ON COMPLEX MULTIPLICATIVE INTEGRATION

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ABSTRACT. In the present paper, we extend the multiplicative integral to complex-valued functions of complex variable. The main difficulty in this way, that is, the multi-valued nature of the complex logarithm is avoided by division of the interval of integration to a finite number of local intervals, in each of which the complex logarithm can be localized in one of its branches. Interestingly, the complex multiplicative integral became a multi-valued function. Some basic properties of this integral are considered. In particular, it is proved that this integral and the complex multiplicative derivative are bonded in a kind of fundamental theorem.

Keywords: Complex calculus, complex integral, multiplicative calculus, fundamental theorem of calculus.

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1. INTRODUCTION

In 1972, Grossman and Katz [13] proposed different alternative calculi to the classical calculus of Newton and Leibnitz. The multiplicative and bigeometric calculi are the most popular of them. This pioneering work initiated numerous studies on multiplicative and bigeometric calculi. In the literature, sometimes, bigeometric calculus is also referred as proportional or product calculus. Bigeometric calculus was pushed forward by the contributions of Grossman [12], Cordova-Lepe [7, 8], etc. Furthermore, elements of stochastic integration of bigeometric nature are introduced in Karandikar [14], and Daletskii and Teterina [9]. On the other hand, multiplicative calculus and its applications were promoted in Bashirov et al. [4, 5] and Stanley [20]. Moreover, Riza et al. [18], Mısırlı and Gürefe [15], and Özyapıcı et al. [16, 17] used multiplicative calculus for advancement of numerical methods; Florack and Van Assen [10] applied multiplicative calculus to biomedical image analysis; and Bashirov and Bashirova [3] used multiplicative calculus for the derivation of a mathematical model of literary texts etc.

The basic difference between different calculi is that they have different reference functions. In the case of Newtonian calculus, this reference function is linear. Therefore, the statements and proofs of facts, which are not perfectly described in terms of linear functions become complicated. For example, Newtonian calculus is suitable for Taylor series, but not for Fourier series. At the same time, the reference function of multiplicative calculus is exponential. This makes the study of exponent related problems such as

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growth suitable in multiplicative calculus. In particular, complex Fourier series, expressed in terms of exponent, are seen to be suitable for multiplicative calculus.

Nevertheless, the capacity of real multiplicative calculus is restricted within the class of positive functions of real variable and, hence, does not accept sine and cosine functions. Therefore, studying Fourier series by means of real multiplicative calculus is not possible. This suggests a creation of complex multiplicative calculus.

A proper complex multiplicative differentiation was prompted in Bashirov and Riza [6], where it was also demonstrated that the complex multiplicative differentiation accepts the functions with positive and as well as negative values whenever they are nowhere-vanishing. This point is unlike to real multiplicative differentiation and very important since the terms of complex Fourier series are also nowhere-vanishing. Continuing this study with the aim of further application to Fourier series, in this paper we deal with complex multiplicative integration by taking into consideration all branches of the complex logarithm.

Many books on complex analysis and calculus are available. We refer to Ahlfors [1], Greene and Krantz [11], and Sarason [19] which are used during this study.

The paper is organized in the following way: In Section 2, we briefly review basic points of multiplicative differentiation and line multiplicative integrals. The basic difficulty for a proper definition of the complex multiplicative integral is a multi-valued nature of the complex logarithm. In Section 3, complex multiplicative integral is defined locally, which allows to work with only one branch of the complex logarithm. Next, in Section 4, the multiplicative complex integral is defined in general form. Finally, in Section 5, we study the properties of complex multiplicative integral.

One major remark about the notation is that the multiplicative versions of the concepts of ordinary calculus are called as *concepts, for example; a *derivative means a multiplicative derivative. We denote by $\mathbb{R}$ and $\mathbb{C}$ the fields of real and complex numbers, respectively. Arg z is the principal value of arg z, noticing that $-\pi < \mathrm{Arg}\, z \leq \pi$. Always $\ln x$ refers to the natural logarithm of the real number $x > 0$ whereas $\log z$ to the same of the complex number $z \neq 0$. By Log z, we denote the value at z of the principal branch of the complex logarithm, that is, $\mathrm{Log}\, z = \ln |z| + i\mathrm{Arg}\, z$, where i denotes the imaginary unit and $|z|$ the modulus of z.

2. Preliminaries

The *derivative $f^*(x)$ of a purely positive or purely negative differentiable function f of a real variable is defined as the limit

$$f^*(x) = \lim_{h \to 0} (f(x+h)/f(x))^{1/h}, \tag{1}$$

showing how many times $|f(x)|$ changes at x. It differs from the derivative $f'(x)$, which shows to the rate of change of f at x. These two derivative concepts are related to each other by the formula

$$f^*(x) = e^{(\ln |f(x)|)'} = e^{\frac{f(x)'}{f(x)}}. \tag{2}$$

The appropriateness of the *derivative, especially, in modeling growth related processes has been demonstrated in various papers, for example; [3, 4, 5, 10, 15, 18].

One can observe that the limit in (1) can not be applied to differentiable functions with values changing the sign. The reason is that such a function certainly has zeros. This lack of integrity is removed by complex *derivative.

Following to Bashirov and Riza [6], let f be nowhere-vanishing differentiable complex function on an open set D in $\mathbb{C}$. To extend formulae (1)–(2) to the complex case, note that in general a branch of $\log f$ may not exist. Even if it exists, it can not be represented as

a composition of a branch of log-function and f. These rigorously can be avoided locally; since for a sufficiently small neighborhood $U \subseteq D$ of the point $z \in D$, the branches of $\log f$ on U exist. They are composition of branches of the log-function and the restriction of f to U, and the log-differentiation formula $(\log f)' = f'/f$ is valid for $\log f$ on U (see Sarason [19]). Taking this into consideration, the *derivative of f can be defined just as

$$f^*(z) = e^{(\ln f(z))'} = e^{\frac{f(z)'}{f(z)}}, \tag{3}$$

noticing that it is independent of the branches of log-function. In [6], it is proved that if $z = x + iy$ and

$$R(z) = R(x, y) = |f(z)| \quad \text{and} \quad \Theta(z) = \Theta(x, y) = \text{Arg} f(z), \tag{4}$$

then

$$\begin{cases} |f^*(z)| = R_x^*(z) = [e^{\Theta}]_y^*(z), \\ \arg f^*(z) = \Theta_x'(z) + 2\pi n = -[\ln R]_y'(z) + 2\pi n, \ n = 0, \pm 1, \pm 2, \ldots, \end{cases} \tag{5}$$

where Θ_x' and $[\ln R]_y'$ are partial derivatives as well as R_x^* and $[e^{\Theta}]_y^*$ are partial *derivatives of the real-valued functions of two real variables.

We say that a complex-valued function f of complex variable is *differentiable* at $z \in \mathbb{C}$, if it is differentiable at z and $f(z) \neq 0$. We also say that f is *holomorphic or *analytic on an open connected set D, if $f^*(z)$ exists for every $z \in D$.

The following examples demonstrate some features of complex *differentiation.

Example 2.1. The function $f(z) = e^{cz}$, $z \in \mathbb{C}$, where $c = \text{const} \in \mathbb{C}$, is an entire function and its *derivative

$$f^*(z) = e^{f'(z)/f(z)} = e^{ce^{cz}/e^{cz}} = e^c, \ z \in \mathbb{C},$$

is again an entire function, taking identically the nonzero value e^c. Thus, in complex *calculus $f(z) = e^{cz}$ plays the role of the linear function $g(z) = az$ with $a = e^c$ from Newtonian calculus.

Example 2.2. For another entire function $f(z) = e^{ce^z}$, $z \in \mathbb{C}$, with $c = \text{const} \in \mathbb{C}$, we have

$$f^*(z) = e^{f'(z)/f(z)} = e^{ce^z e^{ce^z}/e^{ce^z}} = e^{ce^z}, \ z \in \mathbb{C}.$$

Hence, f is a solution of the equation $f^* = f$. Thus, in complex *calculus $f(z) = e^{ce^z}$ plays the role of the exponential function $g(z) = ce^z$ from ordinary calculus.

Example 2.3. The function $f(z) = z$, $z \in \mathbb{C}$, is also entire; but its *derivative

$$f^*(z) = e^{f'(z)/f(z)} = e^{1/z}, \ z \in \mathbb{C} \setminus \{0\},$$

accounts an essential singularity at $z = 0$. This is because *differentiation is applicable to functions with the range in $\mathbb{C} \setminus \{0\}$. Thus, the *derivative of an entire function may not be entire.

In order to develop complex *integration, we need also in line *integrals as well as a fundamental theorem of calculus for them. Following to Bashirov [2], let f be a positive function of two variables, defined on an open connected set in $\mathbb{R}^2$, and let C be a piecewise smooth curve in the domain of f. Take a partition $\mathcal{P} = \{P_0, \ldots, P_m\}$ on C and let (ξ_k, η_k) be a point on C between P_{k-1} and P_k. Denote by Δs_k the arclength of C from the point P_{k-1} to P_k. Define the integral product

$$P(f, \mathcal{P}) = \prod_{k=1}^{m} f(\xi_k, \eta_k)^{\Delta s_k}.$$

The limit of this product when $\max\{\Delta s_1, \ldots, \Delta s_m\} \to 0$ independently on selection of the points (ξ_k, η_k) will be called a *line *integral of f in ds along C*, for which we will use the symbol

$$\int_C f(x, y)^{ds}.$$

The line *integral of f along C exist if f is a positive function and the line integral of $\ln f$ along C exists, and they are related as

$$\int_C f(x, y)^{ds} = e^{\int_C \ln f(x,y) \, ds}.$$

In a similar way, the line *integrals in dx and in dy can be defined and their relation to the respective line integrals can be established in the form

$$\int_C f(x, y)^{dx} = e^{\int_C \ln f(x,y) \, dx} \quad \text{and} \quad \int_C f(x, y)^{dy} = e^{\int_C \ln f(x,y) \, dy}. \tag{6}$$

Clearly, all three kinds of line *integrals exist if f is a positive continuous function. It is also suitable to denote

$$\int_C f(x, y)^{dx} g(x, y)^{dy} = \int_C f(x, y)^{dx} \cdot \int_C g(x, y)^{dy}.$$

In cases when C is a closed curve we write $\oint_C$ instead of $\int_C$.

Example 2.4. Let $c > 0$ and let $C = \{(x(t), y(t)) : a \le t \le b\}$ be a piecewise smooth curve. Then,

$$\int_C c^{dx} = e^{\int_C \ln c \, dx} = e^{(x(b)-x(a)) \ln c} = c^{x(b)-x(a)}.$$

Theorem 2.1 (*Fundamental theorem of calculus for line *integrals*)**.** *Let $D \subseteq \mathbb{R}^2$ be an open connected set and let $C = \{(x(t), y(t)) : a \le t \le b\}$ be a piecewise smooth curve in D. Assume that f is a continuously differentiable positive function on D. Then,*

$$\int_C f_x^*(x, y)^{dx} f_y^*(x, y)^{dy} = \frac{f(x(b), y(b))}{f(x(a), y(a))}.$$

Proof. See Bashirov [2]. □

3. Complex multiplicative integration (local)

Let f be a continuous nowhere-vanishing complex-valued function of complex variable and let $z(t) = x(t) + iy(t)$, $a \le t \le b$, be a complex-valued function of real variable, tracing a piecewise smooth simple curve C in the open connected domain D of f. The complex *integral of f along C will heavily use $\log f$. In order to represent $\log f$ as the composition of branches of log and f along the whole curve C, we will use a "method of localization" from Sarason [19]. In this section, we will consider a simple case assuming that the length of the interval $[a, b]$ is sufficiently small so that all the values of $f(z(t))$ for $a \le t \le b$ fall into an open half plane bounded by a line through the origin. Under this condition, the restriction of $\log f$ to C can be treated as a composition of the branches of log and the restriction of f to C. Moreover, we can select one of the multi-values of $\log f(z(a))$ and consider a branch $\mathcal{L}$ of log so that $\mathcal{L}(f(z(a)))$ equals to this preassigned value. Thus

$$\log f(z(t)) = \mathcal{L}(f(z(t))) + 2\pi n i, \ a \le t \le b, \ n = 0, \pm 1, \pm 2, \ldots.$$

Now, take a partition $\mathcal{P} = \{z_0, \ldots, z_m\}$ on C and let ζ_k be a point on C between z_{k-1} and z_k. Denote $\Delta z_k = z_k - z_{k-1}$. Consider the integral product $\prod_{k=1}^m e^{\Delta z_k \log f(\zeta_k)}$. It can be evaluated as

$$\prod_{k=1}^m e^{\Delta z_k \log f(\zeta_k)} = e^{\sum_{k=1}^m (\mathcal{L}(f(\zeta_k)) + 2\pi n i)\Delta z_k}$$

$$= e^{2\pi n(z(b) - z(a))i} e^{\sum_{k=1}^m \mathcal{L}(f(\zeta_k))\Delta z_k}, \ n = 0, \pm 1, \pm 2, \ldots,$$

showing that $\prod_{k=1}^m e^{\Delta z_k \log f(\zeta_k)}$ has more than one value. Let

$$P_0(f, \mathcal{P}) = e^{\sum_{k=1}^m \mathcal{L}(f(\zeta_k))\Delta z_k} \tag{7}$$

and

$$P_n(f, \mathcal{P}) = e^{2\pi n(z(b) - z(a))i} P_0(f, \mathcal{P}), \ n = 0, \pm 1, \pm 2, \ldots. \tag{8}$$

The limit of $P_0(f, \mathcal{P})$ as $\max\{|\Delta z_1|, \ldots, |\Delta z_m|\} \to 0$ independently on selection of the points ζ_k will be called a *branch value of the complex *integral of f along C* and it will be denoted by $I_0^*(f, C)$. Then *the complex *integral of f along C* can be defined as the multiple values

$$I_n^*(f, C) = e^{2\pi n(z(b) - z(a))i} I_0^*(f, C), \ n = 0, \pm 1, \pm 2, \ldots, \tag{9}$$

which will be denoted by

$$\int_C f(z)^{dz}.$$

Note that if $z(b) - z(a)$ is an integer, then all the values of $\int_C f(z)^{dz}$ equal to $I_0^*(f, C)$, i.e., $I^*(f, C)$ become single-valued. If $z(b) - z(a)$ is a rational number in the form p/q, where p and q are irreducible integers with $q > 0$, then $\int_C f(z)^{dz}$ has q distinct values

$$e^{2\pi n p i / q} I_0^*(f, C), \ n = 0, 1, \ldots, q - 1.$$

Generally, $\int_C f(z)^{dz}$ has countably many distinct values. In case if $z(b) - z(a)$ is a real number, we also have $|I_n^*(f, C)| = |I_0^*(f, C)|$ for all n. Similarly, if $z(b) - z(a)$ is an imaginary number, $\operatorname{Arg} I_n^*(f, C) = \operatorname{Arg} I_0^*(f, C)$ for all n.

The existence of the complex *integral of f can be reduced to the existence of line *integrals in the following way. Let $R(z) = |f(z)|$ and $\Theta(z) = \operatorname{Im} \mathcal{L}(f(z))$ for $z \in C$. Denote $z = x + iy$ and $\Delta z_k = \Delta x_k + i\Delta y_k$. Then from (7),

$$P_0(f, \mathcal{P}) = e^{\sum_{k=1}^m \mathcal{L}(f(\zeta_k))\Delta z_k}$$

$$= e^{\sum_{k=1}^m (\ln R(\zeta_k) + i\Theta(\zeta_k))(\Delta x_k + i\Delta y_k)}$$

$$= e^{\sum_{k=1}^m (\ln R(\zeta_k)\Delta x_k - \Theta(\zeta_k)\Delta y_k) + i\sum_{k=1}^m (\Theta(\zeta_k)\Delta x_k + \ln R(\zeta_k)\Delta y_k)}.$$

If the limits of the sums in the last expression exist, then they are line integrals, producing

$$I_0^*(f, C) = e^{\int_C (\ln R(z)\,dx - \Theta(z)\,dy) + i\int_C (\Theta(z)\,dx + \ln R(z)\,dy)}. \tag{10}$$

Additionally,

$$e^{2\pi n(z(b) - z(a))i} = e^{2\pi n(-(y(b) - y(a)) + i(x(b) - x(a)))} = e^{-\int_C 2\pi n\,dy + i\int_C 2\pi n\,dx}.$$

By (9)–(10), this implies

$$I_n^*(f, C) = e^{\int_C (\ln R(z)\,dx - (\Theta(z) + 2\pi n)\,dy) + i\int_C ((\Theta(z) + 2\pi n)\,dx + \ln R(z)\,dy)} \tag{11}$$

for $n = 0, \pm 1, \pm 2, \ldots$ or, in the multi-valued form,

$$\int_C f(z)^{dz} = e^{\int_C \log f(z)\,dz},$$

in which

$$I_0^*(f, C) = e^{\int_C \mathcal{L}(f(z))\, dz} \quad \text{and} \quad I_n^*(f, C) = e^{2\pi n(z(b) - z(a))i} I_0^*(f, C). \tag{12}$$

To write (11) in terms of line *integrals, note that by (6)

$$|I_n^*(f, C)| = e^{\int_C (\ln R(z)\, dx - (\Theta(z) + 2\pi n)\, dy)} = \int_C R(z)^{dx} \left(e^{-\Theta(z) - 2\pi n}\right)^{dy}$$

and

$$\arg I_n^*(f, C) = \int_C \left((\Theta(z) + 2\pi n)\, dx + \ln R(z)\, dy\right) + 2\pi m$$

$$= \ln \int_C \left(e^{\Theta(z) + 2\pi n}\right)^{dx} R(z)^{dy} + 2\pi m.$$

Hence,

$$I_n^*(f, C) = \int_C R(z)^{dx} \left(e^{-\Theta(z) - 2\pi n}\right)^{dy} e^{i \ln \int_C \left(e^{\Theta(z) + 2\pi n}\right)^{dx} R(z)^{dy}} \tag{13}$$

for $n = 0, \pm 1, \pm 2, \ldots$. Thus, the conditions imposed at the beginning of this section, namely (a) f is nowhere-vanishing and continuous on the open connected set D, (b) C is piecewise smooth and simple curve in D, and (c) $\{f(z(t)) : a \le t \le b\}$ falls into an open half plane bounded by a line through the origin and guarantee the existence of $\int_C f(z)^{dz}$ as multiple values.

The following proposition will be used for justifying the correctness of the definition of the complex *integral for arbitrary interval $[a, b]$ in the next section.

Proposition 3.1 (*1st multiplicative property, local*)**.** *Let f be a nowhere-vanishing continuous function, defined on an open connected set D, and let $C = \{z(t) = x(t) + iy(t) : a \le t \le b\}$ be a piecewise smooth simple curve in D with the property that the set $\{f(z(t)) : a \le t \le b\}$ falls into an open half plane bounded by a line through origin. Take any $a < c < b$ and let $C_1 = \{z(t) = x(t) + iy(t) : a \le t \le c\}$ and $C_2 = \{z(t) = x(t) + iy(t) : c \le t \le b\}$. Then*

$$\int_C f(z)^{dz} = \int_{C_1} f(z)^{dz} \int_{C_2} f(z)^{dz},$$

where the equality is understood in the sense that

$$I_n^*(f, C) = I_n^*(f, C_1) I_n^*(f, C_2) \quad \text{for all} \quad n = 0, \pm 1, \pm 2, \ldots$$

with the same branch $\mathcal{L}$ of log used for $I_0(f, C)$, $I_0(f, C_1)$ and $I_0(f, C_2)$.

Proof. This follows immediately from (11) and the respective property of line integrals. □

Next, we consider an *analog of the fundamental theorem of complex calculus in a local form.

Proposition 3.2 (*Fundamental theorem of complex *calculus, local*)**.** *Let f be a nowhere-vanishing *holomorphic function, defined on an open connected set D, and let $C = \{z(t) = x(t) + iy(t) : a \le t \le b\}$ be a piecewise smooth simple curve in D with the property that the set $\{f(z(t)) : a \le t \le b\}$ falls into an open half plane bounded by a line through origin. Then*

$$\int_C f^*(z)^{dz} = \{e^{2\pi n(z(b) - z(a))i} f(z(b)) / f(z(a)) : n = 0, \pm 1, \pm 2, \ldots\}.$$

Proof. From (13),

$$\int_C f^*(z)^{dz} = \int_C |f^*(z)|^{dx} \left(e^{-\arg f^*(z)}\right)^{dy} \; e^{i\ln \int_C \left(e^{\arg f^*(z)}\right)^{dx}} |f^*(z)|^{dy}.$$

Using (5),

$$I_n^*(f^*, C) = \int_C R_x^*(z)^{dx} \left(e^{[\ln R]_y'(z) - 2\pi n}\right)^{dy} \; e^{i\ln \int_C \left(e^{\Theta_x'(z) + 2\pi n}\right)^{dx}} [e^\Theta]_y^*(z)^{dy}$$

$$= \int_C R_x^*(z)^{dx} R_y^*(z)^{dy} \; e^{i\ln \int_C [e^\Theta]_x^*(z)^{dx} [e^\Theta]_y^*(z)^{dy}}$$

$$\times \int_C \left(e^{-2\pi n}\right)^{dy} \; e^{i\ln \int_C (e^{2\pi n})^{dx}}.$$

By Theorem 2.1,

$$\int_C R_x^*(z)^{dx} R_y^*(z)^{dy} \; e^{i\ln \int_C [e^\Theta]_x^*(z)^{dx} [e^\Theta]_y^*(z)^{dy}} = \frac{R(z(b))e^{i\Theta(z(b))}}{R(z(a))e^{i\Theta(z(a))}} = \frac{f(z(b))}{f(z(a))},$$

and, by Example 2.4,

$$\int_C \left(e^{-2\pi n}\right)^{dy} \; e^{i\ln \int_C (e^{2\pi n})^{dx}} = e^{2\pi n(-(y(b) + y(a)) + i(x(b) - x(a)))} = e^{2\pi n(z(b) - z(a))i}.$$

Thus, the proposition is proved. $\qquad\square$

4. Complex multiplicative integration (general)

The following lemma is crucial for a general definition of complex *integral.

Lemma 4.1. *Let f be a continuous nowhere-vanishing function, defined on an open connected set D, and let $C = \{z(t) = x(t) + iy(t) : a \leq t \leq b\}$ be a piecewise smooth curve in D. Then there exists a partition $\mathcal{P} = \{t_0, t_1, \ldots, t_m\}$ of $[a, b]$ such that each of the sets $\{f(z(t)) : t_{k-1} \leq t \leq t_k\}$, $k = 1, \ldots, m$, falls into an open half plane bounded by a line through origin.*

Proof. To every $t \in [a, b]$, consider $\theta_t = \text{Arg}\, f(z(t))$ and the line L_t formed by the rays $\theta = \theta_t + \pi/2$ and $\theta = \theta_t - \pi/2$. Since f is continuous and nowhere-vanishing, there is an interval $(t - \varepsilon_t, t + \varepsilon_t) \subseteq [a, b]$ such that the set

$$\{f(z(s)) : t - \varepsilon_t < s < t + \varepsilon_t\}$$

falls into one of the open half planes bounded by L_t if $t \in (a, b)$. In the case of $t = a$, such an interval can be selected in the form $[a, a + \varepsilon_a)$ and in the case $t = b$ as $(b - \varepsilon_b, b]$. The collection of all such intervals forms an open cover of the compact subspace $[a, b]$ of $\mathbb{R}$. Therefore, there is a finite number of them covering $[a, b]$. Writing the end points of these intervals in an increasing order $a = t_0 < t_1 < \cdots < t_m = b$ produces a required partition. $\qquad\square$

This lemma determines a way for definition of $\int_C f(z)^{dz}$ in the general case. Assume again that f is a continuous nowhere-vanishing complex-valued function of complex variable and $z(t) = x(t) + iy(t)$, $a \leq t \leq b$, is a complex-valued function of real variable, tracing a piecewise smooth simple curve C in the open connected domain D of f. Let $\mathcal{P} = \{t_0, t_1, \ldots, t_m\}$ be a partition of $[a, b]$ from Lemma 4.1 and let $C_k = \{z(t) : t_{k-1} \leq t \leq t_k\}$. Choose any branch $\mathcal{L}_1$ of log and consider $\int_{C_1} f(z)^{dz}$ as defined in the previous section. Then, select a branch $\mathcal{L}_2$ of log with $\mathcal{L}_2(f(z(t_1))) = \mathcal{L}_1(f(z(t_1)))$ and consider $\int_{C_2} f(z)^{dz}$. Next, select a branch $\mathcal{L}_3$ of log with $\mathcal{L}_3(f(z(t_2))) = \mathcal{L}_2(f(z(t_2)))$ and consider $\int_{C_3} f(z)^{dz}$,

etc. In this process, the selection of the starting branch $\mathcal{L}_1$ is free, but the other branches $\mathcal{L}_2, \ldots, \mathcal{L}_m$ are selected accordingly to construct a continuous single-valued function g on $[a, b]$ so that the value of g at fixed $t \in [a, b]$ equals to one of the branch values of $\log f(z(t))$. Now, following to (12), the *complex *integral of f over C*, that will again be denoted by $\int_C f(z)^{dz}$, can be defined as the multiple values

$$I_n^*(f, C) = \prod_{k=1}^{m} e^{2\pi n(z(t_k) - z(t_{k-1}))i + \int_{C_k} \mathcal{L}_k(f(z))\, dz}, \ \ n = 0, \pm 1, \pm 2, \ldots,$$

or

$$I_n^*(f, C) = e^{2\pi n(z(b) - z(a))i} e^{\sum_{k=1}^{m} \int_{C_k} \mathcal{L}_k(f(z))\, dz}, \ \ n = 0, \pm 1, \pm 2, \ldots. \tag{14}$$

This definition is independent on the selection of the partition $\mathcal{P}$ of $[a, b]$. Indeed, if $\mathcal{Q}$ is another partition, being a refinement of the previous one, then the piece C_k from $z(t_{k-1})$ to $z(t_k)$ of the curve C became departed into smaller non-overlapping pieces C_{ki}, $i = 1, \ldots, l_k$, each over the partition intervals of $\mathcal{Q}$ falling into $[t_{k-1}, t_k]$. Since the range of f over C_k falls into an open half plane bounded by a line through origin, the range of f over each C_{ki} falls into the same half plane. Therefore, by Proposition 3.1,

$$I_n^*(f, C_k) = \prod_{i=1}^{l_k} I_n^*(f, C_{ki})$$

with the same branch $\mathcal{L}_k$ of log used for all $I_0(f, C_k)$ and $I_0(f, C_{k1}), \ldots, I_0(f, C_{kl_k})$. Then

$$I_n^*(f, C_k) = \prod_{k=1}^{m} I_n^*(f, C_k)) = \prod_{k=1}^{m} \prod_{i=1}^{l_k} I_n^*(f, C_{ki}),$$

i.e., both $\mathcal{P}$ and $\mathcal{Q}$ produce the same multiple values. In case if $\mathcal{P}$ and $\mathcal{Q}$ are two arbitrary partitions of $[a, b]$ from Lemma 4.1, we can compare the integral for selections $\mathcal{P}$ and $\mathcal{Q}$ with the same for their refinement $\mathcal{P} \cup \mathcal{Q}$ and deduce that $I_n^*(f, C)$ is independent on selection of $\mathcal{P}$ and $\mathcal{Q}$. Actually, the described method can be seen as a kind of *gluing* method, where we match the values for the partitions $\mathcal{P}$ and $\mathcal{Q}$.

5. Properties of Complex Multiplicative Integrals

Theorem 5.1 (*1st multiplicative property*). *Let f be a nowhere-vanishing continuous function, defined on an open connected set D, and let $C = \{z(t) = x(t) + iy(t) : a \le t \le b\}$ be a piecewise smooth simple curve in D. Take any $a < c < b$ and let $C_1 = \{z(t) = x(t) + iy(t) : a \le t \le c\}$ and $C_2 = \{z(t) = x(t) + iy(t) : c \le t \le b\}$. Then*

$$\int_C f(z)^{dz} = \int_{C_1} f(z)^{dz} \int_{C_2} f(z)^{dz}$$

in the sense that

$$I_n^*(f, C) = I_n^*(f, C_1) I_n^*(f, C_2) \ \text{for all } n = 0, \pm 1, \pm 2, \ldots,$$

where $\mathcal{L}_{01}(f(z(a))) = \mathcal{L}_{11}(f(z(a)))$ and $\mathcal{L}_{1m_1}(f(z(c))) = \mathcal{L}_{21}(f(z(c)))$ if $\mathcal{L}_{01}, \ldots, \mathcal{L}_{0m}$, $\mathcal{L}_{11}, \ldots, \mathcal{L}_{1m_1}$ and $\mathcal{L}_{21}, \ldots, \mathcal{L}_{2m_2}$ are the sequences of branches of $\log$ used in definition of $I_0^*(f, C)$, $I_0^*(f, C_1)$ and $I_0^*(f, C_2)$, respectively.

Proof. This follows from the the way of definition of complex *integral for general interval $[a, b]$ and its independence on the selection of partition $\mathcal{P}$ from Lemma 4.1. $\square$

Theorem 5.2 (*2nd multiplicative property*)**.** *Let f and g be nowhere-vanishing continuous functions, defined on an open connected set D, and let $C = \{z(t) = x(t) + iy(t) : a \le t \le b\}$ be a piecewise smooth simple curve in D. Then*

$$\int_C (f(z)g(z))^{dz} = \int_C f(z)^{dz} \int_C g(z)^{dz}$$

as a set equality, where the product of the sets A and B is treated as $AB = \{ab : a \in A, b \in B\}$.

Proof. This follows from (14) and the set equality $\log(z_1 z_2) = \log z_1 + \log z_2$, where the sum of the sets A and B is treated as $A + B = \{a + b : a \in A, b \in B\}$. $\qquad\square$

Theorem 5.3 (*Division property*)**.** *Let f and g be nowhere-vanishing continuous functions, defined on an open connected set D, and let $C = \{z(t) = x(t) + iy(t) : a \le t \le b\}$ be a piecewise smooth simple curve in D. Then*

$$\int_C (f(z)/g(z))^{dz} = \int_C f(z)^{dz} / \int_C g(z)^{dz}$$

as a set equality, where the ratio of the sets A and B is treated as $A/B = \{a/b : a \in A, b \in B\}$.

Proof. This follows from (14) and the set equality $\log(z_1/z_2) = \log z_1 - \log z_2$, where the difference of the sets A and B is treated as $A - B = \{a - b : a \in A, b \in B\}$. $\qquad\square$

Theorem 5.4 (*Reversing the curve*)**.** *Let f be a nowhere-vanishing continuous functions, defined on an open connected set D, let $C = \{z(t) = x(t) + iy(t) : a \le t \le b\}$ be a piecewise smooth simple curve in D and let $-C$ be the curve C with opposite orientation. Then*

$$\int_C f(z)^{dz} = \left(\int_{-C} f(z)^{dz} \right)^{-1}$$

in the sense that $I_n^*(f, C) = I_n^*(f, -C)^{-1}$ for all $n = 0, \pm 1, \pm 2, \dots$, where $\mathcal{L}_{11}(f(z(a))) = \mathcal{L}_{2m_2}(f(z(a)))$ or $\mathcal{L}_{1m_1}(f(z(b))) = \mathcal{L}_{21}(f(z(b)))$ if $\mathcal{L}_{11}, \dots, \mathcal{L}_{1m_1}$ and $\mathcal{L}_{21}, \dots, \mathcal{L}_{2m_2}$ are the sequences of branches of $\log$ used in definition of $I_0^*(f, C)$ and $I_0^*(f, -C)$, respectively.

Proof. This follows from (14). $\qquad\square$

Theorem 5.5 (*Raising to a natural power*)**.** *Let f be a nowhere-vanishing continuous functions, defined on an open connected set D and let $C = \{z(t) = x(t) + iy(t) : a \le t \le b\}$ be a piecewise smooth simple curve in D. Then for $n = 0, 1, 2, \dots,$*

$$\left(\int_C f(z)^{dz} \right)^n \subseteq \int_C (f(z)^n)^{dz}.$$

Proof. This follows from multiple application of Theorem 5.2 and the fact that $A^n \subseteq AA \cdots A$ (n times), where $A^n = \{a^n : a \in A\}$ and $AA \cdots A$(n times)$= \{a_1 a_2 \cdots a_n : a_i \in A, i = 1, \dots, n\}$. $\qquad\square$

Theorem 5.6 (*Fundamental theorem of calculus for complex *integrals*)**.** *Let f be a nowhere-vanishing *holomorphic function, defined on an open connected set D, and let $C = \{z(t) = x(t) + iy(t) : a \le t \le b\}$ be a piecewise smooth simple curve in D. Then*

$$\int_C f^*(z)^{dz} = \{e^{2\pi n(z(b) - z(a))i} f(z(b))/f(z(a)), \ n = 0, \pm 1, \pm 2, \dots\}. \qquad (15)$$

Proof. Let $\mathcal{P} = \{t_0, t_1, \ldots, t_m\}$ be a partition from Lemma 4.1 and let $C_k = \{z(t) : t_{k-1} \leq t \leq t_k\}$. Then

$$\int_C f(z)^{dz} = \int_{C_1} f(z)^{dz} \cdots \int_{C_m} f(z)^{dz}.$$

Hence, by Proposition 3.2,

$$I_n^*(f, C) = e^{2\pi n(z(t_1)-z(t_0))i + \cdots + 2\pi n(z(t_m)-z(t_{m-1}))i} \frac{f(z(t_1)) \cdots f(z(t_m))}{f(z(t_0)) \cdots f(z(t_{m-1}))}$$

$$= e^{2\pi n(z(b)-z(a))i} \frac{f(z(b))}{f(z(a))}.$$

This proves the theorem. $\qquad\square$

This theorem demonstrates that $\int_C f^*(z)^{dz}$ is independent on the shape of the piecewise smooth curve C, but depends on its initial point $z(a) = x(a) + iy(a)$ and end point $z(b) = x(b) + iy(b)$ on the curve C. Therefore, this integral can be denoted by

$$\int_{z(a)}^{z(b)} f^*(z)^{dz}.$$

Corollary 5.1. *Let f be a nowhere-vanishing *holomorphic function, defined on an open connected set D, and let $C = \{z(t) = x(t) + iy(t) : a \leq t \leq b\}$ be a piecewise smooth simple closed curve in D. Then*

$$\oint_C f^*(z)^{dz} = 1. \tag{16}$$

Proof. Simply, write $z(a) = z(b)$ in (15). $\qquad\square$

Note that in (16) all the values of $\oint_C f^*(z)^{dz}$ are equal to 1, i.e., $\oint_C f^*(z)^{dz}$ becomes single-valued.

Example 5.1. By Example 2.1, the function $f(z) = e^{cz}$, $z \in \mathbb{C}$, where $c = \text{const} \in \mathbb{C}$, has the *derivative $f^*(z) = e^c$. Respectively,

$$\int_C (e^c)^{dz} = e^{2\pi n(z(b)-z(a))i} e^{c(z(b)-z(a))} = e^{(z(b)-z(a))(c+2\pi ni)},$$

where $C = \{z(t) : a \leq t \leq b\}$ is a piecewise smooth curve.

Example 5.2. By Example 2.2, the function $f(z) = e^{ce^z}$, $z \in \mathbb{C}$, where $c = \text{const} \in \mathbb{C}$, has the *derivative $f^*(z) = f(z)$. Respectively,

$$\int_C (e^{ce^z})^{dz} = e^{2\pi n(z(b)-z(a))i} e^{c(e^{z(b)}-e^{z(a)})},$$

where again $C = \{z(t) : a \leq t \leq b\}$ is a piecewise smooth curve.

Example 5.3. The analog of the integral

$$\oint_{|z|=1} \frac{dz}{z} = 2\pi i$$

in complex *calculus is

$$\oint_{|z|=1} (e^{1/z})^{dz}.$$

Assuming that the orientation on the unit circle $|z| = 1$ is positive, we informally have

$$\oint_{|z|=1} (e^{1/z})^{dz} = e^{\oint_{|z|=1} \log e^{1/z} dz} = e^{\oint_{|z|=1} \left(\frac{1}{z} + 2\pi ni\right) dz} = e^{2\pi n(z(b)-z(a))i} e^{2\pi i} = 1.$$

Formally, we use Example 2.3 and calculate the same:

$$\oint_{|z|=1} \left(e^{1/z}\right)^{dz} = e^{2\pi n(z(b)-z(a))i} \frac{z(b)}{z(a)} = \frac{e^{\pi i}}{e^{-\pi i}} = e^{2\pi i} = 1.$$

Thus, this example also fits to Corollary 5.1. The main idea of this is that $e^0 = e^{2\pi i} = 1$ though $2\pi i \neq 0$. In other words, the discontinuity of the branches of log on $(-\infty, 0]$, that creates the Cauchy formula $\int_{|z|=1} \frac{dz}{z} = 2\pi i$, appears in a smooth form in complex *calculus because now $\log z$ is replaced by $e^{\log z} = z$, where the discontinuity of log is compensated by periodicity of the exponential function.

6. Conclusion

In continuation of [6], evidently, the extension of multiplicative calculus to complex valued functions of complex variable eliminates the restriction to positive valued functions caused by multiplicative calculus. The complex multiplicative integral was defined firstly using the "method of localization" in the sense of [19] so that the values of the function are restricted to one half-plane bounded by a line through the origin, so we can decompose the restriction of $\log f$ on C into the branches of the complex logarithm and the restriction of f to C. The general definition of the complex multiplicative integral removes the restriction by the so-called *gluing* method by matching the values of the functions at the branch-cuts. Finally, based on this definition, the properties of complex multiplicative integrals are given and illustrated by application to certain standard examples.

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