew\geq\alpha$ if and only if $|w+1-\alpha|\geq|w-1-\alpha|$

$$w=(1-\beta)(1-t)\frac{D^{\lambda}f(z)}{z}+(\beta+t)\frac{(D^{\lambda}f(z))'}{z'}+\beta t\frac{(D^{\lambda}f(z))''}{z''}-2\beta t$$

It is enough to show that $|w+1-\alpha|-|w-1-\alpha|\geq 0$.

Now, we have

$$|w+1-\alpha|=\Bigg|(1-\beta)(1-t)\left(1+\sum_{n=2}^{\infty}\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)}a_nz^{n-1}+\sum_{n=1}^{\infty}\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)}b_n(\overline{z})^{n-1}\right)$$
$$+(\beta+t)\left(1+\sum_{n=2}^{\infty}\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)}na_nz^{n-1}-\sum_{n=1}^{\infty}\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)}nb_n(\overline{z})^{n-1}\right)$$
$$+\beta t\Bigg(1+\sum_{n=2}^{\infty}\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)}na_nz^{n-1}+\sum_{n=2}^{\infty}\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)}n(n-1)a_nz^{n-1}$$
$$+\sum_{n=1}^{\infty}\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)}nb_n(\overline{z})^{n-1}+\sum_{n=1}^{\infty}\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)}n(n-1)b_n(\overline{z})^{n-1}\Bigg)-2\beta t+1-\alpha\Bigg|$$
$$\geq 2-\alpha-\sum_{n=2}^{\infty}|1+(\beta+t)(n-1)+\beta t(1+n^2)|\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)}|a_n||z^{n-1}|$$
$$-\sum_{n=1}^{\infty}|1-(\beta+t)(n+1)+\beta t(1+n^2)|\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)}|b_n||z^{n-1}|$$

and

$$|w-1-\alpha|=\Bigg|(1-\beta)(1-t)\left(1+\sum_{n=2}^{\infty}\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)}a_nz^{n-1}+\sum_{n=1}^{\infty}\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)}b_n(\overline{z})^{n-1}\right)$$

$$+(\beta+t)\left(1+\sum_{n=2}^{\infty}\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)}na_nz^{n-1}-\sum_{n=1}^{\infty}\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)}nb_n(\overline{z})^{n-1}\right)$$

$$+\beta t\left(1+\sum_{n=2}^{\infty}\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)}na_nz^{n-1}+\sum_{n=2}^{\infty}\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)}n(n-1)a_nz^{n-1}\right.$$

$$\left.+\sum_{n=1}^{\infty}\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)}nb_n(\overline{z})^{n-1}+\sum_{n=1}^{\infty}\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)}n(n-1)b_n(\overline{z})^{n-1}\right)-2\beta t-1-\alpha\Bigg|$$

$$\leq \alpha+\sum_{n=2}^{\infty}|1+(\beta+t)(n-1)+\beta t(1+n^2)|\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)}|a_n||z^{n-1}|$$

$$+\sum_{n=1}^{\infty}|1-(\beta+t)(n+1)+\beta t(1+n^2)|\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)}|b_n||z^{n-1}|.$$

So by using (14) we have

$$|w+1-\alpha|-|w-1-\alpha|\geq 2\left[1-\alpha-\sum_{n=2}^{\infty}\left|(\beta+t)(n-1)+\beta t(1+n^2)+1\right|\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)}|a_n|\right.$$

$$\left.-\sum_{n=1}^{\infty}\left|(\beta+t)(n+1)-\beta t(1+n^2)-1\right|\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)}|b_n|\right]\geq 0$$

and so the proof is completed. □

Remark 1 The coefficient bound (14) in previous theorem is sharp for the function

$$f(z)=z+\sum_{n=2}^{\infty}\frac{u_n}{|(\beta+t)(n-1)+\beta t(1+n^2)+1|}\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)}z^n$$

$$\sum_{n=1}^{\infty}\frac{\overline{w}_n}{|(\beta+t)(n+1)-\beta t(1+n^2)-1|}\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)}(\overline{z})^n, \tag{15}$$

where

$$\frac{1}{1-\alpha}\left(\sum_{n=2}^{\infty}|u_n|+\sum_{n=1}^{\infty}|w_n|\right)=1.$$

Theorem 2.2. *Let $f=h+\overline{g}\in\overline{HW}$. $f(z)\in\overline{HW}(\lambda,\beta,t,\alpha)$ if and only if*

$$\sum_{n=2}^{\infty}\left|(\beta+t)(n-1)+\beta t(1+n^2)+1\right|\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)}|a_n|$$

$$+\sum_{n=1}^{\infty}\left|(\beta+t)(n+1)-\beta t(1+n^2)-1\right|\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)}|b_n|\leq 1-\alpha. \tag{16}$$

Proof. From Theorem 2.1 and since $\overline{HW}(\lambda,\beta,t,\alpha) \subset HW(\lambda,\beta,t,\alpha)$ we conclude the "if" part. For the "only if" part, assume that $f(z) \in \overline{HW}(\lambda,\beta,t,\alpha)$. Therefore, for $z = re^{i\theta} \in U$, we have

$$\Re\left\{(1-\beta)(1-t)\frac{D^\lambda f(z)}{z} + (\beta+t)\frac{(D^\lambda f(z))'}{z'} + \beta t\frac{(D^\lambda f(z))''}{z''} - 2\beta t\right\}$$

$$= \Re\left\{(1-\beta)(1-t)\left(1+\sum_{n=2}^{\infty}\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)}a_n z^{n-1} + \sum_{n=1}^{\infty}\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)}b_n(\overline{z})^{n-1}\right)\right.$$

$$+(\beta+t)\left(1-\sum_{n=2}^{\infty}\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)}na_n z^{n-1} - \sum_{n=1}^{\infty}\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)}nb_n(\overline{z})^{n-1}\right)$$

$$+\beta t\left(1-\sum_{n=2}^{\infty}\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)}na_n z^{n-1} - \sum_{n=2}^{\infty}\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)}n(n-1)a_n z^{n-1}\right.$$

$$\left.\left.+\sum_{n=1}^{\infty}\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)}nb_n(\overline{z})^{n-1} - \sum_{n=1}^{\infty}\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)}n(n-1)b_n(\overline{z})^{n-1}\right) - 2\beta t\right\}$$

$$= \Re\left\{1-\sum_{n=2}^{\infty}[(1-\beta)(1-t)+(\beta+t)n+n\beta t+n(n-1)\beta t]\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)}a_n z^{n-1}\right.$$

$$\left.+\sum_{n=1}^{\infty}[(1-\beta)(1-t)-(\beta+t)n+n\beta t+n(n-1)\beta t]\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)}b_n(\overline{z})^{n-1}\right\}$$

$$\geq 1-\sum_{n=2}^{\infty}|(\beta+t)(n-1)+\beta t(1+n^2)+1|\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)}|a_n|r^{n-1}$$

$$-\sum_{n=1}^{\infty}|(\beta+t)(n+1)-\beta t(1+n^2)-1|\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)}|b_n|r^{n-1} \geq \alpha.$$

The last inequality holds for all $z \in U$. So if $z = r \to 1$ we obtain the required result given by (16). So the proof of the Theorem 2.2 is completed.

As special cases of Theorem 2.2, we can obtain the following two corollaries:

□

Corollary 2.1. $f = h + \overline{g} \in \overline{HG}(\lambda,t,\alpha) \equiv HG(\lambda,t,\alpha) \cap \overline{HW}$ *if and only if*

$$\sum_{n=2}^{\infty}|t(n-1)+1|\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)}|a_n| + \sum_{n=1}^{\infty}|t(n+1)-1|\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)}|b_n| \leq 1-\alpha.$$

Especially for $\lambda = 0$

$$\sum_{n=2}^{\infty}|t(n-1)+1||a_n| + \sum_{n=1}^{\infty}|t(n+1)-1||b_n| \leq 1-\alpha.$$

Corollary 2.2. *$f = h + \overline{g} \in \overline{HA}(\lambda, t, \alpha) \equiv HA(\lambda, t, \alpha) \cap \overline{HW}$ if and only if*

$$\sum_{n=2}^{\infty} |n(1+t+nt)| \frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)} |a_n| + \sum_{n=1}^{\infty} |n(1+t-nt)| \frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)} |b_n| \leq 1-\alpha.$$

Especially for $\lambda = 0$

$$\sum_{n=2}^{\infty} |n(1+t+nt)||a_n| + \sum_{n=1}^{\infty} |n(1+t-nt)||b_n| \leq 1-\alpha.$$

3. Extreme Points

In the following theorem, we represent extreme points of $\overline{HW}(\lambda, \beta, t, \alpha)$.

Theorem 3.1. *$f = h + \overline{g} \in \overline{HW}(\lambda, \beta, t, \alpha)$ if and only if it can be expressed as*

$$f(z) = X_1 z + \sum_{n=2}^{\infty} X_n h_n(z) + \sum_{n=1}^{\infty} Y_n g_n(z), z \in U \tag{17}$$

where

$$h_n(z) = z - \frac{1-\alpha}{|(\beta+t)(n-1) + \beta t(1+n^2) + 1|} \frac{\Gamma(n+1-\lambda)}{\Gamma(2-\lambda)\Gamma(n+1)} z^n \qquad (n = 2, 3, ...) \tag{18}$$

$$g_n(z) = z + \frac{1-\alpha}{|(\beta+t)(n+1) - \beta t(1+n^2) - 1|} \frac{\Gamma(n+1-\lambda)}{\Gamma(2-\lambda)\Gamma(n+1)} (\overline{z})^n \qquad (n = 1, 2, ...) \tag{19}$$

$$X_1 \geq 0, \quad Y_1 \geq 0, \qquad X_1 z + \sum_{n=2}^{\infty} X_n + \sum_{n=1}^{\infty} Y_n = 1 \qquad X_n \geq 0, \quad Y_n \geq 0 \quad for \quad n = 2, 3, ... \quad .$$

Proof. If $f(z)$ be given by (17), then

$$f(z) = z - \sum_{n=2}^{\infty} \frac{1-\alpha}{|(\beta+t)(n-1) + \beta t(1+n^2) + 1|} \frac{\Gamma(n+1-\lambda)}{\Gamma(2-\lambda)\Gamma(n+1)} X_n z^n$$
$$+ \sum_{n=1}^{\infty} \frac{1-\alpha}{|(\beta+t)(n+1) - \beta t(1+n^2) - 1|} \frac{\Gamma(n+1-\lambda)}{\Gamma(2-\lambda)\Gamma(n+1)} Y_n (\overline{z})^n.$$

Since by (16), we have

$$\sum_{n=2}^{\infty} |(\beta+t)(n-1) + \beta t(1+n^2) + 1| \frac{1-\alpha}{|(\beta+t)(n-1) + \beta t(1+n^2) + 1|} \frac{\Gamma(n+1-\lambda)}{\Gamma(2-\lambda)\Gamma(n+1)} |X_n|$$
$$+ \sum_{n=1}^{\infty} |(\beta+t)(n+1) - \beta t(1+n^2) - 1| \frac{1-\alpha}{|(\beta+t)(n+1) - \beta t(1+n^2) - 1|} \frac{\Gamma(n+1-\lambda)}{\Gamma(2-\lambda)\Gamma(n+1)} |Y_n|$$
$$= (1-\alpha) \left(\sum_{n=2}^{\infty} |X_n| + \sum_{n=1}^{\infty} |Y_n| \right) = (1-\alpha)(1-X_1) \leq 1-\alpha.$$

So $f(z) \in \overline{HW}(\lambda,\beta,t,\alpha)$. Conversely, assume $f(z) \in \overline{HW}(\lambda,\beta,t,\alpha)$ by setting

$$X_1 = 1 - \left(\sum_{n=2}^{\infty}|X_n| + \sum_{n=1}^{\infty}|Y_n|\right), \qquad where \quad X_1 \geq 0. \ \ Then$$

$$X_n = \frac{|(\beta+t)(n-1)+\beta t(1+n^2)+1|}{1-\alpha}\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)}|a_n|, \qquad (n \geq 2)$$

$$Y_n = \frac{|(\beta+t)(n+1)-\beta t(1+n^2)-1|}{1-\alpha}\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)}|b_n| \qquad (n \geq 1),$$

we obtain

$$f(z) = z - \sum_{n=2}^{\infty}|a_n|z^n + \sum_{n=1}^{\infty}|b_n|(\overline{z})^n$$

$$= \sum_{n=2}^{\infty}\frac{(1-\alpha)X_n}{|(\beta+t)(n-1)+\beta t(1+n^2)+1|}\frac{\Gamma(n+1-\lambda)}{\Gamma(2-\lambda)\Gamma(n+1)}z^n$$

$$+ \sum_{n=1}^{\infty}\frac{(1-\alpha)Y_n}{|(\beta+t)(n+1)-\beta t(1+n^2)-1|}\frac{\Gamma(n+1-\lambda)}{\Gamma(2-\lambda)\Gamma(n+1)}(\overline{z})^n$$

$$= \left[1 - \left(\sum_{n=2}^{\infty}X_n + \sum_{n=1}^{\infty}Y_n\right)\right]z + \sum_{n=2}^{\infty}h_n(z)X_n + \sum_{n=1}^{\infty}g_n(z)Y_n$$

$$= X_1 z + \sum_{n=2}^{\infty}X_n h_n(z) + \sum_{n=1}^{\infty}Y_n g_n(z),$$

that is the required representation. □

4. Convex Combination

Now we introduce $\overline{HW}(\lambda,\beta,t,\alpha)$ is closed under convex combination.

Theorem 4.1. *If $f_{n_i}(i = 1,2,...)$ belongs to $\overline{HW}(\lambda,\beta,t,\alpha)$, then the function*

$$\phi(z) = \sum_{i=1}^{\infty}\sigma_i f_{n_i}(z) \ \ is\ also\ in\ \ \overline{HW}(\lambda,\beta,t,\alpha), \ \ where\ \ f_{n_i}(z)\ \ is\ defined\ by$$

$$f_{n_i}(z) = z - \sum_{n=2}^{\infty}a_{n_i}z^n + \sum_{n=1}^{\infty}b_{n_i}(\overline{z})^n \quad (i = 1,2,..., \quad 0 \leq \sigma_i \leq 1, \quad \sum_{i=1}^{\infty}\sigma_i = 1). \tag{20}$$

Proof. Since $f_{n_i}(z) \in \overline{HW}(\lambda,\beta,t,\alpha)$, by (16) we have

$$\sum_{n=2}^{\infty}|(\beta+t)(n-1)+\beta t(1+n^2)+1|\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)}|a_{n_i}|$$

$$+\sum_{n=1}^{\infty}|(\beta+t)(n+1)-\beta t(1+n^2)-1|\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)}|b_{n_i}|\leq 1-\alpha, \quad (i=1,2,...).$$

Also

$$\phi(z)=\sum_{i=1}^{\infty}\sigma_i f_i(z)=z-\sum_{n=2}^{\infty}\left(\sum_{i=1}^{\infty}\sigma_i a_{n_i}\right)z^n+\sum_{n=1}^{\infty}\left(\sum_{i=1}^{\infty}\sigma_i b_{n_i}\right)(\overline{z})^n.$$

Now according to Theorem 2.2 we have

$$\sum_{n=2}^{\infty}|(\beta+t)(n-1)+\beta t(1+n^2)+1|\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)}\left|\sum_{i=1}^{\infty}\sigma_i a_{n_i}\right|$$

$$+\sum_{n=1}^{\infty}|(\beta+t)(n+1)-\beta t(1+n^2)-1|\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)}\left|\sum_{i=1}^{\infty}\sigma_i b_{n_i}\right|$$

$$=\sum_{i=1}^{\infty}\sigma_i\left\{\sum_{n=2}^{\infty}|(\beta+t)(n-1)+\beta t(1+n^2)+1|\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)}|a_{n_i}|\right.$$

$$\left.+\sum_{n=1}^{\infty}|(\beta+t)(n+1)-\beta t(1+n^2)-1|\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)}|b_{n,i}|\right\}$$

$$\leq(1-\alpha)\sum_{i=1}^{\infty}\sigma_i=1-\alpha.$$

Thus $\phi(z)\in\overline{HW}(\lambda,\beta,t,\alpha)$.

So, we note that $\overline{HW}(\lambda,\beta,t,\alpha)$ is a convex set.

□

5. Distortion Bounds

In the following theorem we obtain distortion bounds for $f\in\overline{HW}(\lambda,\beta,t,\alpha)$.

Theorem 5.1. *Let $f=h+\overline{g}\in\overline{HW}(\lambda,\beta,t,\alpha)$, $\beta\geq 1$, $|z|=r<1$. Then, we have*

$$|f(z)|\geq(1-|b_1|)\,r-\left(\frac{1-\alpha}{1+\beta+t(1+5\beta)}-\frac{-1+2(\beta+t-\beta t)}{1+\beta+t(1+5\beta)}|b_1|\right)\frac{2-\lambda}{2}r^2, \qquad (21)$$

and

$$|f(z)|\leq(1+|b_1|)\,r+\left(\frac{1-\alpha}{1+\beta+t(1+5\beta)}-\frac{-1+2(\beta+t-\beta t)}{1+\beta+t(1+5\beta)}|b_1|\right)\frac{2-\lambda}{2}r^2. \qquad (22)$$

Proof. Assume $f(z) \in \overline{HW}(\lambda, \beta, t, \alpha)$, then by (16), we have

$$f(z) = \left| z - \sum_{n=2}^{\infty} |a_n| z^n + \sum_{n=1}^{\infty} |b_n| (\overline{z})^n \right| = \left| z + |b_1| (\overline{z}) - \sum_{n=2}^{\infty} \left(|a_n| z^n - |b_n| (\overline{z})^n \right) \right|$$

$$\geq (1 - |b_1|)\, r - \sum_{n=2}^{\infty} (|a_n| + |b_n|)\, r^2$$

$$\geq (1 - |b_1|)\, r - \frac{1-\alpha}{1+\beta+t(1+5\beta)} . \frac{2-\lambda}{2} \times$$

$$\sum_{n=2}^{\infty} \frac{1+\beta+t(1+5\beta)}{1-\alpha} \frac{2}{2-\lambda} (|a_n| + |b_n|)\, r^2$$

$$\geq (1 - |b_1|)\, r - \frac{1-\alpha}{1+\beta+t(1+5\beta)} . \frac{2-\lambda}{2} \times$$

$$\left(\sum_{n=2}^{\infty} \frac{(\beta+t)(n-1)+\beta t(1+n^2)+1}{1-\alpha} \frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)} |a_n| \right.$$

$$\left. + \frac{(\beta+t)(n+1)-\beta t(1+n^2)-1}{1-\alpha} \frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)} |b_n| \right) r^2$$

$$\geq (1 - |b_1|)\, r - \frac{1-\alpha}{1+\beta+t(1+5\beta)} . \frac{2-\lambda}{2} \times \left(1 - \frac{-1+2(\beta+t-\beta t)}{1-\alpha} |b_1| \right) r^2$$

$$= r - |b_1| r - \left(\frac{1-\alpha}{1+\beta+t+5\beta t} - \frac{-1+2(\beta+t-\beta t)}{1+\beta+t+5\beta t} |b_1| \right) \frac{2-\lambda}{2} r^2.$$

Relation (22) can be proved by using the similar statements. Therefore, it is omitted. So the proof is completed.

□

Corollary 5.1. *If* $f \in \overline{HG}(\lambda, t, \alpha)$ *then*

$$|f(z)| \geq (1 - |b_1|) r - \left(\frac{1-\alpha}{t+1} - \frac{2t-1}{t+1} |b_1| \right) \frac{2-\lambda}{2} r^2$$

and

$$|f(z)| \leq (1 + |b_1|) r + \left(\frac{1-\alpha}{t+1} - \frac{2t-1}{t+1} |b_1| \right) \frac{2-\lambda}{2} r^2.$$

Especially for $\lambda = 0$*, we obtain inequalities as follows*

$$|f(z)| \geq (1 - |b_1|) r - \left(\frac{1-\alpha}{t+1} - \frac{2t-1}{t+1} |b_1| \right) r^2$$

and

$$|f(z)| \leq (1 + |b_1|) r + \left(\frac{1-\alpha}{t+1} - \frac{2t-1}{t+1} |b_1| \right) r^2.$$

Corollary 5.2. *If $f \in \overline{HA}(\lambda, t, \alpha)$ then*

$$|f(z)| \geq (1 - |b_1|)r - \left(\frac{1-\alpha}{2(1+3t)} - \frac{1}{2(1+3t)}|b_1| \right) \frac{2-\lambda}{2} r^2$$

and

$$|f(z)| \leq (1 + |b_1|)r + \left(\frac{1-\alpha}{2(1+3t)} - \frac{1}{2(1+3t)}|b_1| \right) \frac{2-\lambda}{2} r^2.$$

Especially for $\lambda = 0$, we obtain inequalities as follows

$$|f(z)| \geq (1 - |b_1|)r - \left(\frac{1-\alpha}{2(1+3t)} - \frac{1}{2(1+3t)}|b_1| \right) r^2$$

and

$$|f(z)| \leq (1 + |b_1|)r + \left(\frac{1-\alpha}{2(1+3t)} - \frac{1}{2(1+3t)}|b_1| \right) r^2.$$

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