TWMS J. App. Eng. Math. V.6, N.2, 2016, pp. 251-263

NECESSARY AND SUFFICIENT CONDITIONS FOR THE WAVE PACKET FRAMES ON POSITIVE HALF-LINE

ABDULLAH[1], §

Abstract. In this paper, we consider wave packet systems as special cases of generalized shift-invariant systems, a concept studied by Hernández, Lebate and Weiss. The objective of the paper is to construct wave packet frames on positive half line. We establish necessary and sufficient conditions for the wave packet frames on positive half-line using Walsh-Fourier transform.

Keywords: frame, wave packet system, sufficient condition and necessary condition.

AMS Subject Classification: 42C15 and 42C40

1. Introduction

The concept of frames in a Hilbert space was originally introduced by Duffin and Schaeffer [8] in the context of non-harmonic Fourier series. In signal processing, this concept has become very useful in analyzing the completeness and stability of linear discrete signal representations. Frames did not seem to generate much interest until the ground-breaking work of Daubechies et al. [7]. They combined the theory of continuous wavelet transforms with the theory of frames to introduce wavelet (affine) frames for $L^2(\mathbb{R})$. Since then the theory of frames began to be more widely investigated, and now it is found to be useful in signal processing, image processing, harmonic analysis, sampling theory, data transmission with erasures, quantum computing, and medicine. Recently, more applications of the theory of frames are found in diverse areas including optics, filter banks, signal detection and in the study of Bosev spaces and Banach spaces. We refer [3, 11] for an introduction to frame theory and its applications.

Wave packet systems are countable collections of dilations, translations, and modulations of a single function $\psi \in L^2(\mathbb{R})$. In [5], Córdoba and Fefferman introduced this form of wave packet system. Wave packet system have been considered and extended by several authors, (see [1], [2], [5], [6], [13], [14], [17]). Czaja, Kutyniok, and Speegle proved that certain geometric conditions on the set of parameters in a wave packet systems are necessary in order for the system to form a frame in [6]. Recently, Shah and Abdullah [17] have established a necessary condition for the wave packet system to be frame for $L^2(\mathbb{R}^+)$ by considering that these systems are special cases of generalized shift-invariant systems whereas the later author has given the general characterization of all tight wave packet frames for $L^2(\mathbb{R}^+)$ and $H^2(\mathbb{R}^+)$ by imposing some mild conditions on the generator

[1] Department of Mathematics, Zakir Husain Delhi College, University of Delhi, JLN Marg, New Delhi-110 002, India.
e-mail: abd.zhc.du@gmail.com;

§ Manuscript received: September 04, 2015; accepted: April 22, 2016.

in [1] and first author introduced Gabor frames on a half-line in which he established the necessary and sufficient conditions for the existence of Gabor frames on $\mathbb{R}^+$ in [18].

Farkov [9] has given general construction of compactly supported orthogonal p-wavelets in $L^2(\mathbb{R}^+)$. Farkov et al. [10] gave an algorithm for biorthogonal wavelets related to Walsh functions on positive half line. On the other hand, Shah and Debnath [21] have constructed dyadic wavelet frames on the positive half-line $\mathbb{R}^+$ using the Walsh-Fourier transform and have established a necessary condition and a sufficient condition for the system $\{\psi_{j,k}(x) = 2^{j/2}\psi(2^j x \ominus k) : j \in \mathbb{Z}, k \in \mathbb{Z}^+\}$ to be a frame for $L^2(\mathbb{R}^+)$ and in [22], they constructed p-Wavelet frame packets on a half-line using the Walsh–Fourier transform. Further, a constructive procedure for constructing tight wavelet frames on positive half-line using extension principles was recently considered by Shah in [20], in which he has pointed out a method for constructing affine frames in $L^2(\mathbb{R}^+)$. Moreover, the author has established sufficient conditions for a finite number of functions to form a tight affine frames for $L^2(\mathbb{R}^+)$. Recently, Meenakshi et al. [15] have introduced the notion of non-uniform multiresolution analysis (NUMRA) on a half-line $\mathbb{R}^+$ and have also established the necessary and sufficient condition for the existence of corresponding wavelets on $\mathbb{R}^+$.

In the present paper, we consider wave packet systems as special cases of generalized shift-invariant systems, a concept studied by Hernández, Lebate, and Weiss in [12]. The objective of the paper is to construct wave packet frames on positive half line. We establish necessary and sufficient conditions for the wave packet frames on positive half line. The paper is structured as follows. In Section 2, we give a brief introduction to Walsh-Fourier analysis on positive half-line. In Section 3 and 4, we present a necessary condition and sufficient condition for the wave packet system $\{D_{p^j}T_{bk}M_{mb}\psi\}_{j,m\in\mathbb{Z},\, k\in\mathbb{Z}^+}$ to be a frame for $L^2(\mathbb{R}^+)$.

2. Notations and preliminaries on Walsh-Fourier Analysis

Let p be a fixed natural number greater than 1. As usual, let $\mathbb{R}^+ = [0,\infty)$ and $\mathbb{Z}^+ = \{0,1,...\}$. Denote by $[x]$ the integer part of x. For $x \in \mathbb{R}^+$ and for any positive integer j, we set

$$x_j = [p^j x](\operatorname{mod} p),\ x_{-j} = [p^{1-j}x](\operatorname{mod} p), \tag{2.1}$$

where $x_j,\ x_{-j} \in \{0,1,...,p-1\}$.

Consider the addition defined on $\mathbb{R}^+$ as follows:

$$x \oplus y = \sum_{j<0} \xi_j p^{-j-1} + \sum_{j>0} \xi_j p^{-j} \tag{2.2}$$

with

$$\xi_j = x_j + y_j (\operatorname{mod} p),\ j \in \mathbb{Z}\backslash\{0\}, \tag{2.3}$$

where $\xi_j \in \{0,1,2,...,p-1\}$ and $x_j,\ y_j$ are calculated by (2.1). Moreover, we write $z = x \ominus y$ if $z \oplus y = x$.

For $x \in [0,1)$, let $r_0(x)$ be given by

$$r_0(x) = \begin{cases} 1,\ x \in \left[0, \dfrac{1}{p}\right), \\ \\ \varepsilon_p^j,\ x \in \left[jp^{-1}, (j+1)p^{-1}\right),\ j = 1,2,...,p-1, \end{cases} \tag{2.4}$$

where $\varepsilon_p = \exp\left(\frac{2\pi i}{p}\right)$. The extension of the function r_0 to $\mathbb{R}^+$ is defined by the equality $r_0(x+1) = r_0(x)$, $x \in \mathbb{R}^+$. Then, the generalized Walsh functions $\{\omega_m(x)\}_{m\in\mathbb{Z}^+}$ are defined by

$$\omega_0(x) = 1,\ \omega_m(x) = \prod_{j=0}^{k} \left(r_0\left(p^j x\right)\right)^{\mu_j},$$

where $m = \sum_{j=0}^{k} \mu_j p^j,\ \mu_j \in \{0,1,2,...,p-1\},\ \mu_k \neq 0.$

For $x,\ \omega \in \mathbb{R}^+$, let

$$\chi(x,\omega) = \exp\left(\frac{2\pi i}{p}\sum_{j=1}^{\infty}(x_j\omega_{-j} + x_{-j}\omega_j)\right), \tag{2.5}$$

where x_j and ω_j are calculated by (2.1).

We observe that

$$\chi\left(x,\ \frac{m}{p^{n-1}}\right) = \chi\left(\frac{x}{p^{n-1}},\ m\right) = \omega_m\left(\frac{x}{p^{n-1}}\right)\ \forall\, x \in [0,\ p^{n-1}),\ m \in \mathbb{Z}^+.$$

The Walsh-Fourier transform of a function $f \in L^1(\mathbb{R}^+)$ is defined by

$$\hat{f}(\omega) = \int_{\mathbb{R}^+} f(x)\overline{\chi(x,\omega)}dx, \tag{2.6}$$

where $\chi(x,\omega)$ is given by (2.5).

If $f \in L^2(\mathbb{R}^+)$ and

$$J_a f(\omega) = \int_0^a f(x)\overline{\chi(x,\,\omega)}dx\ (a < 0), \tag{2.7}$$

then $\hat{f}$ is defined as limit of $J_a f$ in $L^2(\mathbb{R}^+)$ as $a \to \infty$.

The properties of Walsh-Fourier transform are quite similar to the classical Fourier transform. It is known that systems $\{\chi(\alpha,.)\}_{\alpha=0}^{\infty}$ and $\{\chi(.,\alpha)\}_{\alpha=0}^{\infty}$ are orthonormal bases in $L^2(0,1)$. Let us denote by $\{\omega\}$ the fractional part of ω. For $l \in \mathbb{Z}^+$, we have $\chi(l,\omega) = \chi(l,\{\omega\})$.

If $x,y,\omega \in \mathbb{R}^+$ and $x \oplus y$ is p-adic irrational, then

$$\chi(x\oplus y,\omega) = \chi(x,\omega)\chi(y,\omega),\ \chi(x\ominus y,\omega) = \chi(x,\omega)\overline{\chi(y,\omega)}, \tag{2.8}$$

For $a \in \mathbb{R}^+$ and $n \in \mathbb{Z}^+$, the translation operator T_{na} on $L^2(\mathbb{R}^+)$ is defined by

$$(T_{na}f)(x) = f(x \ominus na),\ x \in \mathbb{R}^+.$$

For $b \in \mathbb{R}^+$ and $m \in \mathbb{Z}^+$, the modulation operator M_{mb} is defined by

$$(M_{mb}f)(x) = \chi(x,mb)f(x),\ x \in \mathbb{R}^+.$$

The dilation operator associated with a non-negative integer p is

$$(D_p f)(x) = \sqrt{p}f(px),\ x \in \mathbb{R}^+.$$

The Plancherel theorem asserts that

$$\langle f, g\rangle = \langle \hat{f}, \hat{g}\rangle$$

For $p > 0,\ a,\ b \in \mathbb{R}^+$ and $f \in L^2(\mathbb{R}^+)$

$$\big(D_{p^j} f\big)^\wedge = D_{p^{-j}} \hat{f},\ (M_{mb} f)^\wedge = T_{mb}\hat{f},\ (T_{na} f)^\wedge = M_{-na}\hat{f}$$

and

$$\big(D_{p^j} T_{na} M_{mb} f\big)^\wedge = D_{p^{-j}} M_{-na} T_{mb} \hat{f}$$

Definition 2.1. A countable family $\{e_\alpha : \alpha \in \mathcal{A}\}$ of elements in a separable Hilbert space $\mathcal{H}$ is a frame if there exist constants $0 < A \leq B < \infty$ satisfying

$$A\|v\|^2 \leq \sum_{\alpha\in\mathcal{A}} |\langle v, e_\alpha\rangle|^2 \leq B\|v\|^2 \tag{2.9}$$

for all $v \in \mathcal{H}$. If only the right hand side inequality holds, we say that $\{e_\alpha : \alpha \in \mathcal{A}\}$ is a Bessel system with constant B. A frame is a tight frame if A and B can be chosen so that $A = B$ and is a normalized tight frame (NTF) if $A = B = 1$. Thus, if $\{e_\alpha : \alpha \in \mathcal{A}\}$ is a NTF in $\mathcal{H}$, then

$$\|v\|^2 = \sum_{\alpha\in\mathcal{A}} |\langle v, e_\alpha\rangle|^2. \tag{2.10}$$

Definition 2.2. A system of the form $\{D_{p^j} T_{na} M_{mb}\psi : j \in \mathbb{Z},\ m, n \in \mathbb{Z}^+\}$ is called a wave packet system on $\mathbb{R}^+$, where p is integer such that $p > 2,\ a, b \in \mathbb{R}^+$ and ψ is a fixed function in $L^2(\mathbb{R}^+)$.

The wave packet system $\{D_{p^j} T_{na} M_{mb}\psi : j \in \mathbb{Z},\ m, n \in \mathbb{Z}^+\}$ is said to be frame of $L^2(\mathbb{R}^+)$ if there exist two positive constants A and B such that $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{j\in\mathbb{Z}} \sum_{m\in\mathbb{Z}^+} \sum_{\in\mathbb{Z}^+} \big|\big\langle f, D_{p^j} T_{na} M_{mb}\psi\big\rangle\big|^2 \leq B\|f\|^2,\ f \in L^2(\mathbb{R}^+). \tag{2.11}$$

3. Necessary Condition for Wave Packet Frames

Theorem 3.1. Suppose that the wave packet systems $\{D_{p^j} T_{na} M_{mb}\psi : j \in \mathbb{Z},\ m, n \in \mathbb{Z}^+\}$ is a frame of $L^2(\mathbb{R}^+)$ with frame bounds A and B. Then, we have

$$A \leq \frac{1}{a} \sum_{j\in\mathbb{Z}} \sum_{m\in\mathbb{Z}^+} \Big|\hat{\psi}\left(p^{-j}\xi \ominus mb\right)\Big|^2 \leq B,\ \text{a.e. } \xi \in \mathbb{R}^+ \tag{3.1}$$

Proof. Since $\{D_{p^j} T_{na} M_{mb}\psi : j \in \mathbb{Z},\ m, n \in \mathbb{Z}^+\}$ is a frame for $L^2(\mathbb{R}^+)$ with frame bounds A and B, we have

$$A\|f\|^2 \leq \sum_{j\in\mathbb{Z}} \sum_{m\in\mathbb{Z}^+} \sum_{n\in\mathbb{Z}^+} \big|\langle f, D_{p^j} T_{na} M_{mb}\psi\rangle\big|^2 \leq B\|f\|^2. \tag{3.2}$$

For any $f \in L^2(\mathbb{R}^+)$, we have

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}^+}\sum_{n\in\mathbb{Z}^+}\left|\langle f, D_{p^j}T_{na}M_{mb}\psi\rangle\right|^2 \\
&\qquad = \sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}^+}\sum_{n\in\mathbb{Z}^+}\left|\langle \mathcal{F}f, \mathcal{F}D_{p^j}T_{na}M_{mb}\psi\rangle\right|^2 \\
&\qquad = \sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}^+}\sum_{n\in\mathbb{Z}^+}\left|\left\langle \hat{f}, D_{p^{-j}}M_{-na}T_{mb}\hat{\psi}\right\rangle\right|^2 \\
&\qquad = \sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}^+}\sum_{n\in\mathbb{Z}^+}\left|\left\langle \hat{f}, M_{-nap^j}D_{p^{-j}}T_{mb}\hat{\psi}\right\rangle\right|^2 \\
&\qquad = \sum_{j\in\mathbb{Z}}p^{-j}\sum_{m\in\mathbb{Z}^+}\sum_{n\in\mathbb{Z}^+}\left|\int_{\mathbb{R}^+}\hat{f}(\xi)\overline{\hat{\psi}(p^{-j}\xi\ominus mb)}\,\overline{\chi(na,(p^{-j}\xi\ominus mb))}d\xi\right|^2 \\
&\qquad = \sum_{j\in\mathbb{Z}}p^{j}\sum_{m\in\mathbb{Z}^+}\sum_{n\in\mathbb{Z}^+}\left|\int_{\mathbb{R}^+}\hat{f}(p^j(\xi\oplus mb))\overline{\hat{\psi}(\xi)}\,\overline{\chi(na,\xi)}d\xi\right|^2. \qquad (3.3)
\end{aligned}$$

For fixed $j \in \mathbb{Z}$ and $m \in \mathbb{Z}^+$, we have

$$\begin{aligned}
&\int_0^{a^{-1}}\sum_{l\in\mathbb{Z}^+}\left|\hat{f}\left(p^j\left(\xi\ominus\frac{l}{a}\oplus mb\right)\right)\overline{\hat{\psi}\left(\xi\ominus\frac{l}{a}\right)}\right|d\xi \\
&\qquad = \sum_{l\in\mathbb{Z}^+}\int_0^{a^{-1}}\left|\hat{f}\left(p^j\left(\xi\ominus\frac{l}{a}\oplus mb\right)\right)\overline{\hat{\psi}\left(\xi\ominus\frac{l}{a}\right)}\right|d\xi \\
&\qquad = \sum_{l\in\mathbb{Z}^+}\int_{la^{-1}}^{la^{-1}+a^{-1}}\left|\hat{f}\left(p^j\left(\xi\oplus mb\right)\right)\overline{\hat{\psi}\left(\xi\right)}\right|d\xi \\
&\qquad = \int_{\mathbb{R}^+}\left|\hat{f}\left(p^j\left(\xi\oplus mb\right)\right)\overline{\hat{\psi}\left(\xi\right)}\right|d\xi. \qquad (3.4)
\end{aligned}$$

By using Cauchy-Schwarz's inequality, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^+}\left|\hat{f}\left(p^j\left(\xi\oplus mb\right)\right)\overline{\hat{\psi}\left(\xi\right)}\right|d\xi \\
&\qquad \le \left(\int_{\mathbb{R}^+}\left|\hat{f}\left(p^j\left(\xi\oplus mb\right)\right)\right|^2 d\xi\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^+}\left|\overline{\hat{\psi}\left(\xi\right)}\right|^2 d\xi\right)^{\frac{1}{2}} < \infty. \qquad (3.5)
\end{aligned}$$

Thus, we can define a function F_j by

$$F_j(\xi) = \sum_{l\in\mathbb{Z}^+}\hat{f}\left(p^j\left(\xi\ominus\frac{l}{a}\oplus mb\right)\right)\overline{\hat{\psi}\left(\xi\ominus\frac{l}{a}\right)} \text{ a.e. } \xi\in\mathbb{R}^+. \qquad (3.6)$$

$F_j(\xi)$ is $\frac{1}{a}$-periodic and the above argument gives that $F_j(\xi) \in L^1(0,\, a^{-1})$. In fact, we even have $F_j(\xi) \in L^2(0,\, a^{-1})$. To see this we first note that

$$|F_j(\xi)|^2 \le \sum_{l\in\mathbb{Z}^+} \left|\hat{f}\left(p^j\left(\xi \ominus \frac{l}{a} \oplus mb\right)\right)\right|^2 \sum_{l\in\mathbb{Z}^+} \left|\hat{\psi}\left(\xi \ominus \frac{l}{a}\right)\right|^2. \tag{3.7}$$

Since $\hat{f} \in C_c(\mathbb{R}^+)$, the function $\xi \to \sum_{l\in\mathbb{Z}^+} \left|\hat{f}\left(p^j\left(\xi \ominus \frac{l}{a} \oplus mb\right)\right)\right|^2$ is bounded. Then, there exists a constant k_1, such that $\sum_{l\in\mathbb{Z}^+} \left|\hat{f}\left(p^j\left(\xi \ominus \frac{l}{a} \oplus mb\right)\right)\right|^2 < k_1$, which implies that

$$\begin{aligned}
\int_0^{a^{-1}} |F_j(\xi)|^2 d\xi &\le \int_0^{a^{-1}} \sum_{l\in\mathbb{Z}^+} \left|\hat{f}\left(p^j\left(\xi \ominus \frac{l}{a} \oplus mb\right)\right)\right|^2 \sum_{l\in\mathbb{Z}^+} \left|\hat{\psi}\left(\xi \ominus \frac{l}{a}\right)\right|^2 d\xi \\
&\le k_1 \int_0^{a^{-1}} \sum_{l\in\mathbb{Z}^+} \left|\hat{\psi}\left(\xi \ominus \frac{l}{a}\right)\right|^2 d\xi \\
&= k_1 \sum_{l\in\mathbb{Z}^+} \int_{la^{-1}}^{la^{-1}+a^{-1}} \left|\hat{\psi}(\xi)\right|^2 d\xi \\
&= k_1 \int_{\mathbb{R}^+} \left|\hat{\psi}(\xi)\right|^2 d\xi < \infty.
\end{aligned}$$

So, $F_j(\xi) \in L^2(0, a^{-1})$. Then, according to the definition of $F_j(\xi)$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^+} \hat{f}\left(p^j\left(\xi \oplus mb\right)\right) \overline{\hat{\psi}(\xi)}\,\overline{\chi(na,\xi)} d\xi \\
&= \sum_{l\in\mathbb{Z}^+} \int_{\frac{l}{a}}^{\frac{l}{a}+\frac{1}{a}} \hat{f}\left(p^j\left(\xi \oplus mb\right)\right) \overline{\hat{\psi}(\xi)}\,\overline{\chi(na,\xi)} d\xi \\
&= \sum_{l\in\mathbb{Z}^+} \int_0^{a^{-1}} \hat{f}\left(p^j\left(\xi \ominus \frac{l}{a} \oplus mb\right)\right) \overline{\hat{\psi}\left(\xi \ominus \frac{l}{a}\right)}\,\overline{\chi(na,\xi)} d\xi \\
&= \int_0^{a^{-1}} \left(\sum_{l\in\mathbb{Z}^+} \hat{f}\left(p^j\left(\xi \ominus \frac{l}{a} \oplus mb\right)\right) \overline{\hat{\psi}\left(\xi \ominus \frac{l}{a}\right)}\right) \overline{\chi(na,\xi)} d\xi \\
&= \int_0^{a^{-1}} F_j(\xi)\overline{\chi(na,\xi)} d\xi.
\end{aligned} \tag{3.8}$$

Parseval's equality shows that

$$\sum_{n\in\mathbb{Z}^+} \left|\int_0^{a^{-1}} F_j(\xi)\overline{\chi(na,\xi)} d\xi\right|^2 = \frac{1}{a} \int_0^{a^{-1}} |F_j(\xi)|^2 d\xi. \tag{3.9}$$

Combining (3.8) and (3.9) and the definition of $F_j(\xi)$, we obtain that

$$\sum_{n\in\mathbb{Z}^+}\left|\int_{\mathbb{R}^+}\hat{f}\left(p^j\left(\xi\oplus mb\right)\right)\overline{\hat{\psi}\left(\xi\right)}\overline{\chi(na,\xi)}d\xi\right|^2$$

$$=\frac{1}{a}\int_0^{a^{-1}}\left|\sum_{l\in\mathbb{Z}^+}\hat{f}\left(p^j\left(\xi\ominus\frac{l}{a}\oplus mb\right)\right)\overline{\hat{\psi}\left(\xi\ominus\frac{l}{a}\right)}\right|^2 d\xi. \tag{3.10}$$

$$\sum_{j\in\mathbb{Z}}p^j\sum_{m\in\mathbb{Z}^+}\sum_{n\in\mathbb{Z}^+}\left|\int_{\mathbb{R}^+}\hat{f}\left(p^j\left(\xi\oplus mb\right)\right)\overline{\hat{\psi}\left(\xi\right)}\overline{\chi(na,\xi)}d\xi\right|^2$$

$$=\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}^+}\frac{p^j}{a}\int_0^{a^{-1}}\left|\sum_{l\in\mathbb{Z}^+}\hat{f}\left(p^j\left(\xi\ominus\frac{l}{a}\oplus mb\right)\right)\overline{\hat{\psi}\left(\xi\ominus\frac{l}{a}\right)}\right|^2 d\xi. \tag{3.11}$$

Let $\xi_0\in\mathbb{R}^+$ and also consider $\hat{f}_\varepsilon=\frac{1}{\sqrt{2\varepsilon}}\chi_{[\xi_0-\varepsilon,\xi_0+\varepsilon]}$, $\varepsilon>0$ and ε be sufficiently small.

Therefore, we obtain

$$\|f_\varepsilon\|^2=\|\hat{f}_\varepsilon\|^2=1 \tag{3.12}$$

Thus, we have

$$\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}^+}\left|\hat{\psi}\left(p^{-j}\xi_0\ominus mb\right)\right|^2=\lim_{\varepsilon\to 0}\int_{\xi_0-\varepsilon}^{\xi_0+\varepsilon}\frac{1}{2\varepsilon}\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}^+}\left|\hat{\psi}\left(p^{-j}\xi\ominus mb\right)\right|^2 d\xi. \tag{3.13}$$

From the definition of f, (3.2), (3.3) and (3.11); we have

$$\int_{\xi_0-\varepsilon}^{\xi_0+\varepsilon}\frac{1}{2\varepsilon}\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}^+}\left|\hat{\psi}\left(p^{-j}\xi\ominus mb\right)\right|^2 d\xi$$

$$=\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}^+}\int_0^{a^{-1}}|\hat{f}_\varepsilon(\xi)|^2\left|\hat{\psi}\left(p^{-j}\xi\ominus mb\right)\right|^2 d\xi$$

$$=\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}^+}p^j\int_0^{a^{-1}}\left|\sum_{l\in\mathbb{Z}^+}\hat{f}_\varepsilon\left(p^j\left(\xi\ominus\frac{l}{a}\oplus mb\right)\right)\overline{\hat{\psi}\left(\xi\ominus\frac{l}{a}\right)}\right|^2 d\xi$$

$$=a\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}^+}\sum_{n\in\mathbb{Z}^+}\left|\langle f_\varepsilon,D_{p^j}T_{na}M_{mb}\psi\rangle\right|^2$$

$$\leq aB. \tag{3.14}$$

We may take $\varepsilon\to 0$, we get

$$\frac{1}{a}\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}^+}\left|\hat{\psi}\left(p^{-j}\xi_0\ominus mb\right)\right|^2\leq B \tag{3.15}$$

On the other hand, for any ξ_0, $\eta>0$, a positive integer M may be chosen so that

$$\sum_{j\leq -M}\sum_{m\in\mathbb{Z}^+}\int_{p^{-j}(\xi_0-\varepsilon)}^{p^{-j}(\xi_0+\varepsilon)}\left|\hat{\psi}\left(\xi\ominus mb\right)\right|^2 d\xi<\eta. \tag{3.16}$$

Also, for $0 < \varepsilon < \frac{p^j}{2a}$, the function $\hat{f}_\varepsilon = \left(\frac{1}{\sqrt{2\varepsilon}}\right)\chi_{[\xi_0-\varepsilon,\xi_0+\varepsilon]}$, $\varepsilon > 0$ satisfies

$$\hat{f}_\varepsilon\left(\xi \ominus p^j \frac{l}{a}\right) = 0 \ \forall\ l \in \mathbb{Z}^+$$

with $l \geq \left(\frac{\varepsilon a}{p^j}\right) + 1$ and for all $\xi \in \left[\xi_0 - \left(\frac{p^j}{2a}\right),\ \xi_0 + \left(\frac{p^j}{2a}\right)\right]$.

Hence, for this $\hat{f}_\varepsilon$, we have

$$\begin{aligned}
&\frac{1}{a}\sum_{j\leq -M}\sum_{m\in\mathbb{Z}^+}\int_{\xi_0-\frac{p^j}{2a}}^{\xi_0+\frac{p^j}{2a}}\left|\sum_{l\in\mathbb{Z}^+}\hat{f}_\varepsilon\left(\xi\ominus p^j\frac{l}{a}\right)\overline{\hat{\psi}\left(p^{-j}\xi\ominus\frac{l}{a}\ominus mb\right)}\right|^2 d\xi\\
&\quad\leq \frac{1}{2\varepsilon a}\sum_{j\leq -M}\sum_{m\in\mathbb{Z}^+}\int_{\xi_0-\frac{p^j}{2a}}^{\xi_0+\frac{p^j}{2a}}\left[\sum_{l\in\mathbb{Z}^+}\left|\hat{\psi}\left(p^{-j}\xi\ominus\frac{l}{a}\ominus mb\right)\right|^2\chi_{[\xi_0-\varepsilon,\,\xi_0+\varepsilon]}\left(\xi\ominus p^j\frac{l}{a}\right)\right]\\
&\qquad\qquad\times\left(\frac{\varepsilon a}{p^j}+1\right)d\xi\\
&\quad\leq \sum_{j\leq -M}\sum_{m\in\mathbb{Z}^+}\int_{(\xi_0-\varepsilon)}^{(\xi_0+\varepsilon)}\left\{\frac{1}{p^j}\left|\hat{\psi}\left(p^{-j}\xi\ominus mb\right)\right|^2+\frac{1}{2\varepsilon a}\left|\hat{\psi}\left(p^{-j}\xi\ominus mb\right)\right|^2\right\}d\xi. \qquad (3.17)
\end{aligned}$$

Now, since

$$\sum_{j\leq -M}\sum_{m\in\mathbb{Z}^+}\frac{1}{p^j}\int_{(\xi_0-\varepsilon)}^{(\xi_0+\varepsilon)}\left|\hat{\psi}\left(p^{-j}\xi\ominus mb\right)\right|^2 d\xi = \sum_{j\leq -M}\sum_{m\in\mathbb{Z}^+}\int_{p^{-j}(\xi_0-\varepsilon)}^{p^{-j}(\xi_0+\varepsilon)}\left|\hat{\psi}\left(\xi\ominus mb\right)\right|^2 d\xi$$

Since $\varepsilon < \frac{p^j}{2a}$, the intervals $\left[p^{-j}(\xi_0-\varepsilon), p^{-j}(\xi_0+\varepsilon)\right]$, $j \in \mathbb{Z}$, are mutually disjoint; and hence, by (3.16), we have

$$\sum_{j\leq -M}\sum_{m\in\mathbb{Z}^+}\int_{p^{-j}(\xi_0-\varepsilon)}^{p^{-j}(\xi_0+\varepsilon)}\left|\hat{\psi}\left(\xi\ominus mb\right)\right|^2 d\xi < \eta.$$

So that it follows from (3.17) that

$$\begin{aligned}
I &= \sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}^+}\frac{p^j}{a}\int_0^{a^{-1}}\left|\sum_{l\in\mathbb{Z}^+}\hat{f}_\varepsilon\left(p^j\left(\xi\ominus\frac{l}{a}\oplus mb\right)\right)\overline{\hat{\psi}\left(\xi\ominus\frac{l}{a}\right)}\right|^2 d\xi\\
&= \sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}^+}\frac{1}{a}\int_0^{\frac{p^j}{a}}\left|\sum_{l\in\mathbb{Z}^+}\hat{f}_\varepsilon\left(\xi\ominus p^j\frac{l}{a}\right)\overline{\hat{\psi}\left(p^{-j}\xi\ominus\frac{l}{a}\ominus mb\right)}\right|^2 d\xi\\
&\leq \sum_{j>-M}\sum_{m\in\mathbb{Z}^+}\frac{1}{a}\int_0^{\frac{p^j}{a}}\left|\sum_{l\in\mathbb{Z}^+}\hat{f}_\varepsilon\left(\xi\ominus p^j\frac{l}{a}\right)\overline{\hat{\psi}\left(p^{-j}\xi\ominus\frac{l}{a}\ominus mb\right)}\right|^2 d\xi+\eta\\
&\qquad+\frac{1}{2\varepsilon a}\int_{\xi_0-\varepsilon}^{\xi_0+\varepsilon}\sum_{j\leq -M}\sum_{m\in\mathbb{Z}^+}\left|\hat{\psi}\left(p^{-j}\xi\ominus mb\right)\right|^2 d\xi. \qquad (3.18)
\end{aligned}$$

Therefore, by (3.2), (3.3), (3.11) and (3.18), we have

$$I^* = \sum_{j>-M}\sum_{m\in\mathbb{Z}^+}\frac{1}{a}\int_0^{\frac{p^j}{a}}\left|\sum_{l\in\mathbb{Z}^+}\hat{f}_\varepsilon\left(\xi\ominus p^j\frac{l}{a}\right)\overline{\hat{\psi}\left(p^{-j}\xi\ominus\frac{l}{a}\ominus mb\right)}\right|^2 d\xi$$

$$\geq A-\eta-\frac{1}{2\varepsilon a}\int_{\xi_0-\varepsilon}^{\xi_0+\varepsilon}\sum_{j\leq -M}\sum_{m\in\mathbb{Z}^+}\left|\hat{\psi}\left(p^{-j}\xi\ominus mb\right)\right|^2 d\xi. \tag{3.19}$$

On the other hand, for all sufficiently small $\varepsilon>0$, it is clear that

$$I^* = \sum_{j>-M}\sum_{m\in\mathbb{Z}^+}\frac{1}{a}\int_{\xi_0-\varepsilon}^{\xi_0+\varepsilon}\left|\hat{f}_\varepsilon(\xi)\,\hat{\psi}\left(p^{-j}\xi\ominus mb\right)\right|^2 d\xi$$

$$= \sum_{j>-M}\sum_{m\in\mathbb{Z}^+}\frac{1}{2a\varepsilon}\int_{\xi_0-\varepsilon}^{\xi_0+\varepsilon}\left|\hat{\psi}\left(p^{-j}\xi\ominus mb\right)\right|^2 d\xi$$

where $\hat{f}_\varepsilon = \left(\frac{1}{\sqrt{2\varepsilon}}\right)\chi_{[\xi_0-\varepsilon,\xi_0+\varepsilon]}$. Taking $\varepsilon\to 0$ in (3.19), we get

$$\sum_{j>-M}\sum_{m\in\mathbb{Z}^+}\frac{1}{a}\left|\hat{\psi}\left(p^{-j}\xi_0\ominus mb\right)\right|^2 \geq A-\eta-\sum_{j\leq -M}\sum_{m\in\mathbb{Z}^+}\frac{1}{a}\left|\hat{\psi}\left(p^{-j}\xi_0\ominus mb\right)\right|^2 \tag{3.20}$$

for almost all $\xi_0>0$. Since $\eta>0$ is arbitrary, (3.20) gives

$$\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}^+}\left|\hat{\psi}\left(p^{-j}\xi_0\ominus mb\right)\right|^2 \geq aA. \tag{3.21}$$

for almost all $\xi_0>0$. Hence by (3.15) and (3.21), we have

$$A \leq \frac{1}{a}\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}^+}\left|\hat{\psi}\left(p^{-j}\xi_0\ominus mb\right)\right|^2 \leq B,\ \text{ a.e. } \xi_0\in\mathbb{R}^+$$

Changing variable by $\xi=\xi_0$, we have

$$A \leq \frac{1}{a}\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}^+}\left|\hat{\psi}\left(p^{-j}\xi\ominus mb\right)\right|^2 \leq B,\ \text{ a.e. } \xi\in\mathbb{R}^+$$

4. Sufficient Condition for Wave Packet Frames

Theorem 4.1. Let $\psi\in L^2(\mathbb{R}^+)$ be such that

$$B=\frac{1}{a}\sup_{\xi\in\mathbb{R}^+}\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}^+}\sum_{n\in\mathbb{Z}^+}\left|\hat{\psi}\left(p^{-j}\xi\ominus mb\right)\hat{\psi}\left(p^{-j}\xi\ominus mb\ominus\frac{n}{a}\right)\right|<\infty,$$

then $\{D_{p^j}T_{na}M_{mb}\psi\}_{j\in\mathbb{Z},\,m,n\in\mathbb{Z}^+}$ is a Bessel sequence with bound B. Also, if

$$A=\frac{1}{a}\inf_{\xi\in\mathbb{R}^+}\left(\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}^+}\left|\hat{\psi}\left(p^{-j}\xi\ominus mb\right)\right|^2 -\sum_{j\in\mathbb{Z}}\sum_{0\neq n\in\mathbb{Z}^+}\sum_{m\in\mathbb{Z}^+}\left|\hat{\psi}\left(p^{-j}\xi\ominus mb\right)\hat{\psi}\left(p^{-j}\xi\ominus mb\ominus\frac{n}{a}\right)\right|\right)>0,$$

then $\{D_{p^j}T_{na}M_{mb}\psi\}_{j\in\mathbb{Z},\, m,n\in\mathbb{Z}^+}$ is a frame for $L^2(\mathbb{R}^+)$ with bounds A and B.

Proof. From the definition of F_j and (3.10), we have

$$\sum_{l\in\mathbb{Z}^+}\left|\int_{\mathbb{R}^+}\hat{f}(p^j(\xi\oplus mb))\overline{\hat{\psi}(\xi)}\overline{\chi(la,\xi)}d\xi\right|^2=\frac{1}{a}\int_0^{a^{-1}}|F_j(\xi)|^2\,d\xi.$$

For all $f\in L^2(\mathbb{R}^+)$ such that $\hat{f}$ is continuous and has compact support, we have from (3.3)

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}^+}\sum_{n\in\mathbb{Z}^+}\left|\langle f, D_{p^j}T_{na}M_{mb}\psi\rangle\right|^2\\
&\qquad=\sum_{j\in\mathbb{Z}}p^j\sum_{m\in\mathbb{Z}^+}\sum_{n\in\mathbb{Z}^+}\left|\int_{\mathbb{R}^+}\hat{f}(p^j(\xi\oplus mb))\overline{\hat{\psi}(\xi)}\overline{\chi(na,\xi)}d\xi\right|^2\\
&\qquad=\sum_{j\in\mathbb{Z}}\frac{p^j}{a}\sum_{m\in\mathbb{Z}^+}\int_0^{a^{-1}}\left|\sum_{l\in\mathbb{Z}^+}\hat{f}\left(p^j\left(\xi\ominus\frac{l}{a}\oplus mb\right)\right)\overline{\hat{\psi}\left(\xi\ominus\frac{l}{a}\right)}\right|^2d\xi\\
&\qquad\le\sum_{j\in\mathbb{Z}}\frac{p^j}{a}\sum_{m\in\mathbb{Z}^+}\int_0^{a^{-1}}\left(\sum_{l\in\mathbb{Z}^+}\left|\hat{f}\left(p^j\left(\xi\ominus\frac{l}{a}\oplus mb\right)\right)\overline{\hat{\psi}\left(\xi\ominus\frac{l}{a}\right)}\right|\right)^2d\xi\\
&\qquad=*.
\end{aligned}$$

By using Cauchy-Schwarz inequality, we have

$$\begin{aligned}
*\le&\sum_{j\in\mathbb{Z}}\frac{p^j}{a}\sum_{m\in\mathbb{Z}^+}\int_0^{a^{-1}}\left(\sum_{l\in\mathbb{Z}^+}\left|\hat{f}\left(p^j\left(\xi\ominus\frac{l}{a}\oplus mb\right)\right)\overline{\hat{\psi}\left(\xi\ominus\frac{l}{a}\right)}\right|\right.\\
&\qquad\left.\times\sum_{n\in\mathbb{Z}^+}\left|\overline{\hat{f}\left(p^j\left(\xi\ominus\frac{n}{a}\oplus mb\right)\right)}\hat{\psi}\left(\xi\ominus\frac{n}{a}\right)\right|\right)d\xi\\
=&\sum_{j\in\mathbb{Z}}\frac{p^j}{a}\sum_{m\in\mathbb{Z}^+}\sum_{l\in\mathbb{Z}^+}\int_0^{a^{-1}}\left|\hat{f}\left(p^j\left(\xi\ominus\frac{l}{a}\oplus mb\right)\right)\overline{\hat{\psi}\left(\xi\ominus\frac{l}{a}\right)}\right|\\
&\qquad\times\sum_{n\in\mathbb{Z}^+}\left|\overline{\hat{f}\left(p^j\left(\xi\ominus\frac{n}{a}\oplus mb\right)\right)}\hat{\psi}\left(\xi\ominus\frac{n}{a}\right)\right|d\xi\\
=&\sum_{j\in\mathbb{Z}}\frac{p^j}{a}\sum_{m\in\mathbb{Z}^+}\sum_{l\in\mathbb{Z}^+}\int_{\frac{l}{a}}^{\frac{l}{a}+\frac{1}{a}}\left|\hat{f}\left(p^j\left(\xi\ominus mb\right)\right)\overline{\hat{\psi}\left(\xi\right)}\right|\\
&\qquad\times\sum_{n\in\mathbb{Z}^+}\left|\overline{\hat{f}\left(p^j\left(\xi\ominus\frac{n}{a}\oplus mb\right)\right)}\hat{\psi}\left(\xi\ominus\frac{n}{a}\right)\right|d\xi
\end{aligned}$$

$$= \sum_{j\in\mathbb{Z}} \frac{p^j}{a} \sum_{m\in\mathbb{Z}^+} \int_{\mathbb{R}^+} \left| \hat{f}\left(p^j (\xi \oplus mb)\right) \overline{\hat{\psi}(\xi)} \right| \sum_{n\in\mathbb{Z}^+} \left| \overline{\hat{f}\left(p^j \left(\xi \ominus \frac{n}{a} \oplus mb\right)\right)} \hat{\psi}\left(\xi \ominus \frac{n}{a}\right) \right| d\xi$$

$$= \sum_{j\in\mathbb{Z}} \frac{p^j}{a} \sum_{m\in\mathbb{Z}^+} \sum_{n\in\mathbb{Z}^+} \int_{\mathbb{R}^+} \left| \hat{f}\left(p^j (\xi \oplus mb)\right) \overline{\hat{f}\left(p^j \left(\xi \ominus \frac{n}{a} \oplus mb\right)\right)} \overline{\hat{\psi}(\xi)} \hat{\psi}\left(\xi \ominus \frac{n}{a}\right) \right| d\xi$$

$$= \frac{1}{a} \sum_{j\in\mathbb{Z}} p^j \sum_{m\in\mathbb{Z}^+} \int_{\mathbb{R}^+} \left| \hat{f}\left(p^j (\xi \oplus mb)\right) \right|^2 \left| \hat{\psi}(\xi) \right|^2 d\xi$$

$$+ \frac{1}{a} \sum_{j\in\mathbb{Z}} p^j \sum_{m\in\mathbb{Z}^+} \sum_{0\neq n\in\mathbb{Z}^+} \int_{\mathbb{R}^+} \left| \hat{f}\left(p^j (\xi \oplus mb)\right) \overline{\hat{f}\left(p^j \left(\xi \ominus \frac{n}{a} \oplus mb\right)\right)} \overline{\hat{\psi}(\xi)} \hat{\psi}\left(\xi \ominus \frac{n}{a}\right) \right| d\xi$$

$$= \frac{1}{a} \sum_{j\in\mathbb{Z}} p^j \sum_{m\in\mathbb{Z}^+} \int_{\mathbb{R}^+} \left| \hat{f}\left(p^j (\xi \oplus mb)\right) \right|^2 \left| \hat{\psi}(\xi) \right|^2 d\xi + \frac{1}{a} R(f),$$

where

$$R(f) = \sum_{j\in\mathbb{Z}} p^j \sum_{m\in\mathbb{Z}^+} \sum_{0\neq n\in\mathbb{Z}^+} \int_{\mathbb{R}^+} \left| \hat{f}\left(p^j (\xi \oplus mb)\right) \overline{\hat{f}\left(p^j \left(\xi \ominus \frac{n}{a} \oplus mb\right)\right)} \overline{\hat{\psi}(\xi)} \hat{\psi}\left(\xi \ominus \frac{n}{a}\right) \right| d\xi$$

The Cauchy-Schwarz inequality implies that

$$R(f) \leq \sum_{j\in\mathbb{Z}} p^j \sum_{m\in\mathbb{Z}^+} \sum_{0\neq n\in\mathbb{Z}^+} \left(\int_{\mathbb{R}^+} \left| \hat{f}\left(p^j (\xi \oplus mb)\right) \right|^2 \left| \hat{\psi}(\xi) \hat{\psi}\left(\xi \ominus \frac{n}{a}\right) \right| d\xi \right)^{\frac{1}{2}}$$
$$\times \left(\int_{\mathbb{R}^+} \left| \hat{f}\left(p^j \left(\xi \ominus \frac{n}{a} \oplus mb\right)\right) \right|^2 \left| \hat{\psi}(\xi) \hat{\psi}\left(\xi \ominus \frac{n}{a}\right) \right| d\xi \right)^{\frac{1}{2}}$$

$$\leq \sum_{j\in\mathbb{Z}} p^j \sum_{m\in\mathbb{Z}^+} \left(\sum_{0\neq n\in\mathbb{Z}^+} \int_{\mathbb{R}^+} \left| \hat{f}\left(p^j (\xi \oplus mb)\right) \right|^2 \hat{\psi}(\xi) \hat{\psi}\left(\xi \ominus \frac{n}{a}\right) d\xi \right)^{\frac{1}{2}}$$
$$\times \left(\sum_{0\neq n\in\mathbb{Z}^+} \int_{\mathbb{R}^+} \left| \hat{f}\left(p^j \left(\xi \ominus \frac{n}{a} \oplus mb\right)\right) \right|^2 \hat{\psi}(\xi) \hat{\psi}\left(\xi \ominus \frac{n}{a}\right) d\xi \right)^{\frac{1}{2}}$$

$$= \sum_{j\in\mathbb{Z}} p^j \sum_{m\in\mathbb{Z}^+} \sum_{0\neq n\in\mathbb{Z}^+} \int_{\mathbb{R}^+} \left| \hat{f}\left(p^j (\xi \oplus mb)\right) \right|^2 \left| \hat{\psi}(\xi) \hat{\psi}\left(\xi \ominus \frac{n}{a}\right) \right| d\xi.$$

So,

$$\sum_{j\in\mathbb{Z}} \sum_{m\in\mathbb{Z}^+} \sum_{n\in\mathbb{Z}^+} \left| \langle f, D_{p^j} T_{na} M_{mb} \psi \rangle \right|^2 \leq \frac{1}{a} \sum_{j\in\mathbb{Z}} p^j \sum_{m\in\mathbb{Z}^+} \int_{\mathbb{R}^+} \left| \hat{f}\left(p^j (\xi \oplus mb)\right) \right|^2 \left| \hat{\psi}(\xi) \right|^2 d\xi$$
$$+ \frac{1}{a} \sum_{j\in\mathbb{Z}} p^j \sum_{m\in\mathbb{Z}^+} \sum_{0\neq n\in\mathbb{Z}^+} \int_{\mathbb{R}^+} \left| \hat{f}\left(p^j (\xi \oplus mb)\right) \right|^2 \left| \hat{\psi}(\xi) \hat{\psi}\left(\xi \ominus \frac{n}{a}\right) \right| d\xi$$

$$= \frac{1}{a}\sum_{j\in\mathbb{Z}} p^j \sum_{m\in\mathbb{Z}^+}\sum_{k\in\mathbb{Z}^+}\int_{\mathbb{R}^+}\left|\hat{f}\left(p^j\left(\xi\oplus mb\right)\right)\right|^2\left|\hat{\psi}\left(\xi\right)-\hat{\psi}\left(\xi\ominus\frac{n}{a}\right)\right|d\xi$$

$$= \frac{1}{a}\int_{\mathbb{R}^+}\left|\hat{f}\left(\xi\right)\right|^2\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}^+}\sum_{n\in\mathbb{Z}^+}\left|\hat{\psi}\left(p^{-j}\xi\ominus mb\right)\hat{\psi}\left(p^{-j}\xi\ominus\frac{n}{a}\ominus mb\right)\right|d\xi.$$

If

$$B = \sup_{\xi\in\mathbb{R}^+}\frac{1}{a}\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}^+}\sum_{k\in\mathbb{Z}^+}\left|\hat{\psi}\left(p^{-j}\xi\ominus mb\right)\hat{\psi}\left(p^{-j}\xi\ominus\frac{n}{a}\ominus mb\right)\right| < \infty,$$

then $\{D_{p^j}T_{na}M_{mb}\psi\}_{j\in\mathbb{Z},\, m,n\in\mathbb{Z}^+}$ is a Bessel sequence with bound B, and also

$$\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}^+}\sum_{n\in\mathbb{Z}^+}\left|\langle f, D_{p^j}T_{na}M_{mb}\psi\rangle\right|^2$$

$$\geq \frac{1}{a}\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}^+}\int_{\mathbb{R}^+}\left|\hat{f}\left(\xi\right)\right|^2\left|\hat{\psi}\left(p^{-j}\xi\ominus mb\right)\right|^2 d\xi$$
$$-\left|\frac{1}{a}\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}^+}\sum_{0\neq n\in\mathbb{Z}^+}\int_{\mathbb{R}^+}\left|\hat{f}\left(\xi\right)\hat{f}\left(\xi\ominus p^j\frac{n}{a}\right)\overline{\hat{\psi}(p^{-j}\xi\ominus mb)}\hat{\psi}(p^{-j}\xi\ominus\frac{n}{a}\ominus mb)\right|d\xi\right|$$

$$\geq \int_{\mathbb{R}^+}\left|\hat{f}\left(\xi\right)\right|^2\left(\frac{1}{a}\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}^+}\left|\hat{\psi}\left(p^{-j}\xi\ominus mb\right)\right|^2\right.$$
$$\left.-\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}^+}\sum_{0\neq n\in\mathbb{Z}^+}\frac{1}{a}\left|\hat{\psi}(p^{-j}\xi\ominus mb)\hat{\psi}(p^{-j}\xi\ominus\frac{a}{a}\ominus mb)\right|\right)d\xi.$$

This implies that if

$$A = \frac{1}{a}\inf_{\xi\in\mathbb{R}^+}\left(\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}^+}\left|\hat{\psi}\left(p^{-j}\xi\ominus mb\right)\right|^2\right.$$
$$\left.-\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}^+}\sum_{0\neq n\in\mathbb{Z}^+}\left|\hat{\psi}(p^{-j}\xi\ominus mb)\hat{\psi}(p^{-j}\xi\ominus\frac{n}{a}\ominus mb)\right|\right) > 0.$$

Then, $\{D_{p^j}T_{na}M_{mb}\psi\}_{j\in\mathbb{Z},\, m,n\in\mathbb{Z}^+}$ is a frame for $L^2(\mathbb{R}^+)$ with bounds A and B.

References

[1] Abdullah, (2013), Tight wave packet frames for $L^2(\mathbb{R})$ and $H^2(\mathbb{R})$, Arab J. Math. Sci., 19(2), pp. 151-158.

[2] Christensen,O. and Rahimi,A., (2008), An introduction to wave packet systems in $L^2(\mathbb{R})$, Indian J. Ind. Appl. Math., 1, pp. 42-57.

[3] Christensen,O., (2003), An Introduction to Frames and Riesz Bases, Birkhuser, Boston.

[4] Chui,C.K. and Shi,X., (1993), Inequalities of Littlewood-Paley type for frames and wavelets, SIAM J. Math. Anal., 24 no. 1, pp. 263-277.

[5] Córdoba,A. and Fefferman,C., (1978), Wave packets and Fourier integral operators, Comm. Partial Diff. Eq., 3, pp. 979-1005.

[6] Czaja,W., Kutyniok,C. and Speegle,D., (2006), The Geometry of sets of parameters of wave packet frames, Appl. Comput. Harmon. Anal., 20, pp. 108-125.
[7] Daubechies,I., Grossmann,A. and Meyer,Y., (1986), Painless non-orthogonal expansions, J. Math. Phys. 27 (5), pp. 1271-1283.
[8] Duffin,R.J. and Shaeffer,A.C., (1952), A class of nonharmonic Fourier series, Trans. Am. Math. Soc. 72, pp. 341-366.
[9] Farkov,Y.A., (2005), Orthogonal p-wavelets on R+, in Proceedings of International Conference Wavelets and Splines, St. Petersberg State University, St. Petersberg, pp. 426.
[10] Farkov,Y.A., Maksimov,A.Y. and Stroganov,S.A., (2011), On biorthogonal wavelets related to the Walsh functions, Int. J. Wavelets, Multiresolut. Inf. Process. 9(3), pp. 485-499.
[11] Grchenig,K., (2001), Foundations of TimeFrequency Analysis, Birkhuser, Boston.
[12] Hernández,E., Labate,D. and Weiss,G., (2002), A unified characterization of reproducing systems generated by a finite family, J. Geom. Anal. 12 (4), pp. 615-662.
[13] Hernández,E., Labate,D., Weiss,G. and Wilson,E., (2003), Oversampling, quasi affine frames and wave packets, Appl. Comput. Harmon. Anal. 16, pp. 111-147.
[14] Hernández,E., Weiss,G. and Wilson,E., (2004), An approach to the study of wave packet systems, Contemp. Mathematics, Wavelets, Frames and Operator Theory, 345, pp. 215-235.
[15] Meenakshi, Manchanda,P. and Siddiqi,A.H., (2011), Wavelets associated with Nonuniform multiresolution analysis on positive half line, Int. J. Wavelets, Multiresolut. Inf. Process., 10(2), 1250018, pp. 27.
[16] Ron,A. and Shen,Z., (2005), Generalized shift-invariant systems, Const. Appr., 22, pp. 1-45.
[17] Shah,F.A. and Abdullah, (2012), Necessary condition for the existence of wave packet frames, Southeast Asian Bull. of Math., 36, pp. 287-292.
[18] Shah,F.A., (2012), Gabor frames on a half-line, J. cont. Math. Anal. 5, 47 (2012), pp. 251-260.
[19] Shah,F.A., (2009), Construction of wavelet packets on p-adic field, Int. J. Wavelets, Multiresolut. Inf. Process., 7, pp. 553-565.
[20] Shah,F.A., (2013), Tight wavelet frames generated by the Walsh polynomials, Int. J. Wavelets, Multiresolut. Inf. Process., 11(6), 1350042, pp. 15.
[21] Shah,F.A. and Debnath,L., (2011), Dyadic wavelet frames on a half-line using the Walsh-Fourier transform, Integ. Trans. Special Funct., 22(7), pp. 477-486.
[22] Shah,F.A. and Debnath,L., (2011), p-Wavelet frame packets on a half-line using the Walsh–Fourier transform, Integ. Trans. Special Funct., 22, pp. 907-917.

Dr. Abdullah is currently an assistant professor in Department of Mathematics, Zakir Husain Delhi College, University of Delhi, Delhi, India. He received his Ph.D. degree from Jamia Millia Islamia, New Delhi in 2007 on topic A Study of Function Spaces Using Wavelet Packets. His research interests include wavelet analysis and frame theory. He has to his credit 20 research papers in Arab J Math Sci, Analysis, Applied Mathematics and Computation, Journal of Contemporary Mathematical Analysis, Complex Anal. Oper. Theory, Anal. Theory Appl., etc.