TWMS J. App. Eng. Math. V.7, N.2, 2017, pp. 200-208

MATRIX TRANSFORM OF IRREGULAR WEYL-HEISENBERG WAVE PACKET FRAMES FOR $L^2(\mathbb{R})$

R. KUMAR[1], ASHOK K. SAH[2], §

ABSTRACT. Cordoba and Fefferman [4] introduced wave packet systems by applying certain collections of dilations, modulations and translations to the Gaussian function in the study of some classes of singular integral operators. In this paper, we introduce the concept of matrix transform $M = (\alpha_{p,q,r,j,k,m})$ and with the help of matrix transform we study the action of M on $f \in L^2(\mathbb{R})$ and on its wave packet coefficients. Further, we also obtain the tight frame condition for matrix transform of $f \in L^2(\mathbb{R})$ whose wave packet series expansion is known.

Keywords: frames, matrix transform, wave packet system.

AMS Subject Classification: Primary 42C15; Secondary 42C30, 42B35.

1. Introduction and Preliminaries

The theory of frames for Hilbert spaces were formally introduced by Duffin and Schaeffer [6], to deal with nonharmonic Fourier series.

A system $\{f_k\}$ in a separable Hilbert space $\mathcal{H}$ with inner product $\langle ., . \rangle$ is called *frame* (Hilbert) for $\mathcal{H}$ if there exists positive constants A and B such that

$$A\|f\|^2 \leq \|\langle f, f_k \rangle\|^2_{\ell^2} \leq B\|f\|^2, \text{ for all } f \in \mathcal{H}. \tag{1}$$

If upper inequality in (1) holds, then we say that $\{f_k\}$ is a Bessel sequence for $\mathcal{H}$. The positive constants A and B are called *lower* and *upper frame bounds* of the frame $\{f_n\}$, respectively. They are not unique. The positive constants,

$$\mathrm{A}_0 = \sup\{A : A \text{ satisfy } (1)\}$$
$$\mathrm{B}_0 = \inf\{B : B \text{ satisfy } (1)\}$$

are called *optimal* or *best bounds* of the frame. A frame $\{f_n\}$ for $\mathcal{H}$ is called *tight* if it is possible to choose $A = B$ and *normalized tight* if $A = B = 1$. If removal of one f_j renders the collection $\{f_k\}$ no longer a frame for $\mathcal{H}$, then $\{f_k\}$ is called an *exact frame* for $\mathcal{H}$. The operator $V : \ell^2 \to \mathcal{H}$ given by

$$V(\{c_k\}) = \sum_{k=1}^{\infty} c_k f_k, \{c_k\} \in \ell^2$$

[1] Department of Mathematics, Kirori Mal College, University of Delhi, Delhi-110007, India.
e-mail: rajkmc@gmail.com, ORCID: http://orcid.org/0000-0003-0714-5045;
[2] Department of Mathematics, University of Delhi, Delhi-110007, India.
e-mail: ashokmaths2010@gmail.com, ORCID: http://orcid.org/0000-0002-4651-052X;
§ Manuscript received: July 10, 2016; accepted: October 12, 2016.
TWMS Journal of Applied and Engineering Mathematics Vol.7, No.2;

is called the *synthesis operator* or *pre-frame operator* of the frame. Adjoint of V is the operator $V^* : \mathcal{H} \to \ell^2$ given by

$$V^*(f) = \{\langle f, f_k \rangle\}$$

and is called the *analysis operator* of the frame $\{f_k\}$. Composing V and V^* we obtain the *frame operator* $S = VV^* : \mathcal{H} \to \mathcal{H}$ given by

$$S(f) = \sum_{k=1}^{\infty} \langle f, f_k \rangle f_k, f \in \mathcal{H}.$$

The frame operator S is a positive continuous invertible linear operator from $\mathcal{H}$ onto $\mathcal{H}$. Every vector $f \in \mathcal{H}$ can be written as:

$$f = SS^{-1}f = \sum_{k=1}^{\infty} \langle S^{-1}f, f_k \rangle f_k. \ \ (\textit{Reconstruction formula}) \tag{2}$$

The series given in (2) converges unconditionally for all $f \in \mathcal{H}$ and is called *frame decomposition* or *reconstruction formula* for the frame. Thus, frames are redundant building blocks which have basis like properties. One may observe that the frame decomposition shows that all information about a given signal (vector) $f \in \mathcal{H}$ is contained in the system $\{\langle S^{-1}f, f_k \rangle\}$. The scalars $\langle S^{-1}f, f_k \rangle$ are called *frame coefficients*. Today "frame theory" is a central tool in applied mathematics and engineering. In particular frames are widely used in sampling theory, wavelet theory, wireless communication, signal processing, image processing , differential equations, filter banks, geophysics, quantum computing, wireless sensor network, multiple-antenna code design and many more. An introduction to frames in applied mathematics and engineering can be found in a so nice book by Casazza and Kutynoik [1]. In the theoretical direction, powerful tools from operator theory and Banach spaces are being employed to study frames. For basic theory in frames, an interested reader may be refer to [3, 4, 7, 11].

The wave packet systems were introduced and studied by Cordoba and Fefferman [4] by applying certain collections of dilations, modulations and translations to the Gaussian function in the study of some classes of singular integral operators. Lebate et al. [9] adopted the same expression to describe any collections of functions which are obtained by applying the same operations to a finite family of functions in $L^2(\mathbb{R})$. More precisely, Gabor systems, wavelet systems and the Fourier transform of wavelet systems are special cases of wave packet systems. Wave packet systems have recently been successfully applied to some problems in harmonic analysis and operator theory. The wave packet systems have been studied by several authors, see [2, 5, 8, 10].

Now, we give notations and definitions required in this paper.

We now recall basic notations and definitions. For $1 \leq p < \infty$, let $L^p(\mathbb{R})$ denote the Banach space of complex-valued Lebesgue integrable functions f on $\mathbb{R}$ satisfying

$$\|f\|_p = \left(\int_{\mathbb{R}} |f(t)|^p dt \right)^{\frac{1}{p}} < \infty.$$

For $p = 2$, an inner product on $L^p(\mathbb{R})$ is given by

$$\langle f, g \rangle = \int_R f\overline{g} dt,$$

where $\overline{g}$ denotes the complex conjugate of g.

We consider the unitary operators on $L^2(\mathbb{R})$ which are given by :

Translation $\leftrightarrow T_a f(t) = f(t - a)$, $a \in \mathbb{R}$.

Modulation $\leftrightarrow E_b f(t) = e^{2\pi ibt} f(t),\ b \in \mathbb{R}$.
Dilation $\leftrightarrow D_a f(t) = \frac{1}{\sqrt{|a|}} f(\frac{t}{a}), a \in \mathbb{R},\ a \neq 0$. For $a > 0,\ b, c \in \mathbb{R}$ and $f \in L^2(\mathbb{R})$, it is easy to verify that

$$(D_{a_j} f)\hat{} = D_{a_j^{-1}} \hat{f}, (E_b f)\hat{} = T_b \hat{f}, (T_c f)\hat{} = E_{-c} \hat{f},$$

$$(D_{a_j} T_{bk} E_{c_m} f)\hat{} = D_{a_j^{-1}} E_{-bk} T_{c_m} \hat{f}.$$

2. Matrix transform of *IWH* wave packet frames for $L^2(\mathbb{R})$

Definition 2.1. *Let $\psi \in L^2(\mathbb{R})$, $\{a_j\}_{j\in\mathbb{Z}} \subset \mathbb{R}^+$, $\{c_m\}_{m\in\mathbb{Z}} \subset \mathbb{R}$ and $b \neq 0$. A system of the form $\{D_{a_j} T_{bk} E_{c_m} \psi\}_{j,k,m\in\mathbb{Z}}$ is called an* irregular Weyl-Heisenberg wave packet system *(or IWH wave packet system).*

Definition 2.2. *If IWH wave packet system $\{D_{a_j} T_{bk} E_{c_m} \psi\}_{j,k,m\in\mathbb{Z}}$ constitutes a frame for $L^2(\mathbb{R})$, i.e., if there exists positive constants a_0 and b_0 such that*

$$a_0 \|f\|^2 \leq \sum_{j,k,m\in\mathbb{Z}} |\langle f, D_{a_j} T_{bk} E_{c_m} \psi \rangle|^2 \leq b_0 \|f\|^2, \text{ for all } f \in L^2(\mathbb{R}), \tag{3}$$

then we say that $\{D_{a_j} T_{bk} E_{c_m} \psi\}_{j,k\in\mathbb{Z}}$ is an irregular Weyl-Heisenberg wave packet frame *(or IWH* wave packet frame*).*

The positive constants a_0 and b_0 are called *lower* and *upper frame bounds* of the irregular Weyl-Heisenberg wave packet frame $\{D_{a_j} T_{bk} E_{c_m} \psi\}_{j,k,m\in\mathbb{Z}}$, respectively . If upper inequality in (3) is satisfied, then $\{D_{a_j} T_{bk} E_{c_m} \psi\}_{j,k,m\in\mathbb{Z}}$ is called the *wave packet Bessel sequence for $L^2(\mathbb{R})$* with Bessel bound b_0.

Example 2.1. *Let $\psi = \chi_{[0,1]}$ and let $a_j = 2^j,\ j \in \Lambda = \{1, 2, ..., n\}$. Choose $b = 1$ and $c_m = m$, for all $m \in \mathbb{Z}$.*
Then, for $f \in L^2(\mathbb{R})$ we have

$$\begin{aligned} \sum_{j\in\Lambda,k,m\in\mathbb{Z}} |\langle f, D_{a_j} T_{bk} E_{c_m} \psi \rangle|^2 &= \sum_{j\in\Lambda,k,m\in\mathbb{Z}} |\langle f, D_{2^j} T_k E_m \psi \rangle|^2 \\ &= \sum_{j\in\Lambda,k,m\in\mathbb{Z}} |\langle f, D_{2^j} T_k E_m \chi_{[0,1]} \rangle|^2 \\ &= \sum_{j\in\Lambda,k,m\in\mathbb{Z}} |\langle D_{2^{-j}} f, T_k E_m \chi_{[0,1]} \rangle|^2. \end{aligned}$$

Now, $\{E_m T_k \chi_{[0,1]}\}_{m,k\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{R})$ (see [3]*, p. 71). Therefore,*

$$\sum_{j\in\Lambda,k,m\in\mathbb{Z}} |\langle D_{2^{-j}} f, T_k E_m \chi_{[0,1]} \rangle|^2 = \sum_{j\in\Lambda} \|D_{2^{-j}} f\|^2 = n\|f\|^2.$$

Hence, $\left\{D_{a_j} T_{bk} E_{c_m} \psi\right\}_{j\in\Lambda,k,m\in\mathbb{Z}}$ is a wave packet frame for $L^2(\mathbb{R})$.

Example 2.2. *Let $\psi = \chi_{[0,1]}$ and let $a_j = 2^j,\ j \in \mathbb{Z}$. Choose $b = 1$ and $c_m = m$, for all $m \in \mathbb{Z}$.*
Then, $\left\{D_{a_j} T_{bk} E_{c_m} \psi\right\}_{j,k,m\in\mathbb{Z}}$ is not a wave packet frame for $L^2(\mathbb{R})$.

Indeed, choose $f_o = \chi_{[0,1]}$, *we compute*

$$\begin{aligned}\sum_{j,k,m\in\mathbb{Z}} |\langle f_o, D_{a_j}T_{bk}E_{c_m}\psi\rangle|^2 &= \sum_{j,k,m\in\mathbb{Z}} |\langle f_o, D_{2^j}T_kE_m\psi\rangle|^2 \\ &= \sum_{j,k,m\in\mathbb{Z}} |\langle f_o, D_{2^j}T_kE_m\chi_{[0,1]}\rangle|^2 \\ &= \sum_{j,k,m\in\mathbb{Z}} |\langle D_{2^{-j}}f_o, T_kE_m\chi_{[0,1]}\rangle|^2 \\ &= \sum_{j\in\mathbb{Z}} \|D_{2^{-j}}f_o\|^2 > n\|f_o\|^2, \text{ for any } n.\end{aligned}$$

Hence, $\{D_{a_j}T_{bk}E_{c_m}\psi\}_{j,k,m\in\mathbb{Z}}$ *does not satisfy the upper frame condition for* $L^2(\mathbb{R})$.

For any function $\psi \in L^2(\mathbb{R})$, we consider the system of functions $\{\psi_{j,k,m}\}_{j,k,m\in\mathbb{Z}} \subset L^2(\mathbb{R})$ as

$$\{\psi_{j,k,m}(x) := D_{a_j}T_{bk}E_{c_m}\psi(x) : j,k,m \in \mathbb{Z},\ x \in \mathbb{R}\} \tag{4}$$

By taking Fourier transform to (4) we obtain

$$\hat{\psi}_{j,k,m}(\xi) = a_j^{-1/2}\hat{\psi}(a_j^{-1}\xi - c_m)e^{2\pi ikba_j^{-1}\xi}.$$

Thus, by Plancheral theorem, we have

$$c_{j,k,m} = \langle f, \psi_{j,k,m}\rangle = \int_{\mathbb{R}} f(x)\overline{\psi_{j,k,m}}(x)dx, \quad f \in L^2(\mathbb{R}) \tag{5}$$

We say that the system defined in (4) constitutes a wave packet frame for $L^2(\mathbb{R})$ if there exists positive constants C and D such that

$$C\|f\|^2 \le \sum_{j,k,m\in\mathbb{Z}} |\langle f, \psi_{j,k,m}\rangle|^2 \le D\|f\|^2, \text{ for all } f \in L^2(\mathbb{R}).$$

The constants C and D are known respectively as lower and upper frame bounds. If $C = D$, then we call it tight frame. It is called Parseval frame if $C = D = 1$ and in this case, every function $f \in L^2(\mathbb{R})$ can be written as

$$f(x) = \sum_{j,k,m\in\mathbb{Z}} c_{j,k,m}\psi_{j,k,m}(x) \tag{6}$$

where $c_{j,k,m} = \langle f, \psi_{j,k,m}\rangle$ are given by (5), and we say wave packet co-efficient of the series (6).

Definition 2.3. *Let* $M = (\alpha_{p,q,r,j,k,m})$ *be an infinite matrix of real numbers. Then M-transform of* $\{x_{j,k,m}\}_{j,k,m\in\mathbb{Z}}$ *is defined as* $\sum_{j,k,m\in\mathbb{Z}} \alpha_{p,q,r,j,k,m}x_{j,k,m}$.

The following theorem gives sufficient condition for the matrix transform of the sequences of wave packet coefficients to be belongs to C_0.

Theorem 2.1. *Let* $M = (\alpha_{p,q,r,j,k,m})$ *be an infinite matrix whose elements are of the form* $\alpha_{p,q,r,j,k,m} = \langle \psi_{p,q,r}, \psi_{j,k,m}\rangle$ *and if*

(i) $\sum_{j,k,m} \psi_{j,k,m} \int_{\mathbb{R}} f(y)\overline{\psi_{j,k,m}(y)}dy = 1$
(ii) $\lim_{p,q,r\to\infty} \psi_{p,q,r}(x) = 0$.

Then, M-transform of the sequence of wave packet coefficients $\{c_{j,k,m}\}$ *belong to* C_0.

Proof. Since, the elements of an infinite matrix of the type $\langle\psi_{j,k,m}, \psi_{p,q,r}\rangle$ and the wave packet coefficients $\{c_{j,k,m}\}$ are given by equation (5). We have

$$\begin{aligned}\alpha_{p,q,r,j,k,m}c_{j,k,m} &= \langle\psi_{j,k,m}, \psi_{p,q,r}\rangle\langle f, \psi_{j,k,m}\rangle \\ &= \int_{\mathbb{R}} \psi_{j,k,m}(x)\overline{\psi_{p,q,r}}(x)dx \int_{\mathbb{R}} f(x)\overline{\psi_{j,k,m}}(x)dx \\ &= \int_{\mathbb{R}} f(x)\overline{\psi_{p,q,r}}(x)dx \int_{\mathbb{R}} \psi_{j,k,m}(x)\overline{\psi_{j,k,m}}(x)dx.\end{aligned}$$

Therefore,

$$\sum_{j,k,m\in\mathbb{Z}} \alpha_{p,q,r,j,k,m}c_{j,k,m} = \sum_{j,k,m\in\mathbb{Z}} \int_{\mathbb{R}}\int_{\mathbb{R}} \psi_{p,q,r}(x)\overline{\psi_{j,k,m}}(x)f(y)\overline{\psi_{j,k,m}}(y)dxdy.$$

Using conditions (i) and (ii), we have

$$\lim_{p,q,r\to\infty} \sum_{j,k,m\in\mathbb{Z}} \alpha_{p,q,r,j,k,m}c_{j,k,m} = \lim_{p,q,r\to\infty} \int_{\mathbb{R}} \psi_{p,q,r}(x)dx = 0.$$

This completes the proof. □

The following theorem gives sufficient condition for M-transform of $f \in L^2(\mathbb{R})$ to be satisfy frames condition.

Theorem 2.2. *Let $M = (\alpha_{p,q,r,j,k,m})$ be a non-negative infinite matrix with $\sum_{p,q,r\in\mathbb{Z}} \|\psi_{p,q,r}\|^2 = 1$ and if $c_{j,k,m}$ are the wave packet coefficient associated with the wave packet expansion (6). Then, the frame condition for M-transform of $f \in L^2(\mathbb{R})$ is given by*

$$C_\psi\|f\|^2 \le \sum_{p,q,r\in\mathbb{Z}} |\langle Mf, \psi_{p,q,r}\rangle|^2 \le D_\psi\|f\|^2,$$

Where Mf is the M-transform of $f \in L^2(\mathbb{R})$ and $0 < C_\psi \le D_\psi < \infty$.

Proof. We have, $f(x) = \sum_{j,k,m\in\mathbb{Z}}\langle f, \psi_{j,k,m}\rangle\psi_{j,k,m}(x)$.

Taking M-transform of f, we get

$$Mf(x) = \sum_{p,q,r\in\mathbb{Z}} \langle Mf, \psi_{p,q,r}\rangle\psi_{p,q,r}(x).$$

Therefore,

$$\begin{aligned}\sum_{p,q,r\in\mathbb{Z}} |\langle Mf, \psi_{p,q,r}\rangle|^2 &\le \sum_{p,q,r\in\mathbb{Z}} \int_{\mathbb{R}} |Mf(x)|^2|\overline{\psi_{p,q,r}(x)}|^2dx \\ &\le \|M\|^2\|f\|^2 \sum_{p,q,r\in\mathbb{Z}} \|\psi_{p,q,r}\|^2.\end{aligned}$$

Thus, we have

$$\sum_{p,q,r\in\mathbb{Z}} |\langle Mf, \psi_{p,q,r}\rangle|^2 \le D_\psi\|f\|^2, \tag{7}$$

where D_ψ is a positive constant.

For any, $f \in L^2(\mathbb{R})$, we define

$$g(x) = \left[\sum_{p,q,r\in\mathbb{Z}} |\langle Mf, \psi_{p,q,r}\rangle|^2\right]^{-\frac{1}{2}} f(x).$$

Clearly,

$$\langle Mg, \psi_{p,q,r}\rangle = \left[\sum_{p,q,r\in\mathbb{Z}} |\langle Mf, \psi_{p,q,r}\rangle|^2\right]^{-\frac{1}{2}} \langle Mf, \psi_{p,q,r}\rangle.$$

Hence,

$$\sum_{p,q,r\in\mathbb{Z}} |\langle Mf, \psi_{p,q,r}\rangle|^2 \leq 1.$$

Now, if there exits a constant $K > 0$ such that $\|Mg\|^2 \leq K$, then

$$\left[\sum_{p,q,r\in\mathbb{Z}} |\langle Mf, \psi_{p,q,r}\rangle|^2\right]^{-1} \|f\|^2 \leq \frac{M}{\|A\|^2} = C_\psi > 0.$$

Thus,

$$C_\psi \|f\|_2^2 \leq \sum_{p,q,r\in\mathbb{Z}} |\langle Mf, \psi_{p,q,r}\rangle|^2. \tag{8}$$

Using (7) and (8), we have

$$C_\psi \|f\|^2 \leq \sum_{p,q,r\in\mathbb{Z}} |\langle Mf, \psi_{p,q,r}\rangle|^2 \leq D_\psi \|f\|^2 .$$

This completes the proof. □

The following theorem gives matrix transform of wave packet coefficients in terms of series.

Theorem 2.3. *If* $\{c_{j,k,m}\}_{j,k,m\in\mathbb{Z}}$ *are the wave packet coefficients of* $f \in L^2(\mathbb{R})$ *with* $\|\psi_{p,q,r}\| = 1$*. Then*

$$d_{p,q,r} = \sum_{j,k,m\in\mathbb{Z}} \alpha_{p,q,r,j,k,m} c_{j,k,m},$$

where, $\{d_{p,q,r}\}$ *is defined as the* M*-transform of* $c_{j,k,m}$*.*

Proof. By taking M-transform of equation (5), we have

$$\begin{aligned}\langle Mf, \psi_{j,k,m}\rangle &= \int_{\mathbb{R}} Mf(x)\overline{\psi_{j,k,m}(x)}dx \\ &= \int_{\mathbb{R}} \sum_{j,k,m\in\mathbb{Z}} \alpha_{p,q,r,j,k,m} c_{j,k,m} \psi_{j,k,m}(x)\overline{\psi_{p,q,r}(x)}dx.\end{aligned}$$

Thus,

$$\begin{aligned}\sum_{p,q,r\in\mathbb{Z}} \langle Mf, \psi_{p,q,r}\rangle \psi_{p,q,r}(x) &= \sum_{p,q,r\in\mathbb{Z}} \int_{\mathbb{R}} \sum_{j,k,m\in\mathbb{Z}} \alpha_{p,q,r,j,k,m} c_{j,k,m} \psi_{p,q,r}(x)|\overline{\psi_{p,q,r}(x)}|^2 dx \\ &= \sum_{p,q,r\in\mathbb{Z}} d_{p,q,r}\psi_{p,q,r}(x)\|\psi_{p,q,r}(x)\|^2 dx \\ &= \sum_{p,q,r\in\mathbb{Z}} d_{p,q,r}\psi_{p,q,r}(x).\end{aligned}$$

Hence, $d_{p,q,r} = \langle Mf, \psi_{p,q,r}\rangle$ are the wave packet coefficients of Mf. □

Finally, the authors have given sufficient condition for the sequences $\{\psi_{j,k,m}\}$ to be a tight wave packet frames for $L^2(\mathbb{R})$ in Theorem 2.4.

Theorem 2.4. *Let $M = (\alpha_{p,q,r,j,k,m})$ be an infinite matrix whose elements are $\langle \psi_{j,k,m}, \psi_{p,q,r}\rangle$ and let $\sum_{p,q,r\in\mathbb{Z}} |\hat{\psi}(a_p^{-1}\xi - c_r)|^2 = 1,$ a.e. $\xi \in \mathbb{R}$. Then*

$$\sum_{p,q,r} |d_{p,q,r}|^2 = \sum_{p,q,r\in\mathbb{Z}} |\langle f, \psi_{j,k,m}\rangle|^2 = \frac{1}{b}\|f\|^2.$$

where $d_{p,q,r} = \langle f, \psi_{p,q,r}\rangle$ is the M-transform of the wave packet coefficients $c_{j,k,m}$.

Proof. We compute,

$$\begin{aligned}
\alpha_{p,q,r,j,k,m}c_{j,k,m} &= \langle \psi_{j,k,m}, \psi_{p,q,r}\rangle\langle f, \psi_{j,k,m}\rangle \\
&= \int_{\mathbb{R}} \psi_{j,k,m}(x)\overline{\psi_{p,q,r}}(x)dx \int_{\mathbb{R}} f(x)\overline{\psi_{j,k,m}}(x)dx \\
&= \int_{\mathbb{R}} f(x)\overline{\psi_{p,q,r}}(x)dx \int_{\mathbb{R}} |\psi_{j,k,m}(x)|^2dx \\
&= \int_{\mathbb{R}} f(x)\overline{\psi_{p,q,r}}(x)dx \\
&= \langle f, \psi_{p,q,r}\rangle \\
&= d_{p,q,r}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{p,q,r\in\mathbb{Z}} |d_{p,q,r}|^2 &= \sum_{p,q,r\in\mathbb{Z}} |\alpha_{p,q,r,j,k,m}c_{j,k,m}|^2 \\
&= \sum_{p,q,r\in\mathbb{Z}} |\langle f, \psi_{p,q,r}\rangle|^2 \\
&= \sum_{p,q,r\in\mathbb{Z}} |\langle \hat{f}, \hat{\psi}_{p,q,r}\rangle|^2 \\
&= \sum_{p,q,r\in\mathbb{Z}} |\langle \hat{f}, D_{a_p^{-1}}E_{-bq}T_{c_r}\hat{\psi}\rangle|^2 \\
&= \sum_{p,q,r\in\mathbb{Z}} |\langle \hat{f}, E_{-bqa_p^{-1}}D_{a_p^{-1}}T_{c_r}\hat{\psi}\rangle|^2 \\
&= \sum_{p\in\mathbb{Z}} a_p^{-1} \sum_{r\in\mathbb{Z}}\sum_{q\in\mathbb{Z}} |\int_{\mathbb{R}} \hat{f}(\xi)\overline{\hat{\psi}(a_p^{-1}\xi - c_r)}e^{2\pi iqba_p^{-1}\xi}d\xi|^2 \\
&= \sum_{p\in\mathbb{Z}} \frac{1}{b} \sum_{r\in\mathbb{Z}} \int_0^{\frac{a_p}{b}} |\sum_{l\in\mathbb{Z}} \hat{f}(\xi - \frac{a_pl}{b})\overline{\hat{\psi}(a_p^{-1}\xi - \frac{l}{b} - c_r)}d\xi|^2 \\
&= \frac{1}{b}\sum_{p,r\in\mathbb{Z}} \int_0^{\frac{a_p}{b}} \left(\sum_{l\in\mathbb{Z}} \hat{f}(\xi - \frac{a_pl}{b})\overline{\hat{\psi}(a_p^{-1}\xi - \frac{l}{b} - c_r)}\right)\left(\sum_{l'\in\mathbb{Z}} \overline{\hat{f}(\xi - \frac{a_pl'}{b})}\hat{\psi}(a_p^{-1}\xi - \frac{l'}{b} - c_r)\right)d\xi.
\end{aligned}$$

Let $F(\xi) = \sum_{l' \in \mathbb{Z}} \overline{\hat{f}(\xi - \frac{a_p l'}{b})} \hat{\psi}(a_p^{-1}\xi - \frac{l'}{b} - c_r)$.
Therefore,

$$\begin{aligned}
\sum_{p,q,r\in\mathbb{Z}} |d_{p,q,r}|^2 &= \sum_{p,q,r\in\mathbb{Z}} |\langle f, \psi_{p,q,r}\rangle|^2 \\
&= \frac{1}{b}\sum_{p,r\in\mathbb{Z}} \int_0^{\frac{a_p}{b}} \sum_{l\in\mathbb{Z}} \hat{f}(\xi - \frac{a_p l}{b})\overline{\hat{\psi}(a_p^{-1}\xi - \frac{l}{b} - c_r)} F(\xi) d\xi \\
&= \frac{1}{b}\sum_{p,r\in\mathbb{Z}} \int_{\mathbb{R}} \hat{f}(\xi)\overline{\hat{\psi}(a_p^{-1}\xi - c_r)} F(\xi) d\xi \\
&= \frac{1}{b}\sum_{p,r\in\mathbb{Z}} \left(\sum_{l'\in\mathbb{Z}} \int_{\mathbb{R}} \hat{f}(\xi)\overline{\hat{\psi}(a_p^{-1}\xi - c_r)}\overline{\hat{f}(\xi - \frac{a_p l'}{b})}\hat{\psi}(a_p^{-1}\xi - \frac{l'}{b} - c_r) d\xi \right) \\
&= \frac{1}{b}\sum_{p,r\in\mathbb{Z}} \int_{\mathbb{R}} \hat{f}(\xi)\overline{\hat{f}(\xi)}\hat{\psi}(a_p^{-1}\xi - c_r)\overline{\hat{\psi}(a_p^{-1}\xi - c_r)} d\xi \\
&= \frac{1}{b}\int_{\mathbb{R}} |\hat{f}(\xi)|^2 \sum_{p,r\in\mathbb{Z}} |\hat{\psi}(a_p^{-1}\xi - c_r)|^2 d\xi \\
&= \frac{1}{b}\|f\|^2.
\end{aligned}$$

This completes the proof.

□

Acknowledgement The authors would like to express their sincere thanks to anonymous referee for valuable constructive comments, suggestions which improved this paper.

References

[1] Casazza,P.G. and Kutyniok,G., (2012), Finite frames: Theory and Applications, Birkhäuser.
[2] Christensen,O. and Rahimi,A., (2008), Frame properties of wave packet systems in $L^2(\mathbb{R})$, Adv. Comput. Math., 29, pp.101-111.
[3] Christensen,O., (2008), Frames and Bases, An Introductory Course, Birkhäuser, Boston.
[4] Cordoba,A. and Fefferman,C., (1978), Wave packets and Fourier integral operators, Comm. Partial Differential Equations, 3, 11, pp.979-1005.
[5] Czaja,W., Kutyniok,G., and Speegle,D., (2006), The Geometry of sets of prameters of wave packets, Appl. Comput. Harmon. Anal., 20, pp.108-125.
[6] Duffin,R.J. and Schaeffer,A.C., (1952), A class of nonharmonic Fourier series, Trans. Amer. Math. Soc., 72, pp.341-366.
[7] Heil,C., (2011), A Basis Theory Primer, Expanded edition, Applied and Numerical Harmonic Analysis, Birkhäuser/Springer, New York.
[8] Hernández,E., Labate,D., Weiss,G.,and Wilson,E., (2004), Oversampling, quasi-affine frames and wave packets, Appl. Comput. Harmon. Anal., 16, pp.111-147.
[9] Labate,D., Weiss,G.,and Wilson,E., (2004), An approach to the study of wave packet systems, Contemp. Math., 345, pp.215-235.
[10] Sah,A.K.,and Vashisht,L.K., (2015), Irregular Weyl-Heisenberg wave packet frames in $L^2(\mathbb{R})$, Bull. Sci. Math., 139, pp.61-74.
[11] Young,R., (2001), An introduction to nonharmonic Fourier series, Academic Press, New York.

Raj Kumar got his M.Sc. degree in Mathematics from IIT Delhi, Delhi. He received his Ph.D. degree in mathematics from University of Delhi, Delhi. Currently, he is a senior assistant professor in the Department of Mathematics at the Kirori Mal College, University of Delhi, Delhi. He has given talks at national/international workshops/conferences. His area of research is Theory of Frames for Hilbert and Banach spaces and Wavelets.

Ashok Kumar Sah graduated from L. S. College, B.R.A. Bihar University, Muzaffarpur, in 2007. He got his M.Sc. degree in mathematics from University of Delhi, Delhi, in 2010. He received his M.Phil degree in mathematics from University of Delhi, Delhi, in 2012. Currently, he is an assistant professor in the Department of Mathematics at the Kirori Mal College, University of Delhi, Delhi. He has given talks at national and international workshops/conferences. His area of research are Theory of Frames, wavelets, signal processing.