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ON THE CONSTRUCTION OF GENERAL SOLUTION OF THE GENERALIZED SYLVESTER EQUATION

F.A. ALIEV[1], V.B. LARIN[2], §

Abstract. The problem of construction the general solution of the generalized matrix Sylvester equation is considered. Conditions of existence of solution of this equation are obtained and the algorithm for construction of this solution is given. For construction of the algorithm of this solution and the formulation of the condition of existence of this solution, the standard procedures of MATLAB package are used.

Keywords: generalized matrix Sylvester equation, package of MATLAB, Symbolic Math Toolbox, tensor product, SV Decomposition.

AMS Subject Classification: 15A06,15A24, 15A69.

1. Introduction

At the problems of creating the motion control algorithms of various systems, see for example [1, 2, 5, 6, 8, 10, 12, 13, 19], an important place is occupied by procedures of construction of solutions of different matrix equations, see [3-5, 14, 15] and references therein. It may be noted that the algorithms of construction of solutions of Sylvester equations have attracted the attention of researchers [7, 9, 18, 21]. For example, in [9] the problem of construction of the general solution of a generalized Sylvester equation is considered:

$$\sum_{i=1}^{k} Q_i X R_i + \sum_{i=1}^{\ell} S_i Y T_i = B. \tag{1}$$

Here $X \in R^{\beta\times\gamma}$, $Y \in R^{\mu\times\nu}$, $B \in R^{\alpha\times\delta}$; the other matrices in (1) have corresponding dimensions. In [9] the solvability condition is formed and the algorithm of construction of the general solution of the equation (1) is suggested.

Below these questions are also considered applying to equation (1). However, for the construction of algorithm of solution (1) and formulation of the conditions of existence of the solution the standard procedures of package of MATLAB are used. Thus, calculable procedure is formed so that the used standard procedures of package of MATLAB enter the Symbolic Math Toolbox. For illustration of algorithm an example is considered [9]. In this example the solution is formed which doesn't appear in [9].

[1] Institute of Applied Mathematics, Baku State University, Z.Khalilov str., 23, 1148, Baku, Azerbaijan.
e-mail: f_aliev@yahoo.com;

[2] Institute of Mechanics, Academy of Sciences of Ukraine, Ukraine.
e-mail: larin@gmail.com;

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2. General relations

As is known [16], by the use of Kronecker or tensor product, the equation (1) can be presented [11, 20] as a system of linear algebraic equations :

$$G\begin{bmatrix} x \\ y \end{bmatrix} = b, \tag{2}$$

$$G = \left[\sum_{i=1}^{k} Q_i \otimes R_i' + \sum_{i=1}^{\ell} S_i \otimes T_i'\right],$$

$$x = \begin{bmatrix} x_{1*}' \\ \vdots \\ x_{\beta*}' \end{bmatrix}, y = \begin{bmatrix} y_{1*}' \\ \vdots \\ y_{\mu*}' \end{bmatrix}, b = \begin{bmatrix} b_{1*}' \\ \vdots \\ b_{\alpha*}' \end{bmatrix}.$$

Hereinafter a stroke means the transposition, $\otimes$ is the operation of Kronecker product (procedure kron.m), x_{j*}', y_{j*}', b_{j*}' are the rows of matrices X, Y, B correspondingly. The transition procedure from matrices X, Y, B to the vectors x, y, b is carried out by procedure colon (:) (an inverse transition is carried out by procedure of reshape.m).

Thus, the problem of construction of the general solution of equation (1) is reduced to the problem of construction of the general solution of the linear algebraic equation (2).

Consequently, the condition of existence of the solution (1) can be formulated as follows. For existence of solution (1), the matrices G and $[G\ \ b]$ must have an identical rank [16] (for the calculation of rank of the matrix it is possible to use the procedure of rank.m).

3. The algorithm for construction of the general solution of (1)

Let produce the singular decomposition of the matrix G (procedure svd.m):

$$G = USV'. \tag{3}$$

In [3] U, V are orthogonal matrices, S is the diagonal matrix. The first r (r is the rank of matrix G) elements of diagonal of S are not equal to zero. Let us consider the matrix $U'G = SV'$. In connection with the marked structure of matrix S, only first r rows of the matrix $U'G$ will not be equal to zero. We designate the matrix formed from the first r rows of the matrix $U'G$ by Ag. Multiplying left and right part of equation (2) on a matrix U' and leaving in both parts only first r rows, we will rewrite (2) as follows:

$$A_g\begin{bmatrix} x \\ y \end{bmatrix} = b_u. \tag{4}$$

Here, the vector b_u is formed from the first r components of vector $U'b$.
Note that matrix Ag appearing in (4) is the complete rank matrix. Therefore, for determination of general solution of (2) it is possible to use the relation [17]:

$$\begin{bmatrix} x \\ y \end{bmatrix} = A_g{}'\left(A_g A_g{}'\right)^{-1} b_u + N\xi,$$

$$N = \left(I - A_g{}'\left(A_g A_{\mathrm{g}}{}'\right)^{-1} A_g\right). \tag{5}$$

Here, the first member in the right part determines the particular solution (2) which has a minimum norm, ξ is a vector of free parameters which defines the general solution (2). In (5) and further, I is a identity matrix of corresponding size.

Let us produce the singular decomposition of matrix N, analogically to (3):

$$N = U_n S_n V_n'.$$

Let the first q diagonal elements of the matrix S_n be not equal to zero. Consequently, the matrix $NV_n = U_n S_n$ will have only first q nonzero columns. The matrix which is formed from the first q columns of the matrix NV_n (determined the zero subspace of the matrix A_g) is designated as N_q. The relation (5) is rewritten as follows:

$$\begin{bmatrix} x \\ y \end{bmatrix} = A_g' \left(A_g A_g'\right)^{-1} b_u + N_q \xi_q, \tag{6}$$

where the dimension of vector of free parameters ξ_q is equal to q.

Defining the vector $\begin{bmatrix} x \\ y \end{bmatrix}$ accordingly to (6), i.e., the general solution of (2)(for the given vector ξ_q), then, using the procedure of reshape.m, it is possible to construct the matrices X, Y, which determine the general solution (1) using vectors x, y

A problem of choice of the vector of free parameters is considered. Obviously, in case of choice of other free parameters (different from the parameters determined by the vector ξ_q) the structure of (6) will not change. So, at the choice of new vector of free parameters (vector c) and corresponding matrix N_c (the columns of which determine the zero subspace of matrix A_g) we have:

$$N_q \xi_q = N_c c. \tag{7}$$

The relation (7) allows to set the connection between ξ_q and c. Note that for the calculation of matrix N_c, it is possible to use the procedure of null.m.

Example. The initial data coincide with accepted in the example 1 [9].

$$B(1) k = 2, \ell = 1, Q1 = Q_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, Q_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}, R_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

$$R_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}, S_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, T_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

At these initial data the rank of the matrix G is equal to 4 and coincides with the rank of the matrix $[G,\ b]$. Thus, at these initial data the equation (1) has a solution. In [9] the next solution of (1) is given at the accepted initial data:

$$X = \Phi(c_r) + \Lambda(B), \Phi(c_r) = \begin{bmatrix} 10c_6 + 6c_8 & 61c_2 & -3c_6 - 14c_8 \\ 61c_1 & 61c_3 & 61c_5 \\ -3c_6 - 14c_8 & 61c_4 & 7c_6 - 8c_8 \end{bmatrix},$$

$$Y = \Psi(c_r) + \Pi(B), \Psi(c_r) = \begin{bmatrix} -14c_6 + 16c_8 & -8c_6 + 44c_8 \\ 61c_8 & 61c_9 \end{bmatrix},$$

$$\Lambda(B) = \begin{bmatrix} \frac{19}{61} & 0 & -\frac{24}{61} \\ 0 & 0 & 0 \\ \frac{37}{61} & 0 & -\frac{5}{61} \end{bmatrix}, \Pi(B) = \begin{bmatrix} \frac{10}{61} & -\frac{3}{61} \\ 0 & 0 \end{bmatrix}.$$

Note that a coefficient c_7 does not appear in matrices $\Phi(c_r), \Psi(c_r)$.

Using the relation (6), we will find that the first element of the first part of this relation determines the matrices $\Lambda(B)$, $\Pi(B)$, which coincide with the given ones in [9].

The matrices $\Phi(c_r)$, $\Psi(c_r)$ are determining the matrix N_c, appearing in (7), in which the seventh column is zero:

$$N_c = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 10 & 0 & 6 & 0 \\ 0 & 61 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3 & 0 & -14 & 0 \\ 61 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 61 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 61 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3 & 0 & -14 & 0 \\ 0 & 0 & 0 & 61 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 7 & 0 & -8 & 0 \\ 0 & 0 & 0 & 0 & 0 & -14 & 0 & 16 & 0 \\ 0 & 0 & 0 & 0 & 0 & -8 & 0 & 44 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 61 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 61 \end{bmatrix}.$$

It is possible to change the seventh column of the matrix N_c, i.e. to rewrite this matrix in the form:

$$N_c7 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 10 & 0 & 6 & 0 \\ 0 & 61 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3 & 1 & -14 & 0 \\ 61 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 61 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 61 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3 & 1 & -14 & 0 \\ 0 & 0 & 0 & 61 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 7 & 1 & -8 & 0 \\ 0 & 0 & 0 & 0 & 0 & -14 & -2 & 16 & 0 \\ 0 & 0 & 0 & 0 & 0 & -8 & -4 & 44 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 61 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 61 \end{bmatrix}.$$

We will note that the rank of this matrix is equal to 9 and it satisfies to the condition $A_g N_c 7 = 0$. Consequently, the general solution of (1) given in [9] must be completed by matrices $x7, y7$, which are determined by the seventh column of the matrix N_c7.

$$X = \Phi(c_r) + \Lambda(B) + x7 \cdot c_7,$$

$$Y = \Psi(c_2) + \Pi(B) + y7 \cdot c_7,$$

$$x7 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad y7 = \begin{bmatrix} -2 & -4 \\ 0 & 0 \end{bmatrix}.$$

Let us note that, if in the right part of (1), as in an example 2 [9], the matrix

$$B_0 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

appears, the rank of the matrix $[G\ b]$ is equal to 5 and, consequently, the equation (1) does not have a solution. This conclusion coincides with the conclusion of [9].

Conclusion

The problem of construction of the general solution of the generalized Sylvester matrix equation is considered. The conditions of existence of the solution of this equation are obtained and an algorithm of construction of the solution is given. For the construction of the algorithm of solution and formulation of the condition of existence of the solution the standard procedures of package of MATLAB are used.

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Fikret A. Aliev graduated from Baku State University in 1966. He got his Ph.D. degree from the Institute of Mathematica, Kiev and his Doctor of Sciences degree from Institute of System Investigation, Moscow, in 1975, and 1990 respectively. Since 1993 he is the director of the Research Institute of Applied Mathematics of Baku State University. He is a member of the National Academy of Sciences of Azerbaijan. His current research interests are applied mathematics, system theory, control and numerical analysis. He is the author /co-author of 8 books and more than 150 scientific papers. He is the vice president of Turkic World Mathematical Society Science Foundation (2011).

Vladimir B. Larin is the head of the Department of Dynamics of Complex Systems at the S.P. Timoshenko Institute of Mechanics of the National Academy of Sciences of Ukraine. He is a member of the National Committee of Ukraine on Theoretical and Applied Mechanics, the National Committee of the Ukrainian Association of Automatic Control, and the American Mathematical Society. V.B. Larin is a well-known specialist in the field of control, optimization, system theory, numerical analysis, matrix theory, mechanics, and robust stability analysis.