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Research Article

Numbers with empty rational Korselt sets

Nejib Ghanmi

Preparatory Institute of Engineering Studies of Tunis, Tunis University, Tunisia

Abstract

Let N be a positive integer, and $\alpha = \dfrac{\alpha_1}{\alpha_2} \in \mathbb{Q} \setminus \{0, N\}$ with $\gcd(\alpha_1, \alpha_2) = 1$. N is called an α-Korselt number, equivalently α is said an N-Korselt base, if $\alpha_2 p - \alpha_1$ divides $\alpha_2 N - \alpha_1$ for every prime divisor p of N. The set of N-Korselt bases in $\mathbb{Q}$ is denoted by $\mathbb{Q}\text{-}\mathcal{KS}(N)$ and called the set of rational Korselt bases of N.

In this paper rational Korselt bases are deeply studied, where we give in details their belonging sets and their forms in some cases. This allows us to deduce that for each integer $n \geq 3$, there exist infinitely many squarefree composite numbers N with n prime factors and empty rational Korselt sets.

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1. Introduction

As known, for $\alpha \in \mathbb{Z} \setminus \{0\}$, an integer $N \in \mathbb{N} \setminus \{0, 1, \alpha\}$ is an α-Korselt number if $p - \alpha$ divides $N - \alpha$ for all prime divisors p of N. Korselt numbres are considred by Bouallègue-Echi-Pinch [1] as a natural generalization of Carmichael numbers which are simply characterized by the Korselt criterion as follows.

Korselt's criterion 1.1 ([8]). A squarefree composite integer $N > 1$ is a Carmichael number if and only if $p - 1$ divides $N - 1$ for all prime factors p of N.

Carmichael numbers are exactly the 1-Korselt squarefree composite numbers. Since α-Korselt numbers for $\alpha \in \mathbb{Z}$ (or simply Korselt numbers) were introduced, they have been the subject of intensive study, one may find more details in [1, 2, 6, 7]. Motivated by these facts, Ghanmi [3] introduced the notion of $\mathbb{Q}$-Korselt numbers as extension of Korselt numbers to $\mathbb{Q}$ by setting the following definitions.

Definition 1.2. Let $N \in \mathbb{N} \setminus \{0, 1\}$, $\alpha = \dfrac{\alpha_1}{\alpha_2} \in \mathbb{Q} \setminus \{0\}$ with $\gcd(\alpha_1, \alpha_2) = 1$. Then

(1) N is said to be an *α-Korselt number* (*K_α-number*), if $N \neq \alpha$ and $\alpha_2 p - \alpha_1$ divides $\alpha_2 N - \alpha_1$ for every prime divisor p of N.
(2) By the $\mathbb{Q}$-*Korselt set* of the number N (or the Korselt set of N over $\mathbb{Q}$), we mean the set $\mathbb{Q}\text{-}\mathcal{KS}(N)$ of all $\beta \in \mathbb{Q} \setminus \{0, N\}$ such that N is a K_β-number.

Email addresses: nejib.ghanmi@ipein.rnu.tn , neghanmi@yahoo.fr
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(3) The cardinality of $\mathbb{Q}$-$\mathcal{KS}(N)$ will be called the $\mathbb{Q}$-*Korselt weight* of N; we denote it by $\mathbb{Q}$-$\mathcal{KW}(N)$.

Further, in [4] the author state the notion of Korselt bases as follows.

Definition 1.3. Let $N \in \mathbb{N} \setminus \{0, 1\}$, $\alpha \in \mathbb{Q}$ and $\mathbb{B}$ be a subset of $\mathbb{N}$. Then

(1) α is called an *N-Korselt base* (*K_N-base*), if N is a *K_α-number.*

(2) By the $\mathbb{B}$-*Korselt set* of the base α (or the Korselt set of the base α over $\mathbb{B}$), we mean the set $\mathbb{B}$-$\mathcal{KS}(B(\alpha))$ of all $M \in \mathbb{B}$ such that α is a K_M-base.

(3) The cardinality of $\mathbb{B}$-$\mathcal{KS}(B(\alpha))$ will be called the $\mathbb{B}$-*Korselt weight* of the base α; we denote it by $\mathbb{B}$-$\mathcal{KW}(B(\alpha))$.

The set $\mathbb{Q}$-$\mathcal{KS}(N)$ is simply called the rational Korselt set of N. In this paper we are concerned only with squarefree composite numbers.

It's clear that every nonzero positive integer has finitely many Korselt bases over $\mathbb{Q}$ (see [3, Theorem 2.3]), hence a natural question can be posed about the existence of such numbers with empty rational Korselt set and how many there are. Obviously, this cannot happen for $N = pq$; numbers with two distinct primes factors because $q + p - 1$ lies always in $\mathbb{Q}$-$\mathcal{KS}(N)$. However, when N has more than three prime factors, the answer is affirmative. Moreover, we show in this work that for each integer $n \geq 3$, there exist infinitely many squarefree composite numbers N with exactly n prime factors and empty rational Korselt sets.

We give a brief description of the content of the paper. In Section 2, we provide in details the belonging set of a rational korselt base of a number N, and in some cases we give explicitly its general form. Consequently, in Section 3, we deduce that for each integer $n \geq 3$, there exist infinitely many squarefree composite numbers N with n prime factors and empty rational Korselt sets.

In the rest of this paper, we consider for $n \geq 2$ and $p_1 < p_2 < \ldots < p_n$ prime numbers, the number $N = p_1 p_2 \ldots p_n$ and $\alpha = \dfrac{\alpha_1}{\alpha_2} \in \mathbb{Q}$, where we assume, without loss of generality, that $\alpha_2 > 0$, $\alpha_1 \in \mathbb{Z}$ and $\gcd(\alpha_1, \alpha_2) = 1$. Let $\gcd(\alpha_1, N) = Q$, $\alpha_1 = \alpha_1^{''} Q$ and $P = \dfrac{N}{Q}$. If $Q \neq 1$, we write $Q = \prod_{i=1}^{m} q_i$, where q_i are prime numbers and $q_i < q_j$ for each $i < j$. For $I = \{i_1, i_2, ..., i_s\} \subseteq \{1, 2, ..., m\}$, we set $Q_{i_1 i_2 \ldots i_s} = \dfrac{Q}{\prod_{i \in I} q_i}$.

As known, $\alpha = \dfrac{\alpha_1}{\alpha_2} \in \mathbb{Q}$-$\mathcal{KS}(N)$ is equivalent to

$$\alpha_2 p_i - \alpha_1 \mid \alpha_2 N - \alpha_1; \quad \forall p_i \mid N. \tag{1.1}$$

This allows us to determine easily, with a simple Maple program, the rational Korselt set of any given positive integer N. For example:

$$\mathbb{Q}\text{-}\mathcal{KS}(30) = \left\{4, 6, \frac{15}{8}, \frac{40}{13}, \frac{5}{2}, \frac{10}{3}, \frac{15}{4}, \frac{24}{5}\right\}. \tag{1.2}$$

$$\mathbb{Q}\text{-}\mathcal{KS}(105) = \left\{6, 9, \frac{126}{25}, \frac{35}{6}, \frac{90}{13}, \frac{21}{5}, \frac{35}{12}\right\}. \tag{1.3}$$

2. Korselt rational base properties

We begin by giving the belonging set of the N-Korselt bases in $\mathbb{Q}$ when N is not dividing α_1.

Proposition 2.1. *Let* $\alpha = \dfrac{\alpha_1}{\alpha_2} \in \mathbb{Q}$-$\mathcal{KS}(N)$ *and* p, q *two prime factors of* N*. Then the following properties hold.*

(1) $\alpha < N$.

(2) *If* $\gcd(q, \alpha_1) = 1$ *then* $-\dfrac{N}{q} + q + 1 \leq \alpha \leq \dfrac{N}{q} + q - 1$.

(3) *Suppose that* q *divides* α_1. *If* $\gcd(p, \alpha_1) = 1$ *and* $(N, \alpha) \neq (pq, \dfrac{2qp}{q+1})$ *then* $\dfrac{2pq - N}{2q - 1} \leq \alpha \leq \dfrac{2pq + N}{2q + 1}$.

Proof.

(1) Suppose $N < \alpha$. Then $0 < \alpha - N < \alpha - p_n$, and consequently $0 < |k| = \dfrac{\alpha - N}{\alpha - p_n} < 1$, contradicting $k \in \mathbb{Z}$. Hence,

$$\alpha < N. \tag{2.1}$$

(2) Since $\alpha_2 N - \alpha_1 = \alpha_2 p \left(\dfrac{N}{p} - 1\right) + (\alpha_2 p - \alpha_1)$, $\alpha_2 p - \alpha_1 \mid \dfrac{N}{p} - 1$ by (1.1). Therefore, $-\dfrac{N}{q} + 1 \leq \alpha_2 q - \alpha_1 \leq \dfrac{N}{q} - 1$, so that,

$$1 - \frac{N}{q} \leq \frac{1}{\alpha_2}\left(1 - \frac{N}{q}\right) \leq q - \alpha \leq \frac{1}{\alpha_2}\left(\frac{N}{q} - 1\right) \leq \frac{N}{q} - 1.$$

Consequently, $\dfrac{2pq - N}{2q - 1} \leq \alpha \leq \dfrac{2pq + N}{2q + 1}$.

(3) Let $\alpha_1 = \alpha_1' q$ and $N = pqN_1$. As $\alpha_2 N - \alpha_1 = q(\alpha_2 p N_1 - \alpha_1')$ and $\gcd(\alpha_2 p - \alpha_1, q) = 1$, it follows by (1.1) that $\alpha_2 p - \alpha_1 \mid \alpha_2 p N_1 - \alpha_1'$, hence there exists $k \in \mathbb{Z} \setminus \{0\}$ such that $(\alpha_2 p - \alpha_1)k = \alpha_2 p N_1 - \alpha_1'$. Thus,

$$\alpha_2 p(k - N_1) = \alpha_1'(kq - 1). \tag{2.2}$$

We will prove that $|k| \neq 1$.

• Suppose $k = 1$. Then, by (2.2),

$$\alpha_2 p(1 - N_1) = \alpha_1'(q - 1). \tag{2.3}$$

Consequently, $N_1 \neq 1$ and $\alpha_1' < 0$. Now, since $\gcd(\alpha_2 p, \alpha_1') = 1$, we get $\alpha_2 p \mid q - 1$ by (2.3). Hence,

$$\alpha_2 p < q. \tag{2.4}$$

Let r be a prime factor of N_1. Replacing the factor p of N by r, we deduce, as in the beginning of the proof, that there exists a positive integer l such that $(\alpha_2 r - \alpha_1) = \dfrac{\alpha_2 p N_1 - \alpha_1'}{l}$. Since $r \neq p$, we obtain $l \neq 2$. So, $\alpha_2 r - \alpha_1 \leq \dfrac{\alpha_2 p N_1 - \alpha_1'}{2} = \dfrac{\alpha_2 p - \alpha_1}{2}$ and consequently, $\alpha_2(r - p) \leq \alpha_1 - \alpha_2 r$. As $-\alpha_2 p < \alpha_2(r - p)$ and $\alpha_1 - \alpha_2 r < \alpha_1 < -q$, it follows that $-\alpha_2 p < -q$, so that, $\alpha_2 p > q$, contradicting (2.4).

• Now, suppose $k = -1$. Then, by (2.3),

$$\alpha_2 p(1 + N_1) = \alpha_1'(q + 1). \tag{2.5}$$

Two subcases are to be considered.

(*i*) Assume $N_1 = 1$. Then, $2\alpha_2 p = \alpha_1'(q + 1)$ by (2.5), and so $\alpha = \dfrac{2qp}{q+1}$ which is excluded by hypothesis.

(*ii*) If r is a prime divisor of N_1, then there exists $s \in \mathbb{Z} \setminus \{0\}$ such that

$$\alpha_2 r - \alpha_1 = -\frac{\alpha_2 p N_1 - \alpha_1'}{s}. \tag{2.6}$$

As $r \neq p$, we get $s \neq 1$. Moreover, we claim that $s \geq 3$, indeed if is not true (i.e. $s = 2$), then since $\alpha_2 pN_1 - \alpha_1^{'} > 0$ by (2.1) and $k = -1$,

$$\alpha_2 r - \alpha_1 = -\frac{\alpha_2 pN_1 - \alpha_1^{'}}{2} = \frac{\alpha_2 p - \alpha_1}{2}. \tag{2.7}$$

This implies that $r = \dfrac{p+\alpha}{2}$, so that, $2r > \alpha$. It follows by (2.5) that $\alpha_1^{'}(q+1) = \alpha_2 p(1+N_1) > \alpha_2 pr > \alpha_2 \dfrac{\alpha}{2} p = \alpha_1^{'} \dfrac{qp}{2}$, and so $q(p-2) < 2$. Therefore, as q is prime (hence $q(p-2) \neq 1$), $q(p-2) = 0$ and so $p = 2$. But, as $1 = \gcd(\alpha_1, p) = \gcd(\alpha_1, 2)$ hence α_1 is odd, we should have $\alpha_2 p - \alpha_1 = 2\alpha_2 - \alpha_1$ is also odd, contradicting (2.7).

Now, because $s \geq 3$, $(\alpha_2 r - \alpha_1) \geq -\dfrac{\alpha_2 pN_1 - \alpha_1^{'}}{3} = \dfrac{\alpha_2 p - \alpha_1}{3}$ by (2.6). Therefore, $r \geq \dfrac{p+2\alpha}{3}$, and so $3r > 2\alpha$. Hence, by (2.5), $\alpha_1^{'}(q+1) = \alpha_2 p(1+N_1) > \alpha_2 pr > \alpha_2 \dfrac{2\alpha}{3} p = \alpha_1^{'} \dfrac{2qp}{3}$, and so $q(2p-3) < 3$. This forces $q = p = 2$ as p and q are prime, which is not possible.

So, we conclude that $|k| \neq 1$, hence $-\dfrac{N-\alpha}{2q} \leq p - \alpha \leq \dfrac{N-\alpha}{2q}$. Thus, $\dfrac{2pq-N}{2q-1} \leq \alpha \leq \dfrac{2pq+N}{2q+1}$. □

Remark 2.2.

(1) The optimal bounds for the inequalities in Proposition 2.1(1) are given by letting q be the largest prime divisor of N with $\gcd(q, \alpha_1) = 1$.

(2) For $(N, \alpha) = (pq, \dfrac{2qp}{q+1})$, the inequalities in Proposition 2.1(2) do not hold. For example when $p = 2, q = 5$ we have $N = 2 \times 5$ is a $\dfrac{10}{3}$-Korselt number, however, $\dfrac{2qp+N}{2q+1} = \dfrac{30}{11} < \alpha = \dfrac{10}{3}$.

Now, to prove the next result (Proposition 2.6), we need to state the three following lemmas, whenever $\gcd(\alpha_1, N) = Q = N$. For each $1 \leq i \leq n$, set $r_i = \alpha_2 - Q_i \alpha_1^{''}$. If $Q = N$ (i.e. $P = 1$) then as $\gcd(r_i, Q_i) = 1$ and $q_i r_i = \alpha_2 q_i - \alpha_1 \mid \alpha_2 N - \alpha_1 = Q(\alpha_2 - \alpha_1^{''})$ by (1.1), it follows that $r_i \mid \alpha_2 - \alpha_1^{''}$, and so there exists $m_i \in \mathbb{Z}$ such that

$$m_i r_i = \alpha_2 - \alpha_1^{''}. \tag{2.8}$$

If $m_i < 0$, we set $m_i^{'} = -m_i$.

Lemma 2.3. *Let* $\alpha = \dfrac{\alpha_1}{\alpha_2} \in \mathbb{Q}\text{-}\mathcal{KS}(N)$. *If* N *divides* α_1 *then the following assertions hold.*

(1) $\alpha_2 > \alpha_1^{''} > 0$.
(2) $Q_{ij} \mid m_i - m_j$ *for all* $1 \leq i \neq j \leq n$.
(3) $(r_i)_{1 \leq i \leq n}$ *is increasing and* $(m_i)_i$ *is decreasing on each of sets* $J = \{1 \leq i \leq n;\ \ m_i < 0\}$ *and* $K = \{1 \leq i \leq n;\ \ m_i > 0\}$.

Proof. Assume that $\alpha_1 = \alpha_1^{''} N$ (i.e. $Q = N$).

(1) If we suppose $\alpha_1^{''} < 0$, then $0 < \alpha_2 - \alpha_1^{''} < \alpha_2 - Q_i \alpha_1^{''} = r_i \mid \alpha_2 - \alpha_1^{''}$ which is not possible. So, $\alpha_1^{''} > 0$.

Now, because $\dfrac{\alpha_1^{''} N}{\alpha_2} = \alpha < N$ by (2.1), $\alpha_2 - \alpha_1^{''} > 0$.

(2) Since $m_i r_i = \alpha_2 - \alpha_1^{''}$ by (2.8), it yields that

$$\frac{\alpha_2}{\alpha_1^{''}} = Q_i + \frac{Q_i - 1}{m_i - 1}. \tag{2.9}$$

So, for all $1 \leq i \neq j \leq n$

$$Q_{ij}(q_j - q_i) = Q_i - Q_j = \frac{Q_j - 1}{m_j - 1} - \frac{Q_i - 1}{m_i - 1}, \tag{2.10}$$

hence,

$$Q_i - Q_j = \frac{Q_j - Q_i}{m_j - 1} + (Q_i - 1)\frac{m_i - m_j}{(m_j - 1)(m_i - 1)}. \tag{2.11}$$

As $Q_{ij} \mid Q_i - Q_j$ and $\gcd(Q_{ij}, Q_i - 1) = 1$, it follows that

$$Q_{ij} \mid m_i - m_j.$$

(3) Since $(Q_i)_i$ is decreasing, $(r_i)_i$ is increasing as $\alpha_1^{''} > 0$. Hence, if $i, j \in K$ with $i < j$, then $r_j > r_i > 0$, and so $0 < m_j = \dfrac{\alpha_2 - \alpha_1^{''}}{r_j} < m_i = \dfrac{\alpha_2 - \alpha_1^{''}}{r_i}$. Hence, $(m_i)_{i\in K}$ is decreasing. Similarly, if $i, j \in J$ with $i < j$, then $r_i < r_j < 0$, hence $m_j = \dfrac{\alpha_2 - \alpha_1^{''}}{r_j} < m_i = \dfrac{\alpha_2 - \alpha_1^{''}}{r_i} < 0$. Thus $(m_i)_{i\in J}$ is decreasing. □

Now, in addition to that N divides α_1, we suppose in the next two Lemmas that $n \geq 3$. This yields to the existence of $1 \leq k < l < s \leq n$ such that $q_k q_l q_s \mid Q = N$.

Lemma 2.4. *Let* $\alpha = \dfrac{\alpha_1}{\alpha_2} \in \mathbb{Q}\text{-}\mathcal{KS}(N)$. *Suppose that N divides α_1 and $n \geq 3$. Then*

(1) *If $m_l > m_k^{'} > m_s > 0$, then we have the following.*
  (i) $q_l = m_s + m_k^{'}$ *and so* $N = q_k q_l q_s$.
  (ii) $q_l - 2 = q_k \geq 3$

(3) *If $m_l^{'} > m_s > m_k^{'} > 0$, then we get the following.*
  (i) $q_l = m_s + m_k^{'}$ *and so* $N = q_k q_l q_s$.
  (ii) $m_k^{'} + 3 \leq m_s$.
  (iii) $q_s = q_l + 2$.

Proof. Let the integers $1 \leq k < l < s \leq n$ be such that $q_k q_l q_s \mid Q = N$.

(1) Assume that $m_l > m_k^{'} > m_s > 0$.

(*i*) By (2.10)

$$Q_{kl}(q_l - q_k) < \frac{Q_l - 1}{m_k^{'} - 1} + \frac{Q_k - 1}{m_k^{'} + 1} = \frac{m_k^{'}(Q_l + Q_k) + Q_l - Q_k}{(m_k^{'})^2 - 1}.$$

Multiplying by $(m_k^{'})^2 - 1$ and dividing by Q_{kl}, it follows that

$$((m_k^{'})^2 - 1)(q_l - q_k) < m_k^{'}(q_k + q_l) + q_k - q_l.$$

Therefore,

$$q_l - q_k < \frac{2q_l}{m_k^{'} + 1}. \tag{2.12}$$

As $q_l \mid Q_{ks}$ and $Q_{ks} \mid m_s + m_k^{'}$ by Lemma 2.3 (2), we should have $q_l = Q_{ks} = m_s + m_k^{'}$. Indeed, if is not the case, we get $q_l \leq \dfrac{m_s + m_k^{'}}{2} < m_k^{'}$, hence $q_l - q_k < \dfrac{2q_l}{m_k^{'} + 1} < 2$ by (2.12) and so $q_l = 3$, $q_k = 2$ and $N = 6q_s$. Moreover, as $2(q_s - 3) = Q_l - Q_s < \dfrac{Q_s - 1}{m_s - 1} = \dfrac{6}{m_s - 1}$, necessarily $q_s = 5$ and so $N = 30$. So, $\alpha = \dfrac{30}{\alpha_2} \in \mathbb{Q}\text{-}\mathcal{KS}(30)$ with $\gcd(\alpha_2, 30) = 1$. This is not true by (1.2). Thus, $q_l = m_s + m_k^{'}$ and $N = q_k q_l q_s$.

(*ii*) As $q_l = Q_{ks} = m_s + m_k^{'} \leq 2m_k^{'}$, (2.12) implies that $q_l - q_k < 4$. Hence $q_l = q_k + 2$ with $q_k \neq 2$ since q_l is prime.

(3) Now, suppose that $m_l^{'} > m_s > m_k^{'} > 0$.

(i) By (2.10)

$$Q_{ls}(q_s - q_l) < \frac{Q_s - 1}{m_s - 1} + \frac{Q_l - 1}{m_s + 1} = \frac{m_s(Q_s + Q_l) + Q_s - Q_l}{m_s^2 - 1}.$$

Multiplying by $m_s^2 - 1$ and dividing by Q_{ls}, we get

$$(m_s^2 - 1)(q_s - q_l) < m_s(q_l + q_s) + q_l - q_s.$$

Hence,

$$q_s - q_l < \frac{2q_l}{m_s - 1}. \tag{2.13}$$

As $q_l \mid Q_{ks}$ and $Q_{ks} \mid m_s + m_k^{'}$ by Lemma 2.3 (2) , we claim that $q_l = m_s + m_k^{'}$. Indeed, if is not the case, then $q_l \leq \frac{m_s + m_k^{'}}{2} \leq m_s - 1$, therefore, $2 \leq q_s - q_l < \frac{2q_l}{m_s - 1} \leq 2$ by (2.13), which is not true. So, $q_l = m_s + m_k^{'}$ and $N = q_k q_l q_s$.

(ii) Now, let us show that the case $m_k^{'} = m_s - 1$ does not hold. Suppose the contrary, then $q_l = m_s + m_k^{'} = 2m_s - 1$ by $3(i)$. This implies by (2.10) that

$$Q_{kl}(q_l - q_k) = \frac{Q_k - 1}{m_k^{'} + 1} - \frac{Q_l - 1}{m_l^{'} + 1} < \frac{Q_k - 1}{m_k^{'} + 1}.$$

Therefore, $q_l - q_k < \frac{q_l}{m_k^{'} + 1} = \frac{2m_s - 1}{m_s} < 2$, which forces $q_l = 3$, $q_k = 2$, $m_s = 2$ and $N = 6q_s$. Hence, $\frac{\alpha_2}{\alpha_1^{''}} = Q_s + \frac{Q_s - 1}{m_s - 1} = 6 + 5 = 11$, by (2.9), and so $\alpha_2 = 11$ and $\alpha_1^{''} = 1$. This implies by (2.8) that $m_l^{'}(2q_l - 11) = 10$, hence $m_l^{'}$ is even and as $m_l^{'} > m_s = 2$, we should have $m_l^{'} \geq 4$. Therefore, $2q_l - 11 \leq \frac{10}{4}$ which yields to $q_l = 5$ and $N = 30$, which is not true by (1.2). Thus, $m_k^{'} < m_s - 1$. Now, since $q_l = m_s + m_k^{'}$ is prime, then $m_k^{'} \neq m_s - 2$. Consequently, $m_k^{'} + 3 \leq m_s$.

(iii) As $q_l = m_s + m_k^{'} < 2m_s - 2$, then $q_s = q_l + 2$ by (2.13). □

Lemma 2.5. *Let $\alpha = \frac{\alpha_1}{\alpha_2} \in \mathbb{Q}\text{-}\mathcal{KS}(N)$ such that N divides α_1 and $n \geq 3$. Then*

(1) $\alpha_1^{''} = 1$.

(2) *If $m_l > m_k^{'} > m_s > 0$, then there exists $\beta_{ls} \in \mathbb{Z}$ such that $m_s \mid m_l$ and $m_l \mid \beta_{ls} r_s$.*

(3) *If $m_l^{'} > m_s > m_k^{'} > 0$, then there exists $\beta_{kl} \in \mathbb{Z}$ such that $m_k^{'} \mid m_l^{'}$ and $m_l^{'} \mid \beta_{kl} r_k$.*

Proof. (1) For $i \neq j$, $m_i r_i = m_j r_j = \alpha_2 - \alpha_1^{''}$ by (2.8). Hence,

$$(\alpha_2 - \alpha_1^{''})\left(\frac{1}{m_j} - \frac{1}{m_i}\right) = r_j - r_i,$$

which implies that

$$(\alpha_2 - \alpha_1^{''})\left(\frac{m_i - m_j}{m_i m_j}\right) = \alpha_1^{''}(Q_i - Q_j).$$

By setting $m_i - m_j = \beta_{ij} Q_{ij}$, it follows that

$$(\alpha_2 - \alpha_1^{''})\beta_{ij} = \alpha_1^{''} m_i m_j (q_j - q_i). \tag{2.14}$$

Now, since $q_l = Q_{ks} = m_s - m_k$ by Lemma 2.4, then $\beta_{sk} = 1$ and so $\alpha_2 - \alpha_1^{''} = \alpha_1^{''} m_s m_k^{'}(q_s - q_k)$ by (2.14). Hence, $\alpha_1^{''} = 1$ as $\gcd(\alpha_1^{''}, \alpha_2) = 1$.

(2) Assume that $m_l > m_k^{'} > m_s > 0$. First, since $\beta_{sk} = 1$, we have

$$\alpha_2 - 1 = m_s m_k^{'}(q_s - q_k). \tag{2.15}$$

and (2.14) becomes

$$(\alpha_2 - 1)\beta_{ij} = m_i m_j (q_j - q_i). \tag{2.16}$$

As $q_l - 2 = q_k$ by Lemma 2.4, we get by (2.16)

$$(\alpha_2 - 1)\beta_{kl} = 2m_k^{'} m_l. \tag{2.17}$$

Therefore, $\beta_{kl} m_s \dfrac{q_s - q_k}{2} = m_l$ by (2.15), and so $m_s \mid m_l$.

Now, since $(\alpha_2 - 1)\beta_{ls} = m_s m_l (q_s - q_l)$ by (2.16) and $\alpha_2 - 1 = m_s r_s$, it follows that $r_s \beta_{ls} = m_l (q_s - q_l)$ which means that $m_l \mid \beta_{ls} r_s$.

(3) Now, suppose that $m_l^{'} > m_s > m_k^{'} > 0$. As $q_s = q_l + 2$ by Lemma 2.4, then $(\alpha_2 - 1)\beta_{sl} = 2m_l^{'} m_s$ by (2.16). Therefore, $\beta_{sl} m_k^{'} \dfrac{q_s - q_k}{2} = m_l^{'}$ by (2.15), and so $m_k^{'} \mid m_l^{'}$. Similarly, since $\alpha_2 - 1 = m_k r_k$, $(\alpha_2 - 1)\beta_{kl} = m_k^{'} m_l^{'} (q_l - q_k)$ by (2.16). Hence, $r_k \beta_{kl} = m_l^{'}(q_k - q_l)$ and so $m_l^{'} \mid \beta_{kl} r_k$. □

Proposition 2.6. *Let* $\alpha = \dfrac{\alpha_1}{\alpha_2} \in \mathbb{Q}\text{-}\mathcal{KS}(N)$ *and* n *(resp.,* n_1*) be the number of prime factors of* N *(resp.,* P*). Then*

(1) $0 \le n_1 \le n$.
(2) *If* $n \ge 3$ *then* $1 \le n_1 \le n$.

Proof.
(1) Straightforward.
(2) Suppose $N = Q = \gcd(\alpha_1, N)$ (i.e. $n_1 = 0$) and $n \ge 3$. Depending on $m_i^{'}$ and the signs of the integers r_i for $i \in \{k, l, s\}$, we consider the following cases.

(a) Suppose that $0 < r_k < r_l < r_s$. Then, $Q_{ls} \mid m_l - m_s$ by Lemma 2.3(2) and so $Q_{ls} \le m_l - 1$. Hence, $q_s < Q_{kl}(q_l - q_k) < \dfrac{Q_l - 1}{m_l - 1} < \dfrac{Q_l}{Q_{ls}} = q_s$ by (2.10), which is impossible.

(b) If $r_k < r_l < r_s < 0$ (i.e. $m_s^{'} > m_l^{'} > m_k^{'} > 0$), then by(2.10)

$$Q_l - Q_s = \frac{Q_l - 1}{m_l^{'} + 1} - \frac{Q_s - 1}{m_s^{'} + 1} < \frac{Q_l - 1}{m_l^{'} + 1}.$$

As $Q_{kl} < m_l^{'} + 1$ by Lemma 2.3 (2) , it follows that $q_k < Q_{ls}(q_s - q_l) < \dfrac{Q_l - 1}{m_l^{'} + 1} < \dfrac{Q_l}{Q_{kl}} = q_k$, which is not true.

(c) Now, assume that $r_k < 0 < r_l < r_s$ (i.e. $m_l > m_s > 0$ and $m_k^{'} > 0$). Then the following subcases are to be discussed.

(i) If $m_k^{'} > m_l > m_s > 0$, then by (2.11)

$$Q_{kl}(q_l - q_k) < \frac{Q_l - 1}{m_l - 1} + \frac{Q_k - 1}{m_l + 1} = \frac{m_l(Q_l + Q_k) + Q_l - Q_k}{m_l^2 - 1}.$$

Multiplying by $m_l^2 - 1$ and dividing by Q_{kl}, we obtain

$$(m_l^2 - 1)(q_l - q_k) < m_l(q_k + q_l) + q_k - q_l.$$

As $Q_{ls} \le m_l - 1$ by Lemma 2.3 (2) , it follows that $q_l - q_k < \dfrac{2q_k}{Q_{ls}}$. Therefore, $q_k(q_l - q_k) \le Q_{ls}(q_l - q_k) < 2q_k$, which forces $q_l = 3$, $q_k = 2$ and so $N = 6q_s$. Hence, $2(q_s - 3) = Q_l - Q_s < \dfrac{Q_s - 1}{m_s - 1} = \dfrac{6}{m_s - 1}$ by (2.10), which yields to $q_s = 5$ and so $N = 30$. Impossible by (1.2).

(ii) Assume that $m_l > m_k^{'} > m_s > 0$. Since $m_s \mid m_l$ and $m_l \mid \beta_{ls} r_s = \beta_{ls}(\alpha_2 - 1) + \beta_{ls}(1 - Q_s)$ by Lemma 2.5 and $m_l \mid \alpha_2 - 1$ by (2.17), it follows that $m_l \mid \beta_{ls}(Q_s - 1) = (\beta_{ls} q_k + m_s) q_l - m_s q_l - \beta_{ls}$. Therefore $m_l \mid m_s q_l + \beta_{ls}$, as $m_l = \beta_{ls} q_k + m_s$. Since $m_s \mid m_l$, it follows that $m_s \mid \beta_{ls}$ and we can write

$$\frac{m_l}{m_s} \mid q_l + \frac{\beta_{ls}}{m_s} = q_k + 2 + \frac{\beta_{ls}}{m_s}.$$

As $\dfrac{m_l}{m_s} = 1 + \dfrac{\beta_{ls}}{m_s} q_k$, it follows that

$$1 + \frac{\beta_{ls}}{m_s} q_k \mid q_k + 2 + \frac{\beta_{ls}}{m_s}. \tag{2.18}$$

Hence, $1 + \dfrac{\beta_{ls}}{m_s} q_k \leq q_k + 2 + \dfrac{\beta_{ls}}{m_s}$, and so $\left(\dfrac{\beta_{ls}}{m_s} - 1\right)(q_k - 1) \leq 2$. This implies as $q_k \geq 3$, that ($q_k = 3$ and $\dfrac{\beta_{ls}}{m_s} = 2$) or $\dfrac{\beta_{ls}}{m_s} = 1$.

• If $\dfrac{\beta_{ls}}{m_s} = 1$ then $1 + q_k = 1 + \dfrac{\beta_{ls}}{m_s} q_k \mid q_k + 2 + \dfrac{\beta_{ls}}{m_s} = q_k + 3$ by (2.18), hence $q_k = 1$. Impossible.

• If $q_k = 3$ and $\dfrac{\beta_{ls}}{m_s} = 2$, then $q_l = q_k + 2 = 5$. But, since by (2.10) $3(q_s - 5) = Q_l - Q_s < \dfrac{Q_s - 1}{m_s - 1} = \dfrac{14}{m_s - 1}$, it follows that $q_s = 7$ and so $N = 105$ which is not true by (1.3).

(iii) If $m_l > m_s > m_k^{'} > 0$, then as $Q_{ks} \leq 2m_s$ by Lemma 2.3 (2) , we get $2q_k < Q_l - Q_s < \dfrac{2Q_s}{Q_{ks}} = 2q_k$ by (2.10), which is impossible.

(d) If $r_k < r_l < 0 < r_s$ (i.e. $m_l^{'} > m_k^{'} > 0$ and $m_s > 0$), then the following subcases are to be discussed.

(i) Suppose that $m_s > m_l^{'} > m_k^{'} > 0$. Then by (2.10)

$$Q_{ls}(q_s - q_l) < \frac{Q_s - Q_l}{m_s - 1} + Q_l \frac{(m_l^{'} + m_s - 1)}{m_l^{'}(m_s - 1)} < Q_l \frac{(m_l^{'} + m_s - 1)}{m_l^{'}(m_s - 1)} < \frac{2Q_l}{m_l^{'}}.$$

As $Q_{kl} < m_l^{'}$ by Lemma 2.3 (2) , it follows that $2q_k \leq Q_{ls}(q_s - q_l) < \dfrac{2Q_l}{Q_{kl}} = 2q_k$, which is impossible.

(ii) Assume that $m_l^{'} > m_s > m_k^{'} > 0$. As $m_k^{'} \mid m_l^{'}$ and $m_l^{'} \mid \beta_{kl} r_k = \beta_{kl}(\alpha_2 - 1) + \beta_{kl}(1 - Q_k)$ by Lemma 2.5 and $m_l^{'} \mid \alpha_2 - 1$ by (2.17), it follows that $m_l^{'} \mid \beta_{kl}(Q_k - 1) = (\beta_{kl} q_s + m_k^{'}) q_l - m_k^{'} q_l - \beta_{kl}$. Hence, as $m_l^{'} = \beta_{kl} q_s + m_k^{'}$, we get $m_l^{'} \mid m_k^{'} q_l + \beta_{kl}$. Since $m_k^{'} \mid m_l^{'}$, it follows that $m_k^{'} \mid \beta_{kl}$ and we can write

$$\frac{m_l^{'}}{m_k^{'}} \mid q_l + \frac{\beta_{kl}}{m_k^{'}} = q_s - 2 + \frac{\beta_{kl}}{m_k^{'}}. \tag{2.19}$$

Since $\dfrac{m_l^{'}}{m_k^{'}} = 1 + \dfrac{\beta_{kl}}{m_k^{'}} q_s$, it follows that $1 + \dfrac{\beta_{kl}}{m_k^{'}} q_s \leq q_s - 2 + \dfrac{\beta_{kl}}{m_k^{'}}$, and so $\left(\dfrac{\beta_{kl}}{m_k^{'}} - 1\right)(q_s - 1) \leq -2$. This is impossible since $q_s \geq 5$ and $\dfrac{\beta_{kl}}{m_k^{'}} \geq 1$.

(iii) If $m_l^{'} > m_k^{'} > m_s > 0$, then by (2.10)

$$Q_{kl}(q_l - q_k) < \frac{Q_k - 1}{m_k^{'} + 1} < \frac{Q_k}{m_k^{'}}.$$

As, $q_l < 2m_k'$ by Lemma 2.3 (2) , it follows that $q_l - q_k < \frac{q_l}{m_k'} < 2$. Hence, $q_k = 2$, $q_l = 3$ and so $N = 6q_s$. Therefore, by (2.9) we get

$$Q_l - \frac{Q_l - 1}{m_l' + 1} = Q_s + \frac{Q_s - 1}{m_s - 1} = 6 + \frac{5}{m_s - 1} \leq 11.$$

As $m_l' \geq 4$, it follows that $2q_s Q_{kls} = Q_l < \frac{55}{4}$, hence $q_s = 5$ and so $Q_{kls} = 1$. So $N = 30$, which is not true by (1.2).

Finally, as all cases lead to an absurdity, we conclude that $n_1 \geq 1$. □

Example 2.7. By this example, we show that the inequality bound values given in Proposition 2.6 are all attained. Let us conserve the same notation as in Proposition 2.6.

(1) If $N = 2 \times 3$, then $N \in \mathcal{KS}\left(B\left(\frac{12}{5}\right)\right)$ with $Q = N$ and so $n_1 = 0$.

Also, $N = 2 \times 3 \in \mathcal{KS}\left(B\left(\frac{5}{2}\right)\right)$ with $Q = 1$ and so $n_1 = 2$.

(2) For $N = 2 \times 3 \times 5$, $N \in \mathcal{KS}\left(B\left(\frac{5}{2}\right)\right)$ with $Q = 5$ and so $n_1 = 1$.

Let $N = 2 \times 5 \times 11$. $N \in \mathcal{KS}\left(B\left(\frac{13}{2}\right)\right)$ with $P = N$ and so $n_1 = n = 3$.

By the next result we provide a generalization of the result giving by Proposition 2.1(2) whenever $\gcd(N, \alpha_1) > 1$.

Proposition 2.8. *Suppose that N is an α-Korselt number and $\gcd(\alpha_1, N) = Q > 1$. If $P \neq 1$, let p be a prime factor of P and $R = \frac{P}{p}$. Then*

(1) *Suppose that $P = 1$ (i.e. $n = 2$ and $Q = N$) and $N = q_1 q_2$ with $q_1 < q_2$. Then we have the following.*
  (*a*) *If $q_2 > 2q_1$, then*

$$\alpha \in \left\{\frac{2N}{3q_1 - 1}, \frac{2N}{q_2 + 1}, \frac{N}{2q_1 - 1}\right\}.$$

  (*b*) *If $q_2 < 2q_1$, then*

$$\alpha \in \left\{\frac{kN}{S}, \frac{(k+1)N}{S}\right\},$$

  for some positive integers S where $k = \left\lfloor \frac{S}{q_1} \right\rfloor$.

(2) *If $P \neq 1$, then $\frac{2Qp - N}{2Q - 1} \leq \alpha \leq \frac{Qp + N}{Q + 1}$.*

(3) *Assume that $P \neq 1$ and $(N, \alpha) \neq \left(pQ, \frac{2pQ}{Q+1}\right)$. Then*
  (*a*) $\frac{2Qp - N}{2Q - 1} \leq \alpha \leq \frac{2Qp + N}{2Q + 1}$.
  (*b*) *Suppose that $n_1 \geq 2$. If p_1 and r_1 are respectively the smallest and the largest prime factors of P, then*

$$\frac{2Qr_1 - N}{2Q - 1} \leq \alpha \leq \frac{2Qp_1 + N}{2Q + 1}.$$

Proof.

(1) First, as $P = 1$ (i.e. $Q = N$), we have $n = 2$ by Proposition 2.6.
  (*a*) See [5, Theorem 3.4].
  (*b*) See [5, Proposition 2.18, Lemmas 2.19 , 2.20].

(2) As $\alpha_2 N-\alpha_1=Q(\alpha_2 P-\alpha_1^{''})$ and $\gcd(\alpha_2 p-\alpha_1,Q)=1$, then $\alpha_2 p-\alpha_1 \mid \alpha_2 P-\alpha_1^{''}$ by (1.1). Hence, there exists $k\in\mathbb{Z}\setminus\{0\}$ such that $(\alpha_2 p-\alpha_1)k=\alpha_2 P-\alpha_1^{''}$. Therefore,

$$\alpha_2 p(k-R)=\alpha_1^{''}(kQ-1). \tag{2.20}$$

Let us prove that $k\neq 1$. Suppose $k=1$, then (2.20) becomes

$$\alpha_2 p(1-R)=\alpha_1^{''}(Q-1). \tag{2.21}$$

Consequently, $R\neq 1$ and $\alpha_1^{''}<0$. As, in addition, $\gcd(\alpha_2 p,\alpha_1^{''})=1$, it follows that $\alpha_2 p \mid Q-1$, hence

$$\alpha_2 p<Q. \tag{2.22}$$

Let r be a prime factor of R. Replacing p by r, we deduce as in the beginning of the proof, that there exists a positive integer l such that $\alpha_2 r-\alpha_1=\dfrac{\alpha_2 pR-\alpha_1^{''}}{l}$. As $r\neq p$, then $l\neq 1$ and so $\alpha_2 r-\alpha_1\leq \dfrac{\alpha_2 pR-\alpha_1^{''}}{2}=\dfrac{\alpha_2 p-\alpha_1}{2}$ which yields to $\alpha_2(r-p)\leq \alpha_1-\alpha_2 r$. As, in addition, $-\alpha_2 p<\alpha_2(r-p)$ and $\alpha_1-\alpha_2 r<\alpha_1<-Q$, it follows that $-\alpha_2 p<-Q$, that is to say $\alpha_2 p>Q$. A contradiction with (2.22). Thus $k\neq 1$, and so

$$-\frac{N-\alpha}{Q}\leq p-\alpha\leq \frac{N-\alpha}{2Q}.$$

Consequently, $\dfrac{2Qp-N}{2Q-1}\leq \alpha\leq \dfrac{Qp+N}{Q+1}$.

(3) (a) Let us show that $k\neq -1$. Suppose $k=-1$, then (2.20) gives

$$\alpha_2 p(1+R)=\alpha_1^{''}(Q+1). \tag{2.23}$$

Two cases are to be considered.

• If $R=1$ then $2\alpha_2 p=\alpha_1^{''}(Q+1)$ by (2.23), and so $\alpha=\dfrac{2Qp}{Q+1}$ which is excluded by hypothesis.

• Suppose that $R\neq 1$. Let r be a prime divisor of R. Then there exists an integer $s\in\mathbb{Z}\setminus\{0\}$ such that

$$\alpha_2 r-\alpha_1=-\frac{\alpha_2 pR-\alpha_1^{''}}{s}. \tag{2.24}$$

As $r\neq p$, it's clear that $s\neq 1$. Moreover, we claim that $s\geq 3$, indeed if is not true (i.e. $s=2$), as $\alpha<N$ by (2.1) hence $\alpha_2 pR-\alpha_1^{''}>0$, then

$$\alpha_2 r-\alpha_1=-\frac{\alpha_2 pR-\alpha_1^{''}}{2}=\frac{\alpha_2 p-\alpha_1}{2}. \tag{2.25}$$

This implies that $r=\dfrac{p+\alpha}{2}$ hence $2r>\alpha$. Therefore, by (2.23) we get $\alpha_1^{''}(Q+1)=\alpha_2 p(1+R)>\alpha_2 pr>\alpha_2\dfrac{\alpha}{2}p=\alpha_1^{''}\dfrac{Qp}{2}$, thus $Q(p-2)<2$. This forces $p=2$. As $1=\gcd(\alpha_1,p)=\gcd(\alpha_1,2)$, then α_1 is odd, hence $\alpha_2 p-\alpha_1=2\alpha_2-\alpha_1$ is also an odd integer, contradicting (2.25).

Now, as $s\geq 3$, $\alpha_2 r-\alpha_1\geq -\dfrac{\alpha_2 pR-\alpha_1^{''}}{3}=\dfrac{\alpha_2 p-\alpha_1}{3}$ by (2.24). This yields to $r\geq \dfrac{p+2\alpha}{3}$, hence $3r>2\alpha$. Therefore, $\alpha_1^{''}(Q+1)=\alpha_2 p(1+R)>\alpha_2 pr>\alpha_2\dfrac{2\alpha}{3}p=\alpha_1^{''}\dfrac{3Qp}{3}$ by (2.23), hence $Q(2p-3)<3$. This forces $Q=p=2$, which is not possible. So, we conclude that $k\neq -1$ and as $k\neq 1$, it follows that

$$-\frac{N-\alpha}{2Q}\leq p-\alpha\leq \frac{N-\alpha}{2Q},$$

which is equivalent to

$$\frac{2Qp - N}{2Q - 1} \le \alpha \le \frac{2Qp + N}{2Q + 1}.$$

(b) Straightforward from (a). □

Remark 2.9.

(1) The optimal bounds for the inequalities in Proposition 2.8(2),(3-a) are given by letting q be the largest prime divisor of N such that $\gcd(q, \alpha_1) = 1$.

(2) The case when $(N, \alpha) = (pQ, \frac{2pQ}{Q+1})$ does not verify the inequalities in Proposition 2.8(3); for instance $N = 2 \times 3 \times 5$ is a $\frac{15}{4}$-Korselt number ($Q = 15, p = 2$), however $\frac{2Qp + N}{2Q + 1} = \frac{90}{31} < \alpha = \frac{15}{4}$.

3. Numbers with empty rational Korselt set

The main result of this work is the following theorem.

Theorem 3.1. *Let $n \ge 2$ be an integer and $p_1 < p_2 < \ldots < p_n$ be fixed distinct prime numbers. Then, there exists an integer q_0 such that, for each prime number $q > q_0$, $N = p_1 p_2 \ldots p_n q$ has an empty Korselt set.*

Proof. Let $N_1 = p_1 p_2 \ldots p_n$, q be a prime number such that $q > p_n$ and $N = N_1 q$. Let $\alpha = \frac{\alpha_1}{\alpha_2} \in \mathbb{Q}\text{-}\mathcal{KS}(N)$, $Q = \gcd(\alpha_1, N)$ and $N = PQ$. Then, we consider two cases:

Case 1. Suppose that q does not divide Q. Then, by Proposition 2.1(1), we have

$$q + 1 - \frac{N}{q} \le \alpha \le \frac{N}{q} + q - 1,$$

hence

$$1 - N_1 = 1 - \frac{N}{q} \le \alpha - q \le \frac{N}{q} - 1 = N_1 - 1.$$

So,

$$\alpha_2(1 - N_1) \le \alpha_2 q - \alpha_1 \le \alpha_2(N_1 - 1). \tag{3.1}$$

Now, by (1.1)

$$\alpha_2 p_j - \alpha_1 \mid \alpha_2 N - \alpha_1 + (\alpha_2 p_j - \alpha_1)(N_1 - 1),$$

therefore,

$$\alpha_2 p_j - \alpha_1 \mid (\alpha_2 q - \alpha_1) N_1 + \alpha_2 p_j (N_1 - 1).$$

This implies that

$$\alpha_1 - \alpha_2 p_j \le |\alpha_2 q - \alpha_1| N_1 + \alpha_2 p_j (N_1 - 1),$$

and so

$$\alpha_2 q \le (\alpha_2 q - \alpha_1) + |\alpha_2 q - \alpha_1| N_1 + \alpha_2 p_j N_1 \le |\alpha_2 q - \alpha_1|(N_1 + 1) + \alpha_2 p_j N_1.$$

Therefore, by (3.1)

$$\alpha_2 q \le \alpha_2 (N_1 - 1)(N_1 + 1) + \alpha_2 p_j N_1,$$

hence,

$$q \le 2N_1^2 - 1. \tag{3.2}$$

Case 2. Suppose that q divides Q. First, since N has more than three prime factors, $Q \ne N$ by Proposition 2.6. Let p_j be such that $\gcd(p_j, \alpha_1) = 1$ and $\alpha_1 = \alpha_1'' Q$; by Proposition 2.8, we have

$$\frac{2Qp_j - N}{2Q - 1} \le \alpha \le \frac{Qp_j + N}{Q + 1}.$$

This gives

$$\frac{2p_j - P}{2Q-1} \leq \frac{\alpha_1''}{\alpha_2} \leq \frac{p_j + P}{Q+1}. \tag{3.3}$$

It follows that if q (hence Q) approaches infinity then $\frac{\alpha_1''}{\alpha_2}$ tends to 0, and so α_2 approaches infinity, since $\alpha_1'' \neq 0$. Therefore $\alpha_2 - \alpha_1''\frac{Q}{q} = \alpha_2(1 - \frac{\alpha_1''}{\alpha_2}\frac{Q}{q})$ tends to infinity. However, since $\alpha_2 q - \alpha_1 = q(\alpha_2 - \alpha_1''\frac{Q}{q})$ and $\alpha_2 N - \alpha_1 = \alpha_2 q\left(\frac{N}{q} - 1\right) + (\alpha_2 q - \alpha_1)$, $\alpha_2 - \alpha_1''\frac{Q}{q}$ divides $\frac{N}{q} - 1$ by (1.1), which is impossible. This leads to the existence of an integer $q_0 > 2N_1^2 - 1$ such that for all $q > q_0$ the inequalities (3.2) and (3.3) do not hold. So, we conclude that for each prime number $q > q_0$, $N = N_1 q$ has an empty Korselt set. □

Now, the next result follows immediately.

Theorem 3.2. *For each integer $n \geq 3$, there exist infinitely many squarefree composite numbers N with n prime factors and empty rational Korselt set.*

Remark 3.3. For each two distinct prime numbers p and q, $N = pq$ has a non empty rational Korselt set since $q + p - 1 \in \mathbb{Q}\text{-}\mathcal{KS}(N)$.

References

[1] K. Bouallegue, O. Echi and R. Pinch, *Korselt Numbers and Sets*, Int. J. Number Theory **6**, 257-269, 2010.

[2] O. Echi and N. Ghanmi, *The Korselt Set of pq*, Int. J. Number Theory, **8** (2), 299-309, 2012.

[3] N. Ghanmi, $\mathbb{Q}$-*Korselt Numbers*, Turkish J. Math. **42**, 2752-2762, 2018.

[4] N. Ghanmi, *Korselt Rationel Bases of Prime Powers*, Studia Sci. Math. Hungar. **56** (4), 388-403, 2019.

[5] N. Ghanmi, *The* $\mathbb{Q}$-*Korselt Set of pq*, Period. Math. Hungar. **81** (2), 174-193, 2020.

[6] N. Ghanmi and I. Al-Rassasi, *On Williams Numbers With Three Prime Factors*, Miss. J. Math. Sc. **25** (2), 134-152, 2013.

[7] N. Ghanmi, O. Echi and I. Al-Rassasi, *The Korselt Set of a Squarefree Composite Number*, Math. Rep. Cand. Aca. Sc. **35** (1), 1-15, 2013.

[8] A. Korselt, *Problème chinois*, L'intermediaire des Mathématiciens **6**, 142–143, 1899.