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GENERALIZED WEIGHTED ČEBYSEV AND OSTROWSKI TYPE INEQUALITIES FOR DOUBLE INTEGRALS

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Abstract. In this paper, we firstly establish generalized weighted Montgomery identity for double integrals. Then, some generalized weighted Čebysev and Ostrowski type inequalities for double integrals are given.

Keywords: Čebysev type inequalities, Ostrowski type inequalities, weighted integral inequalities.

AMS Subject Classification: 26D07, 26D10, 26D15, 26A33

1. Introduction

Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) whoose derivative $f':(a,b)\to\mathbb{R}$ is baunded on (a,b), i.e. $\|f'\|_\infty:=\sup\limits_{t\in(a,b)}|f'(t)|<\infty$. Then we have

$$\left|f(x)-\frac{1}{b-a}\int\limits_a^b f(t)dt\right|\leq\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f'\|_\infty,$$

for all $x\in[a,b]$. The constant $\frac{1}{4}$ is the best possible [10]. This inequality is well known in the literature as the Ostrowski inequality. For some results which generalize, improve and extend the inequality (1), see ([2], [5], [18], [19], [21]) and the references therein.

In [4], P. L. Čebysev proved the following important integral inequality

$$|T(f,g)|\leq\frac{1}{12}(b-a)^2\|f'\|_\infty\|g'\|_\infty \tag{1}$$

where $f,g:[a,b]\to\mathbb{R}$ are absolutely continuous functions whose derivatives $f',g'\in L_\infty[a,b]$ and

$$T(f,g)=\frac{1}{b-a}\int\limits_a^b f(x)g(x)dx-\left(\frac{1}{b-a}\int\limits_a^b f(x)dx\right)\left(\frac{1}{b-a}\int\limits_a^b g(x)dx\right) \tag{2}$$

which is called the Čebysev functional, provided the integrals in (2) exist. In recent years many researchers have given the generalization of Čebysev type inequalities, we can mention the works ([1], [3], [6], [9], [12], [13], [14], [16], [20]).

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Let $w_1 : [a,b] \to [0,\infty)$ be a weight function. We define $m_1(a,b) = \int_a^b w_1(s)ds$ and $m_1(a,t) = \int_a^t w_1(s)ds$, so that $m_1(a,t) = 0$ for $t < a$.

In [13], Rafiq et al. proved the following weighted Montgomery's indentity:

Let $f : [a,b] \to \mathbb{R}$ be absolutely continuous, $\varphi_1 : \mathbb{R}_+ \to \mathbb{R}_+$ be a differentiable function on $\mathbb{R}_+$ with $\varphi_1(0) = 0$, $\varphi_1(m_1(a,b)) \neq 0$ and φ_1' is integrable on $\mathbb{R}_+$, then

$$f(x) = \frac{1}{\varphi_1(m_1(a,b))} \int_a^b w_1(t)\varphi_1'(m_1(a,t))f(t)dt + \frac{1}{\varphi_1(m_1(a,b))} \int_a^b P_{w_1,\varphi_1}(x,t)f'(t)dt \tag{3}$$

for all $x \in [a,b]$, where

$$P_{w_1,\varphi_1}(x,t) = \begin{cases} \varphi_1(m_1(a,t)), & a \leq t \leq x \\ \varphi_1(m_1(a,t)) - \varphi_1(m_1(a,b)), & x \leq t \leq b. \end{cases} \tag{4}$$

Recently, many authors have studied on Čebysev inequality for double integrals, please see ([7], [8], [11] [15]). In [8], Guazene-Lakoud and Aissaoui established a weighted Čebysev type inequality for double integrals using the probability density function. In this paper, we obtain a generalized weighted Čebysev type inequality similar to this inequality for double integrals using the weighted funtions which are not necessarily the probability density functions. Moreover, we established an Ostrowski type inequality for double integral which is the generalization of the inequality given in [17].

2. Generalized Weighted Montgomery Identity for Double Integrals

In order to prove our main theorems, we need to prove following identities:

Let $w_2 : [c,d] \to [0,\infty)$ be a weight function. We define $m_2(c,d) = \int_c^d w_2(u)du$ and $m_2(c,s) = \int_c^s w_2(u)du$, so that $m_2(c,s)$ for $s < c$. $\varphi_2 : \mathbb{R}_+ \to \mathbb{R}_+$ be a differentiable function on $\mathbb{R}_+$ with $\varphi_2(0) = 0$, $\varphi_2(m_2(c,d)) \neq 0$ and φ_2' is integrable on $\mathbb{R}_+$.

Theorem 2.1. *Let $f : \Delta = [a,b] \times [c,d] \to \mathbb{R}$ be a partial differentiable function such that second derivative $\frac{\partial^2 f(t,s)}{\partial s \partial t}$ is integrable on Δ. Then for all $(x,y) \in \Delta$ we have*

$$\begin{aligned} f(x,y) &= \frac{1}{\varphi_1(m_1(a,b))\varphi_2(m_2(c,d))} \\ &\quad \times \left[\int_a^b \int_c^d w_1(t)w_2(s)\varphi_1'(m_1(a,t))\varphi_2'(m_2(c,s))f(t,s)dsdt \right. \\ &\quad + \int_a^b \int_c^d w_1(t)\varphi_1'(m_1(a,t))Q_{w_2,\varphi_2}(y,s)\frac{\partial f(t,s)}{\partial s}dsdt \end{aligned} \tag{5}$$

$$+\int_a^b\int_c^d w_2(s)\varphi_2'(m_2(c,s))P_{w_1,\varphi_1}(x,t)\frac{\partial f(t,s)}{\partial t}dsdt$$
$$\left.+\int_a^b\int_c^d P_{w_1,\varphi_1}(x,t)Q_{w_2,\varphi_2}(y,s)\frac{\partial^2 f(t,s)}{\partial s\partial t}dsdt\right]$$

where $P_{w_1,\varphi_1}(x,t)$ is defined as in (4) and $Q_{w_2,\varphi_2}(y,s)$ defined by

$$Q_{w_2,\varphi_2}(y,s)=\begin{cases}\varphi_2(m_2(c,s)), & c\le s\le y\\ \varphi_2(m_2(c,s))-\varphi_2(m_2(c,d)), & y\le s\le d.\end{cases} \tag{6}$$

Proof. Aplying the identity (3) for the partial derivative $\frac{\partial f(t,y)}{\partial t}$, we have

$$f(x,y) = \frac{1}{\varphi(m(a,b))}\int_a^b w_1(t)\varphi_1'(m_1(a,t))f(t,y)dt \tag{7}$$
$$+\frac{1}{\varphi_1(m_1(a,b))}\int_a^b P_{w_1,\varphi_1}(x,t)\frac{\partial f(t,y)}{\partial t}dt$$

for all $(x,y)\in\Delta$. Similarly, applying the identity (3) for the partial derivative $\frac{\partial f(t,s)}{\partial s}$, we get

$$f(t,y) = \frac{1}{\varphi_2(m_2(c,d))}\int_c^d w_2(s)\varphi_2'(m_2(c,s))f(t,s)ds \tag{8}$$
$$+\frac{1}{\varphi_2(m_2(c,d))}\int_c^d Q_{w_2,\varphi_2}(y,s)\frac{\partial f(t,s)}{\partial s}ds$$

for all $(t,y)\in\Delta$. For partial derivative of (8) according to t, we have

$$\frac{\partial f(t,y)}{\partial t} = \frac{1}{\varphi_2(m_2(c,d))}\int_c^d w_2(s)\varphi_2'(m_2(c,s))\frac{\partial f(t,s)}{\partial t}ds \tag{9}$$
$$+\frac{1}{\varphi_2(m_2(c,d))}\int_c^d Q_{w_2,\varphi_2}(y,s)\frac{\partial^2 f(t,s)}{\partial s\partial t}ds$$

for all $(t,y)\in\Delta$. If we subsitute the equalities (8) and (9) in (7), then we obtain the required result. □

Remark 2.1. *If we choose $\varphi_1(u)\equiv\varphi_2(u)\equiv u$ in the Theorem 2.1, then the Theorem 2.1 reduces the Lemma 2 in* [17].

Theorem 2.2. *Let $f:\Delta\to\mathbb{R}$ be a partial differentiable function such that second derivative $\frac{\partial^2 f(t,s)}{\partial s\partial t}$ is integrable on Δ. Then we have the following generalized weighted Montgomery's identity,*

$$\begin{aligned}
& f(x,y)-\frac{1}{\varphi_1(m_1(a,b))}\int_a^b w_1(t)\varphi_1'(m_1(a,t))f(t,y)dt \qquad (10)\\
& -\frac{1}{\varphi_2(m_2(c,d))}\int_c^d w_2(s)\varphi_2'(m_2(c,s))f(x,s)ds\\
& +\frac{1}{\varphi_1(m_1(a,b))\varphi_2(m_2(c,d))}\int_a^b\int_c^d w_1(t)w_2(s)\varphi_1'(m_1(a,t))\varphi_2'(m_2(c,s))f(t,s)dsdt\\
= & \frac{1}{\varphi_1(m_1(a,b))\varphi_2(m_2(c,d))}\int_a^b\int_c^d P_{w_1,\varphi_1}(x,t)Q_{w_2,\varphi_2}(y,s)\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt
\end{aligned}$$

for all $(x,y)\in\Delta$.

Proof. Using the integration by parts we have,

$$\begin{aligned}
& \int_a^b\int_c^d P_{w_1,\varphi_1}(x,t)Q_{w_2,\varphi_2}(y,s)\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt \qquad (11)\\
= & \int_a^b P_{w_1,\varphi_1}(x,t)\left[\int_c^d \varphi_2(m_2(c,s))\frac{\partial^2 f(t,s)}{\partial t\partial s}ds-\varphi_2(m_2(c,d))\int_c^d\frac{\partial^2 f(t,s)}{\partial t\partial s}ds\right]dt\\
= & \int_a^b P_{w_1,\varphi_1}(x,t)\left[\varphi_2(m_2(c,d))\frac{\partial f(t,y)}{\partial t}-\int_c^d w_2(s)\varphi_2'(m_2(c,s))\frac{\partial f(t,s)}{\partial t}ds\right]dt\\
= & \varphi_2(m_2(c,d))\int_a^b P_{w_1,\varphi_1}(x,t)\frac{\partial f(t,y)}{\partial t}dt\\
& -\int_c^d\int_a^b w_2(s)\varphi_2'(m_2(c,s))P_{w_1,\varphi_1}(x,t)\frac{\partial f(t,s)}{\partial t}dtds.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
& \int_a^b P_{w_1,\varphi_1}(x,t)\frac{\partial f(t,y)}{\partial t}dt \qquad (12)\\
= & \int_a^b \varphi_1(m_1(a,t))\frac{\partial f(t,y)}{\partial t}dt-\varphi_1(m_1(a,b))\int_x^b\frac{\partial f(t,y)}{\partial t}dt\\
= & \varphi_1(m_1(a,b))f(x,y)-\int_a^b w_1(t)\varphi_1'(m_1(a,t))f(t,y)dt,
\end{aligned}$$

and

$$\int_c^d \int_a^b w_2(s)\varphi_2'(m_2(c,s))P_{w_1,\varphi_1}(x,t)\frac{\partial f(t,s)}{\partial t}dtds \tag{13}$$

$$= \int_c^d w_2(s)\varphi_2'(m_2(c,s))\left[\int_a^b \varphi_1(m_1(a,t))\frac{\partial f(t,s)}{\partial t}dt - \varphi_1(m(a,b))\int_x^b \frac{\partial f(t,s)}{\partial t}dt\right]ds$$

$$= \int_c^d w_2(s)\varphi_2'(m_2(c,s))\left[\varphi(m_1(a,b))f(x,s) - \int_a^b w_1(t)\varphi_1'(m_1(a,t))f(t,s)dt\right]ds$$

$$= \varphi_1(m_1(a,b))\int_c^d w_2(s)\varphi_2'(m_2(c,s))f(x,s)ds$$

$$-\int_a^b \int_c^d w_1(t)w_2(s)\varphi_1'(m_1(a,t))\varphi_2'(m_2(c,s))f(t,s)dt.$$

If we subsitute the equalities (12) and (13) in (11), then we obtain the required identity (10). □

Remark 2.2. *If we take w_1 and w_2 as two probability density functions in (10), then the identity (10) reduces the identity (6) in* [8].

3. New Generalized Weighted Ostrowski and Čebysev Inequalities

Theorem 3.1. *Let $f : \Delta \to \mathbb{R}$ have continuous partial derivatives $\frac{\partial f(t,s)}{\partial t}$, $\frac{\partial f(t,s)}{\partial s}$ and $\frac{\partial^2 f(t,s)}{\partial s \partial t}$ on Δ. Then we have the following weighted Ostrowski inequality*

$$\left| f(x,y) - \frac{1}{\varphi_1(m_1(a,b))\varphi_2(m_2(c,d))}\int_a^b \int_c^d w_1(t)w_2(s)\varphi_1'(m_1(a,t))\varphi_2'(m_2(c,s))f(t,s)dsdt \right|$$

$$\leq \frac{1}{\varphi_1(m_1(a,b))\varphi_2(m_2(c,d))}\left[m_1(a,b)\left\|\frac{\partial f(t,s)}{\partial s}\right\|_\infty \|\varphi_1'\|_\infty H_2(y)\right.$$

$$\left.+m_2(c,d)\left\|\frac{\partial f(t,s)}{\partial t}\right\|_\infty \|\varphi_2'\|_\infty H_1(x) + \left\|\frac{\partial f(t,s)}{\partial t}\right\|_\infty H_1(x)H_2(y)\right]$$

where

$$H_1(x) = \int_a^b |P_{w_1,\varphi_1}(x,t)|\,dt,$$

and

$$H_2(y) = \int_c^d |Q_{w_2,\varphi_2}(y,s)|\,ds.$$

Proof. Taking modulus in Theorem 2.1, we have

$$\left| f(x,y) - \frac{1}{\varphi_1(m_1(a,b))\varphi_2(m_2(c,d))} \int_a^b \int_c^d w_1(t)w_2(s)\varphi_1'(m_1(a,t))\varphi_2'(m_2(c,s))f(t,s)dsdt \right|$$

$$\leq \frac{1}{\varphi_1(m_1(a,b))\varphi_2(m_2(c,d))} \left[\int_a^b \int_c^d w_1(t) \left|\varphi_1'(m_1(a,t))\right| \left|Q_{w_2,\varphi_2}(y,s)\right| \left|\frac{\partial f(t,s)}{\partial s}\right| dsdt \right.$$

$$+ \int_a^b \int_c^d w_2(s) \left|\varphi_2'(m_2(c,s))\right| \left|P_{w_1,\varphi_1}(x,t)\right| \left|\frac{\partial f(t,s)}{\partial t}\right| dsdt$$

$$\left. + \int_a^b \int_c^d \left|P_{w_1,\varphi_1}(x,t)\right| \left|Q_{w_2,\varphi_2}(y,s)\right| \left|\frac{\partial^2 f(t,s)}{\partial s \partial t}\right| dsdt \right]$$

$$\leq \frac{1}{\varphi_1(m_1(a,b))\varphi_2(m_2(c,d))} \left[\left\|\frac{\partial f(t,s)}{\partial s}\right\|_\infty \left\|\varphi_1'\right\|_\infty \int_a^b \int_c^d w_1(t) \left|Q_{w_2,\varphi_2}(y,s)\right| dsdt \right.$$

$$+ \left\|\frac{\partial f(t,s)}{\partial t}\right\|_\infty \left\|\varphi_2'\right\|_\infty \int_a^b \int_c^d w_2(s) \left|P_{w_1,\varphi_1}(x,t)\right| dsdt$$

$$\left. + \left\|\frac{\partial f(t,s)}{\partial t}\right\|_\infty \int_a^b \int_c^d \left|P_{w_1,\varphi_1}(x,t)\right| \left|Q_{w_2,\varphi_2}(y,s)\right| dsdt \right].$$

Here, we have the equalities

$$\int_a^b \int_c^d w_1(t) \left|Q_{w_2,\varphi_2}(y,s)\right| dsdt = \left(\int_a^b w_1(t)dt \right) \left(\int_c^d \left|Q_{w_2,\varphi_2}(y,s)\right| ds \right) = m_1(a,b)H_2(y),$$

$$\int_a^b \int_c^d w_2(s) \left|P_{w_1,\varphi_1}(x,t)\right| dsdt = \left(\int_c^d w_2(s)ds \right) \left(\int_a^b \left|P_{w_1,\varphi_1}(x,t)\right| dt \right) = m_2(c,d)H_1(x)$$

and

$$\int_a^b \int_c^d \left|P_{w_1,\varphi_1}(x,t)\right| \left|Q_{w_2,\varphi_2}(y,s)\right| dsdt$$

$$= \left(\int_a^b \left|P_{w_1,\varphi_1}(x,t)\right| dt \right) \left(\int_c^d \left|Q_{w_2,\varphi_2}(y,s)\right| ds \right) = H_1(x)H_2(y)$$

which complete the proof. □

Remark 3.1. *If we choose $\varphi_1(u) \equiv \varphi_2(u) \equiv u$ in the Theorem 3.1, then the Theorem 3.1 reduces the Theorem 2 in* [17].

Theorem 3.2. *Let $f, g : \Delta \to \mathbb{R}$ be partial differentiable functions such that their second derivatives $\frac{\partial^2 f(t,s)}{\partial s \partial t}$ and $\frac{\partial^2 g(t,s)}{\partial s \partial t}$ are integrable on Δ. Then we have the weighted Čebysev* *inequality*

$$
\begin{aligned}
& |T(w_1,\varphi_1,w_2,\varphi_2,f,g)| \qquad (14)\\
\leq\; & \frac{1}{\varphi_1^3(m_1(a,b))\varphi_2^3(m_2(c,d))}\left\|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right\|_\infty\left\|\frac{\partial^2 g(t,s)}{\partial t\partial s}\right\|_\infty\|\varphi_1'\|_\infty\|\varphi_2'\|_\infty\\
& \times\int_a^b\int_c^d w_1(x)w_2(y)H_1^2(x)H_2^2(y)dydx
\end{aligned}
$$

where $H_1(x)$ and $H_2(y)$ are defined as in (3.1) and

$$
\begin{aligned}
& T(w_1,\varphi_1,w_2,\varphi_2,f,g)\\
=\; & \frac{1}{\varphi_1(m_1(a,b))\varphi_2(m_2(c,d))}\\
& \times\left[\int_a^b\int_c^d w_1(x)w_2(y)\varphi_1'(m_1(a,x))\varphi_2'(m_2(c,y))f(x,y)g(x,y)dydx\right.\\
& -\frac{1}{\varphi_2(m_2(c,d))}\int_a^b\int_c^d w_1(x)w_2(y)\varphi_1'(m_1(a,x))\varphi_2'(m_2(c,y))f(x,y)\\
& \times\left(\int_c^d w_2(s)\varphi_2'(m_2(c,s))g(x,s)ds\right)dydx\\
& -\frac{1}{\varphi_1(m_1(a,b))}\int_a^b\int_c^d w_1(x)w_2(y)\varphi_1'(m_1(a,x))\varphi_2'(m_2(c,y))g(x,y)\\
& \times\left(\int_a^b w_1(t)\varphi_1'(m_1(a,t))f(t,y)dt\right)dydx\\
& +\frac{1}{\varphi_1(m_1(a,b))\varphi_2(m_2(c,d))}\\
& \times\left(\int_a^b\int_c^d w_1(x)w_2(y)\varphi_1'(m_1(a,x))\varphi_2'(m_2(c,y))f(x,y)dydx\right)\\
& \left.\times\left(\int_a^b\int_c^d w_1(x)w_2(y)\varphi_1'(m_1(a,x))\varphi_2'(m_2(c,y))g(x,y)dydx\right)\right].
\end{aligned}
$$

Proof. From Theorem 2.2, writing again the identity (10) for the function $g(x,y)$, we have

$$\begin{aligned} & g(x,y)-\frac{1}{\varphi_1(m_1(a,b))}\int_a^b w_1(t)\varphi_1'(m_1(a,t))g(t,y)dt \\ & -\frac{1}{\varphi_2(m_2(c,d))}\int_c^d w_2(s)\varphi_2'(m_2(c,s))g(x,s)ds \\ & +\frac{1}{\varphi_1(m_1(a,b))\varphi_2(m_2(c,d))}\int_a^b\int_c^d w_1(t)w_2(s)\varphi_1'(m_1(a,t))\varphi_2'(m_2(c,s))g(t,s)dsdt \\ = & \frac{1}{\varphi_1(m_1(a,b))\varphi_2(m_2(c,d))}\int_a^b\int_c^d P_{w_1,\varphi_1}(x,t)Q_{w_2,\varphi_2}(y,s)\frac{\partial^2 g(t,s)}{\partial t\partial s}dsdt. \end{aligned} \tag{15}$$

After multiplying the identities (10) and (15), multiplying both sides result by $\frac{w_1(x)w_2(y)\varphi_1'(m_1(a,x))\varphi_2'(m_2(c,y))}{\varphi_1(m_1(a,b))\varphi_2(m_2(c,d))}$ and integrating over Δ, we have

$$\begin{aligned} & T(w_1,\varphi_1,w_2,\varphi_2,f,g) \\ = & \frac{1}{\varphi_1^3(m_1(a,b))\varphi_2^3(m_2(c,d))}\int_a^b\int_c^d\left[\int_a^b\int_c^d P_{w_1,\varphi_1}(x,t)Q_{w_2,\varphi_2}(y,s)\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt\right] \\ & \times\left[\int_a^b\int_c^d P_{w_1,\varphi_1}(x,t)Q_{w_2,\varphi_2}(y,s)\frac{\partial^2 g(t,s)}{\partial t\partial s}dsdt\right]dydx. \end{aligned} \tag{16}$$

Taking the modulus in (16), we obtain

$$\begin{aligned} & |T(w_1,\varphi_1,w_2,\varphi_2,f,g)| \\ \leq & \frac{1}{\varphi_1^3(m_1(a,b))\varphi_2^3(m_2(c,d))}\int_a^b\int_c^d w_1(x)w_2(y)\varphi_1'(m_1(a,x))\varphi_2'(m_2(c,y)) \\ & \times\left[\int_a^b\int_c^d |P_{w_1,\varphi_1}(x,t)Q_{w_2,\varphi_2}(y,s)|\left|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right|dsdt\right] \\ & \times\left[\int_a^b\int_c^d |P_{w_1,\varphi_1}(x,t)Q_{w_2,\varphi_2}(y,s)|\left|\frac{\partial^2 g(t,s)}{\partial t\partial s}\right|dsdt\right]dydx \\ \leq & \frac{1}{\varphi_1^3(m_1(a,b))\varphi_2^3(m_2(c,d))}\left\|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right\|_\infty\left\|\frac{\partial^2 g(t,s)}{\partial t\partial s}\right\|_\infty\left\|\varphi_1'\right\|_\infty\left\|\varphi_2'\right\|_\infty \\ & \times\int_a^b\int_c^d w_1(x)w_2(y)\left[\int_a^b\int_c^d |P_{w_1,\varphi_1}(x,t)Q_{w_2,\varphi_2}(y,s)|\,dsdt\right]^2 dydx \end{aligned}$$

$$= \frac{1}{\varphi_1^3(m_1(a,b))\varphi_2^3(m_2(c,d))} \left\| \frac{\partial^2 f(t,s)}{\partial t \partial s} \right\|_\infty \left\| \frac{\partial^2 g(t,s)}{\partial t \partial s} \right\|_\infty \|\varphi_1'\|_\infty \|\varphi_2'\|_\infty$$

$$\times \int_a^b \int_c^d w_1(x) w_2(y) H_1^2(x) H_2^2(y) dy dx.$$

This completes the proof. □

Remark 3.2. *If we take w_1 and w_2 as two probability density functions in (14), then the identity (14) reduces the identity (14) in* [8].

4. Conclusions

In this study, we presented some Čebysev and Ostrowski type inequalities generalized weighted Montgomery identity. A further study could assess similar inequalities by using different types of generalized weighted Montgomery identity.

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