TWMS J. App. Eng. Math. V.7, N.2, 2017, pp. 188-199

B-SPLINE COLLOCATION METHOD FOR NUMERICAL SOLUTION OF NONLINEAR KAWAHARA AND MODIFIED KAWAHARA EQUATIONS

AMIR SABOOR BAGHERZADEH[1], §

Abstract. In this paper, a collocation method is applied for solving the Kawahara and modified Kawahara equations. For the spatial discretization, we use the sextic B-spline collocation (SBSC) method on uniform meshes, finite difference scheme is employed for the time discretization. The stability analysis of the collocation methods are examined by the Von Neumann approach. Numerical results demonstrate the efficiency and accuracy of the proposed methods.

Keywords: Kawahara and modified Kawahara equation, sixth degree B-splines, collocation method, recurrence system of equations, stability analysis, numerical experience.

AMS Subject Classification: 83-02, 99A00

1. Introduction

Nonlinear phenomena appear in many areas such as solid state physics, chemical physics, plasma physics or fluid mechanics. Kawahara type equations are one of the classes of nonlinear evolution equations which occur in many physical phenomena such as the theory of magneto-acoustic waves in plasmas [1] and shallow water waves with surface tension [2].

In this paper, we will focus our attention on the following nonlinear partial differential equation

$$u_t + \mu u_x + \gamma\, u^d\, u_x + p\, u_{3x} + q\, u_{5x} = 0, \quad (x,t) \in \Omega = [a,b] \times [0,T], \tag{1}$$

with the following boundary conditions

$$\begin{cases} u(x,0) = f(x) & x \in [a,b], \\ u_x(a,t) = g_1(t),\ u_{2x}(a,t) = g_2(t),\ u_{3x}(a,t) = g_3(t), & t \in [0,T], \\ u_x(b,t) = h_1(t),\ u_{2x}(b,t) = h_2(t), & t \in [0,T], \end{cases} \tag{2}$$

where $u_{kx} = \frac{\partial^k}{\partial x^k} u(x,t)$ for $k \in \mathbb{N}$ and the nondimensional quantities μ, γ, p and q are non-zero real constants and depend on the physical problem. When $d = 1$, equation (1) is called the Kawahara equation which proposed by Kawahara [1] to make the solitary wave propagation and when $d = 2$, (1) is known as the modified Kawahara equation.

Numerical methods are commonly used to solve (1). For example, Zhang presented the doubly periodic solution of the modified Kawahara equations [4]. The existence and the

[1] Department of Applied Mathematics, School of Mathematical Science, Ferdowsi University of Mashhad, Mashhad, Iran.
e-mail: saboorbagherzadeh.a@gmail.com, ORCID: http://orcid.org/0000-0002-1773-6204;

§ Manuscript received: July 27, 2016; accepted: November 07, 2016.

uniqueness of the solution of (1) are studied by Shuangping et al. [5]. Khan et al. applied the homotopy perturbation and modified Adomian decomposition method [6, 7]. A Dual-Petrov-Galerkin method for the equation (1)–(2) was presented in [8] and Zarebnia et al. showed the numerical solution of the modified Kawahara equation by using Chebyshev polynomials basis functions [9].

B-spline collocation methods are economical alternative ways for solving boundary value problems since they only require the evaluation of the unknown parameters at the grid points. Also, this approximation of the differential equations leads to band matrices which are solvable easily with some low cost algorithms. As is known, proper choice of the B-spline basis functions is the success key of this method. For this reason, sextic B-splines basis functions have been used for solving the Kawahara and modified Kawahara equations. By replacing the time derivative with the first order finite difference scheme and the space derivatives by the sextic B-spline relations, we will obtain an implicit spline-difference scheme. The resulting of this scheme gives a six-diagonal system which can be solved by a six-diagonal solver. Our main aim is to show the efficiency and sufficient accuracy of the SBSC method and state that our method is simple, with reasonable numerical stability and low computational cost.

The layout of this paper is the following. In Section 2, we give some preliminary results about the construction of B-spline basis functions and in Section 3, the SBSC method is used for the equation (1). In Section 4, we discuss theoretically about the stability of the proposed methods and finally in Section 5 we report our numerical experiences and demonstrate the efficiency and accuracy of proposed numerical scheme by considering some examples.

2. Sextic B-Spline Basis Functions

In this section, a sextic spline interpolation $S(x)$ is defined and then we derive some relations to be used in the formulation of the SBSC method for equation (1) with the boundary conditions (2). A detailed description of B-spline functions can be found in [10]. Let partition $\Delta = \{a = x_0 < x_1 < \ldots < x_m = b\}$ be an equally-spaced knots of $[a, b]$ with step size $h = \frac{1}{m}$, where $x_i = a + i\,h$ for $i = 0, 1, \ldots, m$. We consider smooth sextic spline $S(x)$ as an element of

$$\boldsymbol{S}_6(\Delta) = \big\{ f(x) \,|\, f(x) \in \mathcal{C}^5[a, b] \text{ and } \deg\big(f(x)\big) \leq 6 \text{ on } \Delta \big\} \tag{3}$$

The sextic B-splines basis functions $\big\{B_i(x)\big\}_{i=-3}^{m+2}$ are defined as (See [11])

$$B_i(x) = \frac{1}{h^6} \begin{cases} (x - x_i + 3h)^6, & ; \quad x_{i-3} \leq x \leq x_{i-2}, \\ (x - x_i + 3h)^6 - 7(x - x_i + 2h)^6, & ; \quad x_{i-2} \leq x \leq x_{i-1}, \\ (x - x_i + 3h)^6 - 7(x - x_i + 2h)^6 + 21(x - x_i + h)^6, & ; \quad x_{i-1} \leq x \leq x_i, \\ (x - x_i + 3h)^6 - 7(x - x_i + 2h)^6 + 21(x - x_i + h)^6 & \\ -35(x - x_i)^6, & ; \quad x_i \leq x \leq x_{i+1}, \\ (x - x_i - 4h)^6 - 7(x - x_i - 3h)^6 + 21(x - x_i - 2h)^6, & ; \quad x_{i+1} \leq x \leq x_{i+2}, \\ (x - x_i - 4h)^6 - 7(x - x_i - 3h)^6, & ; \quad x_{i+2} \leq x \leq x_{i+3}, \\ (x - x_i - 4h)^6, & ; \quad x_{i+3} \leq x \leq x_{i+4}, \\ 0, & ; \quad \text{otherwise}, \end{cases} \tag{4}$$

see Figure 1. For numerical solution, B_i, B_i', B_i''' and $B_i^{(5)}$ evaluated at the nodal points are needed. These coefficients are summarized in Table 1.

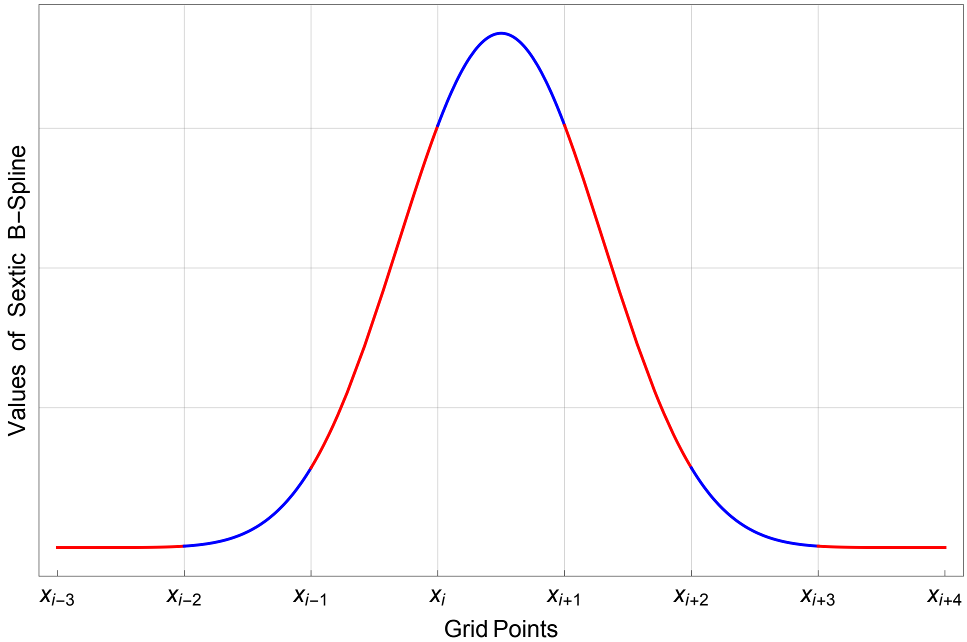

FIGURE 1. Plot of sixth degree B-spline function in $[x_{i-3}, x_{i+4}]$.

TABLE 1. Values of $B_i, B_i', B_i'', B_i''', B_i^{(4)}$ and $B_i^{(5)}$ in nodal points.

	x_{i-3}	x_{i-2}	x_{i-1}	x_i	x_{i+1}	x_{i+2}	x_{i+3}	x_{i+4}
B_i	0	1	57	302	302	57	1	0
$h\,B_i'$	0	6	150	240	-240	-150	-6	0
$h^2\,B_i''$	0	30	270	-300	-300	270	30	0
$h^3\,B_i'''$	0	120	120	-960	960	-120	-120	0
$h^4\,B_i^{(4)}$	0	360	-1080	720	720	-1080	360	0
$h^5\,B_i^{(5)}$	0	720	-3600	7200	-7200	3600	-720	0

3. Discretization and numerical method for the Kawahara type equations

Consider the uniform mesh with grid points $(x_i, t_j) \in \Omega$ to discretize the region Ω, where $x_i \in \Delta$ and $t_j = j\,\Delta t,\ j = 0, 1, \ldots, n$, which Δt is the mesh size in time direction. A global approximation $u(x,t)$ of the analytical solution $U(x,t)$ of (1)–(2) considered as an expansion of B-splines basis function as

$$u(x,t) = \sum_{i=-3}^{m+2} c_i(t)\, B_i(x), \tag{5}$$

where $c_i(t)$ are the time dependent variables to be determined from the sextic B-spline collocation form of the equation (1) together with boundary and initial conditions (2). [1] Now by using (2) and (5), we can obtain the nodal value u and its derivatives at the nodes (x_i, t_j) in the following forms

(i) $u(x_i, t_j) = u_i^j := c_{i-3}^j + 57c_{i-2}^j + 302c_{i-1}^j + 302c_i^j + 57c_{i+1}^j + c_{i+2}^j,$ (6)

[1] Collocation points are selected to coincide with knots.

$$\text{(ii)}\quad u_x(x_i,t_j)=(u_x)_i^j := \tfrac{6}{h}\big(-c_{i-3}^j-25c_{i-2}^j-40c_{i-1}^j+40c_i^j+25c_{i+1}^j+c_{i+2}^j\big), \tag{7}$$

$$\text{(iii)}\quad u_{xx}(x_i,t_j)=(u_{xx})_i^j := \tfrac{30}{h^2}\big(c_{i-3}^j+9c_{i-2}^j-10c_{i-1}^j-10c_i^j+9c_{i+1}^j+c_{i+2}^j\big), \tag{8}$$

$$\text{(iv)}\quad u_{3x}(x_i,t_j)=(u_{3x})_i^j := \tfrac{120}{h^3}\big(-c_{i-3}^j-c_{i-2}^j+8c_{i-1}^j-8c_i^j+c_{i+1}^j+c_{i+2}^j\big), \tag{9}$$

$$\text{(v)}\quad u_{4x}(x_i,t_j)=(u_{4x})_i^j := \tfrac{360}{h^4}\big(-c_{i-3}^j+3c_{i-2}^j-2c_{i-1}^j+2c_i^j-3c_{i+1}^j+c_{i+2}^j\big), \tag{10}$$

$$\text{(vi)}\quad u_{5x}(x_i,t_j)=(u_{5x})_i^j := \tfrac{720}{h^5}\big(c_{i-3}^j-5c_{i-2}^j+10c_{i-1}^j-10c_i^j+5c_{i+1}^j-c_{i+2}^j\big). \tag{11}$$

3.1. **The Kawahara equation.** Consider the following Kawahara equation

$$u_t+u\,u_x+u_{3x}-u_{5x}=0,\quad (x,t)\in\Omega, \tag{12}$$

with initial and boundary conditions (2). At the grid point (x_i,t_j) equation (12) may be discretized as the following form

$$\frac{\partial u}{\partial t}(x_i,t_j)+u(x_i,t_j)\,\frac{\partial u}{\partial x}u(x_i,t_j)+\frac{\partial^3 u}{\partial x^3}(x_i,t_j)-\frac{\partial^5 u}{\partial x^5}(x_i,t_j)=0. \tag{13}$$

By discretizing the time derivative in the usual first order finite difference method and applying the weighted average method (σ-method) to (13) we obtain

$$\begin{aligned}\frac{u_i^{j+1}-u_i^j}{\Delta t} &+ \big(\sigma(u\,u_x)_i^{j+1}+(1-\sigma)(u\,u_x)_i^j\big)+\big(\sigma(u_{3x})_i^{j+1}+(1-\sigma)(u_{3x})_i^j\big)\\ &- \big(\sigma(u_{5x})_i^{j+1}+(1-\sigma)(u_{5x})_i^j\big)=0,\end{aligned} \tag{14}$$

where $\sigma\in(0,1)$ is a parameter. To linearize the nonlinear term $(u\,u_x)_i^{j+1}$ in (14), we can write

$$(u\,u_x)_i^{j+1}=u_i^{j+1}\,(u_x)_i^j+u_i^j\,(u_x)_i^{j+1}-u_i^j\,(u_x)_i^j, \tag{15}$$

for more details about this type of linearization see [12]. Substituting (15) into (14), we obtain

$$\begin{aligned}u_i^{j+1} &+ \Delta t\,\sigma\big(u_i^{j+1}\,(u_x)_i^j+u_i^j\,(u_x)_i^{j+1}+(u_{3x})_i^{j+1}-(u_{5x})_i^j\big)\\ &= u_i^j+\Delta t(2\sigma-1)\,u_i^j(u_x)_i^j+\Delta t\,(1-\sigma)\big((u_{5x})_i^j+(u_{3x})_i^j\big).\end{aligned} \tag{16}$$

By substituting (5) into (16) at the grid points and using (6)–(11), the following difference equation with the variable c is obtained:

$$\alpha_{i1}\,c_{i-3}^{j+1}+\alpha_{i2}\,c_{i-2}^{j+1}+\alpha_{i3}\,c_{i-1}^{j+1}+\alpha_{i4}\,c_i^{j+1}+\alpha_{i5}\,c_{i+1}^{j+1}+\alpha_{i6}\,c_{i+2}^{j+1}$$

$$=\beta_1\,c_{i-3}^j+\beta_2\,c_{i-2}^j+\beta_3\,c_{i-1}^j+\beta_4\,c_i^j+\beta_5\,c_{i+1}^j+\beta_6\,c_{i+2}^j+6\,(2\sigma-1)\,h^4\,\Delta t\,W_i^j\,Z_i^j, \tag{17}$$

for $i=0,1,\dots,m,\ j=0,1,\dots,n$, where,

$$\alpha_{i1} = h^5+6\,\sigma\,h^4\,\Delta t\,\big(Z_i^j-W_i^j\big)-120\,\sigma\,h^2\,\Delta t-720\,\sigma\,\Delta t, \tag{18}$$

$$\alpha_{i2} = 57\,h^5+6\,\sigma\,h^4\,\Delta t\,\big(57\,Z_i^j-25\,W_i^j\big)-120\,\sigma\,h^2\,\Delta t+3600\,\sigma\,\Delta t, \tag{19}$$

$$\alpha_{i3} = 302\,h^5+12\,\sigma\,h^4\,\Delta t\,\big(151\,Z_i^j-20\,W_i^j\big)+960\,\sigma\,h^2\,\Delta t-7200\,\sigma\,\Delta t, \tag{20}$$

$$\alpha_{i4} = 302\,h^5+12\,\sigma\,h^4\,\Delta t\,\big(151\,Z_i^j+20\,W_i^j\big)-960\,\sigma\,h^2\,\Delta t+7200\,\sigma\,\Delta t, \tag{21}$$

$$\alpha_{i5} = 57\,h^5+6\,\sigma\,h^4\,\Delta t\,\big(57\,Z_i^j+25\,W_i^j\big)+120\,\sigma\,h^2\,\Delta t-3600\,\sigma\,\Delta t, \tag{22}$$

$$\alpha_{i6} = h^5+6\,\sigma\,h^4\,\Delta t\,\big(Z_i^j+W_i^j\big)+120\,\sigma\,h^2\,\Delta t+720\,\sigma\,\Delta t, \tag{23}$$

and

$$\begin{aligned}
\beta_1 &= h^5 + 120\,(1-\sigma)\,h^2\,\Delta t + 720\,(1-\sigma)\,\Delta t \quad (24)\\
\beta_2 &= 57\,h^5 + 120\,(1-\sigma)\,h^2\,\Delta t - 3600\,(1-\sigma)\,\Delta t \quad (25)\\
\beta_3 &= 302\,h^5 - 960\,(1-\sigma)\,h^2\,\Delta t + 7200\,(1-\sigma)\,\Delta t \quad (26)\\
\beta_4 &= 302\,h^5 + 960\,(1-\sigma)\,h^2\,\Delta t - 7200\,(1-\sigma)\,\Delta t \quad (27)\\
\beta_5 &= 57\,h^5 - 120\,(1-\sigma)\,h^2\,\Delta t + 3600\,(1-\sigma)\,\Delta t \quad (28)\\
\beta_6 &= h^5 - 120\,(1-\sigma)\,h^2\,\Delta t - 720\,(1-\sigma)\,\Delta t, \quad (29)
\end{aligned}$$

with

$$\begin{aligned}
W_i^j &= c_{i-3}^j + 57\,c_{i-2}^j + 302\,c_{i-1}^j + 302\,c_i^j + 57\,c_{i+1}^j + c_{i+2}^j, \quad (30)\\
Z_i^j &= -c_{i-3}^j - 25\,c_{i-2}^j - 40\,c_{i-1}^j + 40\,c_i^j + 25\,c_{i+1}^j + c_{i+2}^j. \quad (31)
\end{aligned}$$

The system (17) has $m+1$ equations with $m+6$ unknown parameters, therefore, we need five more equations which are obtained from boundary conditions.

3.1.1. *Development of boundary conditions.* Using boundary conditions (2) and (6)-(11), the left boundary conditions are discretized to

$$c_{-3}^j + 57\,c_{-2}^j + 302\,c_{-1}^j + 302\,c_0^j + 57\,c_1^j + c_2^j = g_0(t_j), \quad (32)$$

$$-c_{-3}^j - 25\,c_{-2}^j - 40\,c_{-1}^j + 40\,c_0^j + 25\,c_1^j + c_2^j = \frac{h}{6}\,g_1(t_j), \quad (33)$$

$$c_{-3}^j + 9\,c_{-2}^j - 10\,c_{-1}^j - 10\,c_0^j + 9\,c_1^j + c_2^j = \frac{h^2}{30}\,g_2(t_j), \quad (34)$$

and similarly, for the right boundary conditions we can write

$$c_{m-3}^j + 57\,c_{m-2}^j + 302\,c_{m-1}^j + 302\,c_m^j + 57\,c_{m+1}^j + c_{m+2}^j = h_0(t_j), \quad (35)$$

$$-c_{m-3}^j - 25\,c_{m-2}^j - 40\,c_{m-1}^j + 40\,c_m^j + 25\,c_{m+1}^j + c_{m+2}^j = \frac{h}{6}\,h_1(t_j). \quad (36)$$

Now using (6)-(11), (32)-(34) and (35)-(36), we can determine parameters c_{-3}^ℓ, c_{-2}^ℓ, c_{-1}^ℓ, c_{m+1}^ℓ and c_{m+2}^ℓ for $\ell = j,\ j+1$ as follow

$$c_{-3}^\ell = -\frac{100}{9}\,c_0^\ell + 10\,c_1^\ell + \frac{19}{9}\,c_2^\ell - \frac{31h}{1296}\,g_1(t_\ell) - \frac{h^2}{54}\,g_2(t_\ell) - \frac{61h^3}{5184}\,g_3(t_\ell), \quad (37)$$

$$c_{-2}^\ell = \frac{20}{9}\,c_0^\ell - c_1^\ell - \frac{2}{9}\,c_2^\ell - \frac{h}{1296}\,g_1(t_\ell) + \frac{h^2}{270}\,g_2(t_\ell) + \frac{5h^3}{5184}\,g_3(t_\ell), \quad (38)$$

$$c_{-1}^\ell = -\frac{-360\,c_0^\ell + 3240\,c_1^\ell + 360\,c_2^\ell - 10\,h\,g_1(t_\ell) - 6\,h^2\,g_2(t_\ell) - h^3\,g_3(t_\ell)}{3240}, \quad (39)$$

$$c_{m+1}^\ell = \frac{c_{m-3}^\ell + 17\,c_{m-2}^\ell + 15\,c_{m-1}^\ell - 25\,c_m^\ell}{8} + \frac{h}{96}\,h_1(t_\ell) - \frac{h^2}{480}\,h_2(t_\ell), \quad (40)$$

and

$$c_{m+2}^\ell = \frac{-204\,c_{m-3}^\ell - 2700\,c_{m-2}^\ell - 660\,c_{m-1}^\ell + 3660\,c_m^\ell}{96} + \frac{5\,h^2\,h_2(t_\ell) - 9\,h\,h_1(t_\ell)}{96}. \quad (41)$$

Therefore, by eliminating of the above parameters from (17), the six-banded linear system of equations (17) has a unique solution.

3.1.2. *The initial state.* The proposed scheme (17) is an implicit two-level scheme, for this reason, before starting any computation, it is necessary the initial element parameters c_i^0 for $i = 0, 1, \ldots, m$. By using (6) and taking $t_j = 0$ we have

$$c_{i-3}^0 + 57\, c_{i-2}^0 + 302\, c_{i-1}^0 + 302\, c_i^0 + 57\, c_{i+1}^0 + c_{i+2}^0 = u(x_i, 0) := u_i^0, \tag{42}$$

for $i = 0, 1, \ldots, m$. For eliminating five-additional parameters c_{-3}^0, c_{-2}^0, c_{-1}^0, c_{m+1}^0 and c_{m+2}^0, it is sufficient to use (37)-(41) by choosing $\ell = 0$.

3.2. **The modified Kawahara equation.** Consider the modified Kawahara equation

$$u_t + u_x + u^2\, u_x + p\, u_{3x} + q\, u_{5x} = 0, \quad (x, t) \in \Omega, \tag{43}$$

with boundary and initial conditions (2) where p and q are real constants. Similarly, the weighted average discretization yields

$$\begin{aligned} &\frac{u_i^{j+1} - u_i^j}{\Delta t} + \left(\sigma\,(u_x)_i^{j+1} + (1-\sigma)\,(u_x)_i^j\right) + (u^2)_i^j\left(\sigma\,(u_x)_i^{j+1} + (1-\sigma)\,(u_x)_i^j\right) \\ &+p\left(\sigma\,(u_{3x})_i^{j+1} + (1-\sigma)\,(u_{3x})_i^j\right) + q\left(\sigma\,(u_{5x})_i^{j+1} + (1-\sigma)\,(u_{5x})_i^j\right) = 0, \end{aligned} \tag{44}$$

where $\sigma \in (0, 1)$ is a parameter. Rearranging (44) with respect to time level j and $j+1$, we can obtain the following

$$\begin{aligned} &u_i^{j+1} + \sigma\,\Delta t\left((u_x)_i^{j+1} + (u^2)_i^j\,(u_x)_i^{j+1} + p\,(u_{3x})_i^{j+1} + q\,(u_{5x})_i^{j+1}\right) \\ &= u_i^j - (1-\sigma)\,\Delta t\left((u_x)_i^j + (u^2)_i^j\,(u_x)_i^j + p\,(u_{3x})_i^j + q\,(u_{5x})_i^j\right). \end{aligned} \tag{45}$$

Substituting (5) into (45) at the grid points and using (6)-(11), the following recurrence system of equations with the variable c is obtained:

$$\begin{aligned} &\rho_{i1}\, c_{i-3}^{j+1} + \rho_{i2}\, c_{i-2}^{j+1} + \rho_{i3}\, c_{i-1}^{j+1} + \rho_{i4}\, c_i^{j+1} + \rho_{i5}\, c_{i+1}^{j+1} + \rho_{i6}\, c_{i+2}^{j+1} \\ &\mu_{i1}, c_{i-3}^j + \mu_{i2}, c_{i-2}^j + \mu_{i3}, c_{i-1}^j + \mu_{i4}, c_i^j + \mu_{i5}, c_{i+1}^j + \mu_{i6}, c_{i+2}^j, \end{aligned} \tag{46}$$

for $i = 0, 1, \ldots, m,\ j = 0, 1, \ldots, n$ where,

$$\rho_{i1} = h^5 - 6\,\sigma\,\Psi_i^j\,\Delta t\, h^4 - 120\, p\,\sigma\,\Delta t\, h^2 + 720\, q\,\sigma\,\Delta t, \tag{47}$$

$$\rho_{i2} = 57\, h^5 - 150\,\sigma\,\Psi_i^j\,\Delta t\, h^4 - 120\, p\,\sigma\,\Delta t\, h^2 - 3600\, q\,\sigma\,\Delta t, \tag{48}$$

$$\rho_{i3} = 302\, h^5 - 240\,\sigma\,\Psi_i^j\,\Delta t\, h^4 + 960\, p\,\sigma\,\Delta t\, h^2 + 7200\, q\,\sigma\,\Delta t, \tag{49}$$

$$\rho_{i4} = 302\, h^5 + 240\,\sigma\,\Psi_i^j\,\Delta t\, h^4 - 960\, p\,\sigma\,\Delta t\, h^2 - 7200\, q\,\sigma\,\Delta t, \tag{50}$$

$$\rho_{i5} = 57\, h^5 + 150\,\sigma\,\Psi_i^j\,\Delta t\, h^4 + 120\, p\,\sigma\,\Delta t\, h^2 + 3600\, q\,\sigma\,\Delta t, \tag{51}$$

$$\rho_{i1} = h^5 + 6\,\sigma\,\Psi_i^j\,\Delta t\, h^4 + 120\, p\,\sigma\,\Delta t\, h^2 - 720\, q\,\sigma\,\Delta t, \tag{52}$$

and

$$\mu_{i1} = h^5 + 6\,(1-\sigma)\,\Psi_i^j\,\Delta t\, h^4 + 120\, p\,(1-\sigma)\,\Delta t\, h^2 - 720\, q\,(1-\sigma)\,\Delta t, \tag{53}$$

$$\mu_{i2} = 57\, h^5 + 150\,(1-\sigma)\,\Psi_i^j\,\Delta t\, h^4 + 120\, p\,(1-\sigma)\,\Delta t\, h^2 + 3600\, q\,(1-\sigma)\,\Delta t, \tag{54}$$

$$\mu_{i3} = 302\, h^5 + 240\,(1-\sigma)\,\Psi_i^j\,\Delta t\, h^4 - 960\, p\,(1-\sigma)\,\Delta t\, h^2 - 7200\, q\,(1-\sigma)\,\Delta t, \tag{55}$$

$$\mu_{i4} = 302\, h^5 - 240\,(1-\sigma)\,\Psi_i^j\,\Delta t\, h^4 + 960\, p\,(1-\sigma)\,\Delta t\, h^2 + 7200\, q\,(1-\sigma)\,\Delta t, \tag{56}$$

$$\mu_{i5} = 57\, h^5 - 150\,(1-\sigma)\,\Psi_i^j\,\Delta t\, h^4 - 120\, p\,(1-\sigma)\,\Delta t\, h^2 - 3600\, q\,(1-\sigma)\,\Delta t, \tag{57}$$

$$\mu_{i6} = h^5 - 6\,(1-\sigma)\,\Psi_i^j\,\Delta t\, h^4 - 120\, p\,(1-\sigma)\,\Delta t\, h^2 + 720\, q\,(1-\sigma)\,\Delta t, \tag{58}$$

with $\Psi_i^j = 1 + \Phi_i^j$, where

$$\Phi_i^j = \left(c_{i-3}^j + 57\, c_{i-2}^j + 302\, c_{i-1}^j + 302\, c_i^j + 57\, c_{i+1}^j + c_{i+2}^j\right)^2, \tag{59}$$

The recurrence system of equation (46) has $m+1$ equations with $m+6$ unknown parameters, therefore, we still need five more additional equations which are obtained from boundary conditions.

Remark 3.1. *By choosing $\sigma = \frac{1}{2}$, the σ-average method in* (45) *becomes to the famous Crank-Nicolson method and in this case the coefficients in recurrence system* (46) *have the form $\rho_{i\ell} = \mu_{i\,7-\ell}$ for $\ell = 1, 2, \ldots, 6$.*

Remark 3.2. *To eliminate five additional parameters c^{ℓ}_{-3}, c^{ℓ}_{-2}, c^{ℓ}_{-1}, c^{ℓ}_{0}, c^{ℓ}_{1} and c^{ℓ}_{-3} where $\ell = j,\, j+1$ in recurrence system* (46)*, we can use relations* (37)-(41)*, respectively. In addition, the initial state for the implicit two-level scheme* (46) *is exactly the same as Subsection 3.1.2.*

4. Stability analysis

Von Neumann stability method is applied for analyzing the stability of the proposed scheme. For this reason, consider the trial solution at a given point x_i

$$c_i^j = \eta^j \exp(\imath\theta i h), \tag{60}$$

where $\imath = \sqrt{-1}$, θ is the mode number, h is the element size and η is the amplification factor of the scheme. To investigate the stability of difference scheme (17), the nonlinear term $u\,u_x$ of the Kawahara equation is linearized by making the quantities u and u_x locally constants. Now substituting the Fourier mode into linearized form of equation (17), we obtain

$$\eta = \frac{A_1 - \imath\, B_1}{A_2 - \imath B_2}, \tag{61}$$

where

$$\begin{aligned}
A_1 &= \beta_1 \cos(3\theta\, h) + (\beta_2 + \beta_6)\cos(2\theta\, h) + (\beta_3 + \beta_5)\cos(\theta\, h) + \beta_4, & (62)\\
B_1 &= \beta_1 \sin(3\theta\, h) + (\beta_2 + \beta_6)\sin(2\theta\, h) + (\beta_3 + \beta_5)\sin(\theta\, h), & (63)\\
A_2 &= \alpha_{i1} \cos(3\theta\, h) + (\alpha_{i2} + \alpha_{i6})\cos(2\theta\, h) + (\alpha_{i3} + \alpha_{i1})\cos(\theta\, h) + \alpha_{i4}, & (64)\\
B_2 &= \alpha_{i1} \sin(3\theta\, h) + (\alpha_{i2} + \alpha_{i6})\sin(2\theta\, h) + (\alpha_{i3} + \alpha_{i5})\sin(\theta\, h). & (65)\\
& & (66)
\end{aligned}$$

The stability condition $|\eta| \le 1$ is satisfied by the following inequality:

$$\begin{aligned}
A_1^2 + B_1^2 - A_2^2 - B_2^2 \;&\le\; -230400\Delta t\big(27h^9 q^2\sigma + 81h^8 \Delta t p^2 \sigma^2 - 16\delta t\big(2(h^2-3)\\
&+\; (6+h^2)\cos(\theta h)\big)^2 \sin^2(\frac{x}{2})\big) \le 0. \qquad (67)
\end{aligned}$$

So we can investigate the results above in the following Theorem:

Theorem 4.1. *The scheme* (17) *for solving equation* (12) *is unconditionally stable.*

The next step in this Section is studying the stability of (46). For this reason, we must first linearize the system (46). Set:

$$\Psi_i := 1 + \Phi_i = 1 + (d + 57d + 302d + 302d + 57d + d)^2 = \mathbb{F}. \tag{68}$$

Then, consider again the trial solution (60) at an arbitrary point x_i and substituting (60) into (46) we get

$$\eta = \frac{C_1 - \imath\, D_1}{C_2 - \imath\, D_2}, \tag{69}$$

where,

$$C_1 = \mu_{i1}\,\cos(3\theta h) + (\mu_{i2} + \mu_{i6})\cos(2\theta h) + (\mu_{i3} + \mu_{i5})\,\cos(\theta h) + \mu_{i1}, \quad (70)$$

$$D_1 = \mu_{i1}\,\sin(3\theta h) + (\mu_{i2} + \mu_{i6})\sin(2\theta h) + (\mu_{i3} + \mu_{i5})\,\sin(\theta h), \quad (71)$$

$$C_2 = \rho_{i1}\,\cos(3\theta h) + (\rho_{i2} + \rho_{i6})\cos(2\theta h) + (\rho_{i3} + \rho_{i5})\,\cos(\theta h) + \rho_{i1}, \quad (72)$$

$$D_2 = \rho_{i1}\,\sin(3\theta h) + (\rho_{i2} + \rho_{i6})\sin(2\theta h) + (\rho_{i3} + \rho_{i5})\,\sin(\theta h). \quad (73)$$

For the stability condition, we can write

$$C_1^2 + D_1^2 - C_2^2 - D_2^2 = -\mathbb{A}^* + (1 - 2\sigma)\,\mathbb{B}^*, \quad (74)$$

with

$$\begin{aligned} \mathbb{A}^* &= -48h^5\Delta t\big(8140h^2p + 121080q + 2551h^4\mathbb{F} + (6200h^2p - 35040q \\ &+ 262h^4\mathbb{F})\,\cos(\theta h)\big)\,\sin^2(\theta h), \quad (75) \\ \mathbb{B}^* &= 288\Delta t^2\big(3(400(11h^4p^2 + 168h^2pq + 756q^2) - 40h^4(49h^2p + 276q)\mathbb{F} + 371h^8\mathbb{F}^2) \\ &+ 5(4h^2p + 120q + 5h^4\mathbb{F})\,(-20(7h^2p + 66q) + 41h^4\mathbb{F})\,\cos(\theta h) \\ &+ 40(20h^2p - 120q + h^4\mathbb{F})\,(-4h^2p - 30q + h^4\mathbb{F})\,\cos(2\theta h)\,\sin^2(\frac{\theta h}{2})\big). \quad (76) \end{aligned}$$

The necessary and sufficient condition for $\eta \le 1$ is that $\sigma \ge \frac{1}{2}$. But for $\sigma < \frac{1}{2}$, it is satisfied when the Δt be sufficiently small. We investigate the results above in the following theorem.

Theorem 4.2. *The scheme* (46) *for solving equation* (43) *is unconditionally stable if* $\sigma \ge \frac{1}{2}$, *and conditionally stable if* $\sigma < \frac{1}{2}$.

5. Numerical experiences

The numerical method which is described in this paper is tested for getting solution of the Kawahara and modified Kawahara differential equations for the numerical accuracy and demonstrate the robustness of the methods. For this reason, the L_2 and L_∞ norms define as

$$\|E\|_2 := \Big(h\sum_{j=0}^{m}|(U_j - u_j)^2|\Big)^{\frac{1}{2}}, \quad (77)$$

$$\|E\|_\infty := \max_j |U_j - u_j|. \quad (78)$$

The three conserved quantities for the Kawahara equation are (see [13, 14])

$$I_1 = \int_a^b U\,dx, \quad I_2 = \int_a^b \frac{U^2}{2}\,dx, \quad I_3 = \int_a^b \big(\frac{U_x^2 + U_{xx}^2}{2} - \frac{U^3}{6}\big)\,dx. \quad (79)$$

5.1. **Example 1.** Consider the Kawahara equation (12) with initial and boundary conditions (2) with the exact solution [13]

$$U(x,t) = \frac{105}{169}\mathrm{sech}^4\big(K(x - \frac{36}{169}t)\big), \quad (80)$$

where $K = \frac{1}{2\sqrt{13}}$. For numerical computation we take $[a,b] = [-20,30]$, $h = 1$ and $\Delta t = 0.01$. The error values and conserved quantities are shown in Table 2 and compared with other methods such as Multiquadric (MQ) and Gaussian (GA) radial basis methods [15], polynomial based differential quadrature (PDQ) method [13] and cosine expansion based differential quadrature (CDQ) method [13]. In Figure 2 traveling wave solution of the Kawahara equation in different time steps plotted and the markers show the exact

TABLE 2. Comparison of the error norms for Kawahara equation.

Method	Time	$\|E\|_2$	$\|E\|_\infty$	I_1	I_2	I_3
Present	0	0	0	5.97359	1.27250	-0.16458
	5	3.28879E-5	1.54883E-5	5.97358	1.27250	-0.16458
	15	3.29411E-5	1.55451E-5	5.97366	1.27250	-0.16458
	25	3.31971E-5	1.56284E-5	5.97367	1.27250	-0.16458
MQ	0	0	0	5.97359	1.27250	-0.16458
	5	9.468E-5	4.6697E-5	5.97348	1.27250	-0.16458
	15	1.5362E-4	5.9394E-5	5.97343	1.27250	-0.16458
	25	1.6818E-4	4.6602E-5	5.97355	1.27250	-0.16458
GA	0	0	0	5.973599	1.27250	-0.16458
	5	1.0075E-4	3.4297E-5	5.973662	1.272502	-0.16458
	15	1.0113E-4	3.8304E-5	5.973675	1.272502	-0.16458
	25	1.3160E-4	3.9907E-5	5.973532	1.272502	-0.16458
PDQ	0	0	0	5.97357	1.27250	-0.16458
	5	1.986E-3	9.21E-4	5.97060	1.27250	-0.16458
	15	2.453E-3	1.045E-3	5.97014	1.27250	-0.16458
	25	2.851E-3	8.635E-4	5.97353	1.27250	-0.16458
CDQ	0	0	0	5.97357	1.27250	-0.16458
	5	1.51E-4	4.3E-5	5.97372	1.27250	-0.16458
	15	1.56E-4	4.9E-5	5.97364	1.27250	-0.16458
	25	1.59E-4	7.6E-5	5.97350	1.27250	-0.16458

solution. Error between the numerical and analytical solutions is depicted in different

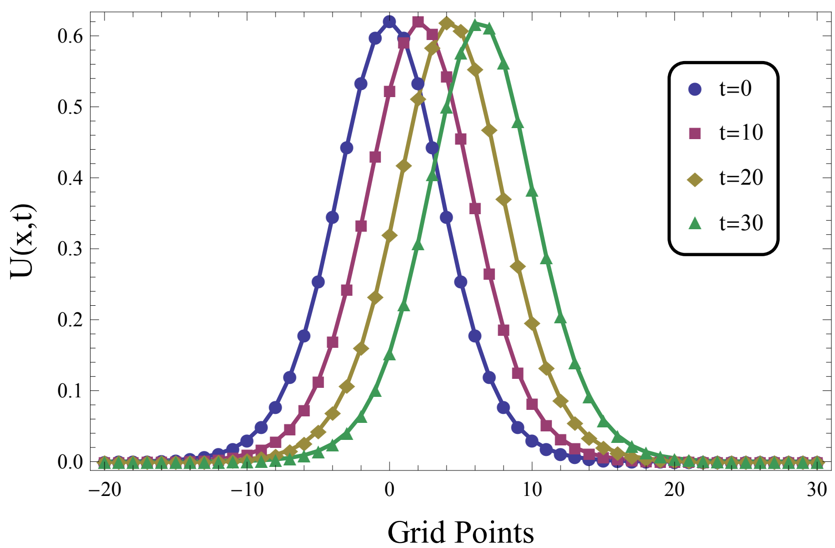

FIGURE 2. The solution of the Kawahara equation in different time steps.

time steps in Figure 3.

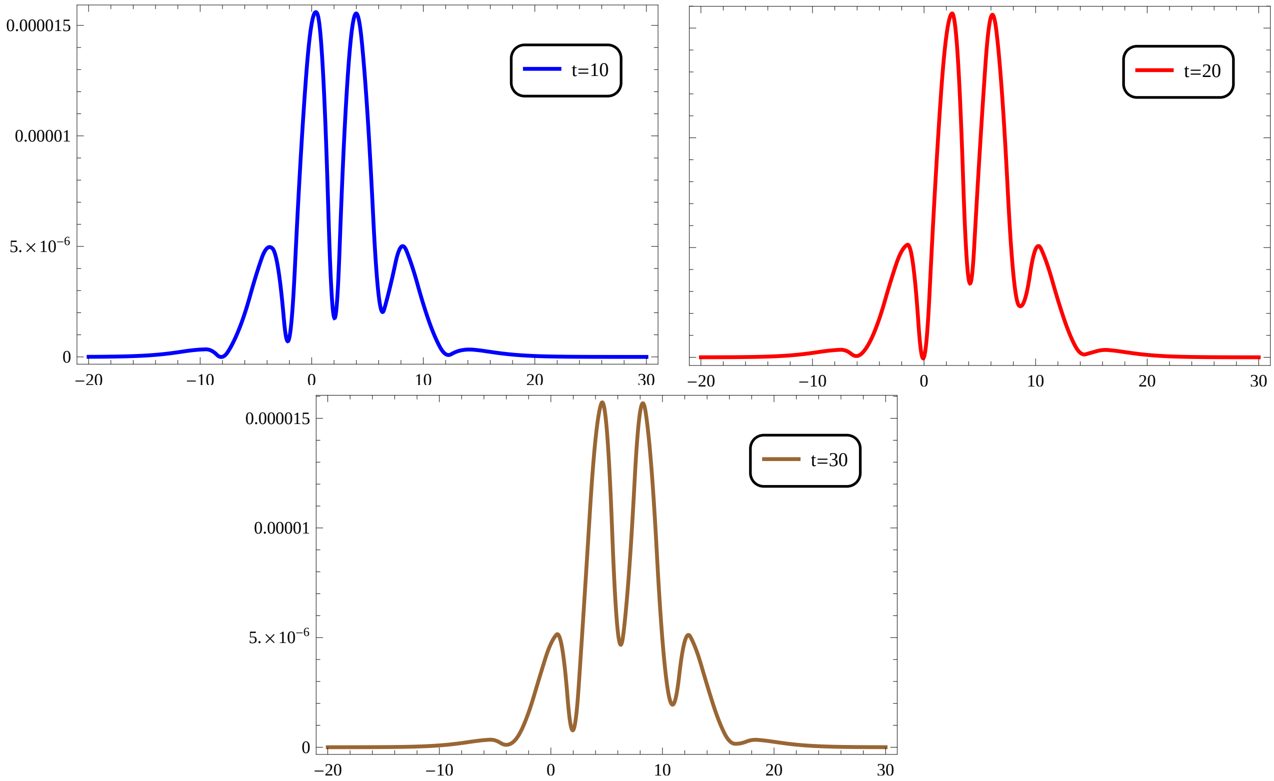

FIGURE 3. The errors (|numerical solution−analytical solution|) at $t = 10,\ t = 20$ and $t = 30$.

5.2. **Example 2.** Consider modified Kawahara equation (43) with boundary and initial conditions (2). The exact solution of this equation is

$$U(x,t) = \frac{3p}{\sqrt{-10q}}\operatorname{sech}^2\big(K(x-\delta t)\big), \tag{81}$$

where $\delta = \frac{25q-4p^2}{25q}$ and $K = 0.5\sqrt{\frac{-p}{5q}}$. For the numerical analysis, we assume the parameters $p = 0.001$ and $q = -1$. Furthermore, let $[a,b] = [-100, 100]$, $T = 100$ and $h = \Delta t = 1$. In Table 3, we observed the errors in our method in some grid points for different time steps. In Table 4, we compare our results with standard radial basis methods such as Multiquadratic (MQ), Gaussian (GA), Inversequadric (IQ) and Inverse Multiquadric (IMQ) (13). In Figure 4, the numerical values of traveling wave solution for modified Kawahara equation and the corresponding errors are plotted, respectively.

TABLE 3. Absolute error values in grid points for different time steps.

Time	$x=-75$	$x=-25$	$x=20$	$x=40$	$x=60$	$x=80$
10	5.12466E-6	3.02662E-6	9.89115E-7	2.72163E-6	4.0583E-6	4.87119E-6
20	5.16457E-6	3.7101E-6	4.74912E-8	1.8924E-6	3.45159E-6	4.53408E-6
40	4.95896E-6	4.68791E-6	1.80594E-6	4.75529E-8	1.89363E-6	3.4536E-6
60	4.49171E-6	5.12952E-6	3.38775E-6	1.80706E-6	4.76148E-8	1.89485E-6

Table 4. Comparison of the error norms at time $t = 100$ for modified Kawahara equation.

Method	$\|E\|_2$	$\|E\|_\infty$	I_1	I_2	I_3
Present	5.17073E-6	5.49144E-5	0.119189	4.16631E-5	-9.13258E-9
MQ	3.120253E-5	4.94228E-4	0.119171	4.16550E-5	-9.13256E-9
GA	3.120140E-5	4.95053E-4	0.119168	4.16552E-5	-9.13256E-9
IQ	3.367458E-5	4.95159E-4	0.119185	4.16550E-5	-9.13256E-9
IMQ	3.248706E-5	5.99387E-4	0.119182	4.16550E-5	-9.13256E-9

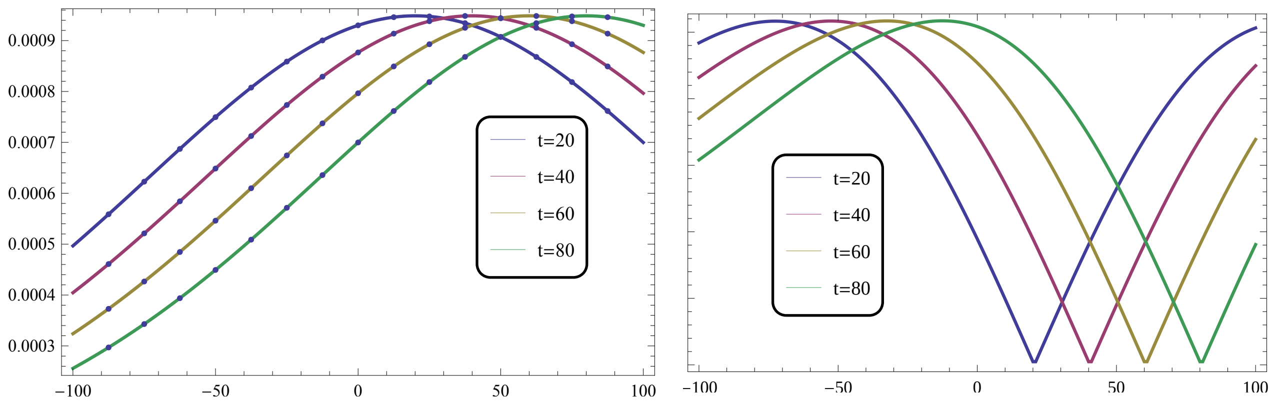

Figure 4. (Left) Traveling wave solution for modified Kawahara equation. (Right) The errors (|numerical solution−analytical solution|) at $t = 20$, $t = 40$, $t = 60$ and $t = 80$.

6. Conclusion

The sixth degree B-spline functions tested on Kawahara and modified Kawahara partial differential equation. The numerical results show that the errors are reasonable and the computed results are in agreement with the reported results in other literatures.

References

[1] Kawahara,T, (1972), Oscillatory solitary waves in dispersive media, J. Phys. Soc. Japan, 33, pp.260-264.

[2] Hunter,J.K. and Scheule,J., (1988), Existence of perturbed solitary wave solution to a model equation for water waves, Physica D., 32, pp.253-268.

[3] Wazwaz,A.M., (2003), Compacton solutions of the Kawahara-type nonlinear dispersive equation, Appl. Math. Comput., 145, pp.133-150.

[4] Zhang,D., (2005), Doubly periodic solutions of the modified Kawahara equations, Chaos Solitons Fractals, 25, pp.1155-1160.

[5] Shuangping,T. and Shuangbin,C., (2002), Existence and uniqueness of solutions to nonlinear Kawahara equations, Chinese Ann. Math. Ser. A, 23, 2, pp.221-228.

[6] Khan,Y. and Faraz,N., (2010), A new approach to differential difference equations, J. Adv. Res. Differ. Equ., 2, 2, pp.1-12.

[7] Khan,Y. and Wu,Q., (2011), Homotopy perturbation transform method for nonlinear equations using He's polynomials, Computers and Mathematics with Applications, 61, 8, pp.1963-1967.

[8] Yuan,J.M., Shen,J., and Wu,J., (2008), A Dual-Petrov-Galerkin method for the Kawahara-type equations, J. Sci. Comput, 34, pp.48-63.

[9] Zarebnia,M. and Jalili,S., (2011), A numerical solution to a modified Kawahara equation, J. of Advanced Research in Differential Equ., 3, pp.65-76.

[10] De Boor,C., (1978), A practical guide to splines, Springer-Verlag, New York.

[11] Prenter,P.M, (1985), Splines and Variational Methods, Wiley, New York, NY.
[12] Rubin,S.G. and Graves,R.A., (1975), Cubic spline approximation for problems in fluid mechanics, Nasa, Washington, D. C., Nasa TR R-436, pp.253-268.
[13] Korkmaz,A. and Dağ,I., (2009), Crank-Nicolson differential quadrature algorithms for the Kawahara equation, Chaos Solutions and Fractals, Vol.42, No.1, pp.65-73.
[14] Malik,R.P., (1997), On fifth order KdV-type equation, solv-int/9710010, JINR, Dubna, October 1997.
[15] Bibi,N., Tirmizi,S.I.A., and Haq,S., (2011), Meshless method of lines for numerical solution of Kawahara type equation, Applied Mathematics, 2, pp.608-618, doi:10.4236/am.2011.25081.

Amir Saboor Bagherzadeh received his PhD from Vienna University of Technology in 2011. His research interests are numerical analysis, error estimation, spline and B-spline approximation, numerical methods for ODEs and PDEs, and engineering mathematics.