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On Generalized Contraction Principles over $S-$metric Spaces with Application to Homotopy

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Abstract − In the present paper, we introduce the concept of a class of generalized contraction mappings called A-contraction on S-metric space and investigate the existence of fixed points over such spaces. Analogue result has been formulated in integral setting over such an S-metric space. Moreover, the result is applied to homotopy theory.

Keywords − *Fixed point, A-contraction, S-metric space.*

1. Introduction and Preliminaries

In sixties, attempts were initiated through the study of 2-metric spaces by S.Gähler [1,2] to generalize the metric space. However, Ha et al. [3] have pointed out that the results over 2-metrics spaces are independent, rather than generalizations, of the corresponding results in metric spaces. Another such generalization is D-metric space introduced by Dhage [4] in 1992 where he proved some results on fixed points of contraction mappings over complete and bounded D-metric spaces. But in 2006, Mustafa and Sims [5] pointed out that Dhage's notion of a D-metric space is fundamentally flawed and most of the results claimed by Dhage and others are invalid. They introduced a more appropriate and robust version of a generalized metric space namely G-metric space in 2006. Sedghi et al. [6,7] improved and modified D-metric space and thus introduced D^*-metric space. They proved some basic properties of D^*-metric spaces and some fixed point theorems on it. In continuation with untiring attempts to find a most appropriate one, Sedghi et al. [8,9] recently introduced and characterized the concept of S-metric space which modifies D-metric and G-metric spaces.

Definition 1.1. **(S-metric space)** Let X be a non-empty set. An S-metric on X is a function $S : X^3 \to [0, \infty)$ that satisfies the following conditions, for each $x, y, z, a \in X$,
(S1) $S(x, y, z) = 0$, if and only if $x = y = z$,
(S2) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.
The pair (X, S) is called an S-metric space.

According to Sedghi et al. [8], some of the examples of such S-metric spaces are:
(1) Let $X = \mathbb{R}^n$ and $\|.\|$ be a norm on X, then $S(x, y, z) = \|y + z - 2x\| + \|y - z\|$ is an S-metric on X.
(2) Let $X = \mathbb{R}^n$ and $\|.\|$ be a norm on X, then $S(x, y, z) = \|x - z\| + \|y - z\|$ is an S-metric on X.

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(3) Let X be a nonempty set, d be a metric on X, then $S(x,y,z) = d(x,z) + d(y,z)$ is an S-metric on X.
(4) [intuitive geometric example for S-metric] Let $X = \mathbb{R}^2$, d be a metric on X, therefore, $S(x,y,z) = d(x,y) + d(x,z) + d(y,z)$ is an S-metric on X. If we connect the points x, y, z by a line, we have a triangle and if we choose a point a within the triangle, then the inequality $S(x,y,z) \leq S(x,x,a) + S(y,y,a) + S(z,z,a)$ holds.
(5) Let $\mathbb{R}$ be the real line. Then $S(x,y,z) = |x-z| + |y-z|$ for all $x,y,z \in \mathbb{R}$ is an S-metric on $\mathbb{R}$. This S-metric on $\mathbb{R}$ is called the usual S-metric on $\mathbb{R}$.

Definition 1.2. [8] Let (X,S) be an S-metric space and $A \subset X$.
(1) A subset A of X is called S-bounded if there exists $r > 0$ such that $S(x,x,y) < r$ for all $x,y \in A$.
(2) A sequence $\{x_n\}$ in X converges to $x \in X$ if and only if $S(x_n,x_n,x) \to 0$ as $n \to \infty$. That is for every $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $S(x_n,x_n,x) < \epsilon$ whenever $n \geq n_0$. We denote this by $\lim_{n\to\infty} x_n = x$ or $\lim_{n\to\infty} S(x_n,x_n,x) = 0$.
(3) A sequence $\{x_n\}$ in X is called a Cauchy sequence if $S(x_n,x_n,x_m) \to 0$ as $n,m \to \infty$. That is for every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S(x_n,x_n,x_m) < \epsilon$ whenever $n,m \geq n_0$.
(4) The S-metric space (X,S) is called complete if every Cauchy sequence is convergent to an element of X.

Lemma 1.3. [8] For a S-metric space X, we have $S(x,x,y) = S(y,y,x)$ $\forall x,y \in X$.

Lemma 1.4. [9] Let (X,S) be an S-metric space. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$, then $S(x_n,x_n,y_n) \to S(x,x,y)$ as $n \to \infty$.

Definition 1.5. [9] Let $T : X \to Y$ be a map from an S-metric space X to an S-metric space Y. Then T is continuous at $x \in X$ if and only if $Tx_n \to Tx$ in Y whenever $x_n \to x$ in X.

A mapping T is continuous at X if and only if it is continuous at all $x \in X$.

Theorem 1.6. [8] Let (X,S) be a complete S-metric space and let $F : X \to X$ be a contraction i.e

$$S\left(F(x),F(x),F(y)\right) \leq LS\left(x,x,y\right) \quad \text{for all } x,y \in X$$

where $0 \leq L < 1$. Then F has a unique fixed point $u \in X$. Furthermore, for any $x \in X$ we have $\lim_{n\to\infty} F^n(x) = u$ with

$$S\left(F^n(x),F^n(x),u\right) \leq \frac{2L^n}{1-L} S\left(x,x,F(x)\right).$$

Theorem 1.7. [8] Let (X,S) be a compact S-metric space and let $F : X \to X$ satisfying

$$S\left(F(x),F(x),F(y)\right) < S\left(x,x,y\right) \quad \text{for all } x,y \in X \text{ and } x \neq y.$$

Then F has a unique fixed point in X.

2. A-contraction and fixed point

Akram et al. [10, 11] have defined A-contractions as follows: Let a nonempty set A consisting of all functions $\alpha : \mathbb{R}^3_+ \to \mathbb{R}_+$ satisfying
(A_1) : α is continuous on the set $\mathbb{R}^3_+$ of all triplets of nonnegative reals (with respect to the Euclidean metric on $\mathbb{R}^3$).
(A_2) : $a \leq kb$ for some $k \in [0,1)$ whenever $a \leq \alpha(a,b,b)$ or $a \leq \alpha(b,a,b)$ or $a \leq \alpha(b,b,a)$, for all $a,b \in \mathbb{R}_+$.

Definition 2.1. [10] A self map T on a metric space X is said to be A-contraction if it satisfies the condition

$$d\left(Tx,Ty\right) \leq \alpha\left(d\left(x,y\right), d\left(x,Tx\right), d\left(y,Ty\right)\right)$$

for all $x,y \in X$ and for some α in A.

Following the definition of A-contraction mappping on a metric space (see [10]- [11]) and over a 2-metric space (see [12]), we now define A-contractions on an S-metric space and prove fixed point theorem on it.

Definition 2.2. A self map T on an S-metric space X is said to be A-contraction if it satisfies the condition

$$S\left(Tx,Tx,Ty\right)\leq\alpha\left(S\left(x,x,y\right),S\left(x,x,Tx\right),S\left(y,y,Ty\right)\right)$$

for all $x,y\in X$ and for some α in A.

Now we state our main theorem.

Theorem 2.3. Let (X,S) be a complete S-metric space and let T be A-contraction mapping on X. Then, T has a unique fixed point in X.

Proof. Let x_0 be an arbitrary element of X and consider the sequence $\{x_n\}$ of iterates $x_{n+1}=Tx_n$, $n\in N$. Now

$$S\left(x_1,x_1,x_2\right)=S\left(Tx_0,Tx_0,Tx_1\right)\leq\alpha\left(S\left(x_0,x_0,x_1\right),S\left(x_0,x_0,x_1\right),S\left(x_1,x_1,x_2\right)\right)$$

implies

$$S\left(x_1,x_1,x_2\right)\leq kS\left(x_0,x_0,x_1\right)\tag{1}$$

for some $k\in[0,1)$ because $\alpha\in A$. By easy iteration one can check that

$$S\left(x_n,x_n,x_{n+1}\right)\leq k^nS\left(x_0,x_0,x_1\right).\tag{2}$$

For all $m>n$ and by using Lemma 1.3 and (S2) we get

$$\begin{aligned}S\left(x_n,x_n,x_m\right)&\leq 2\sum_{i=n}^{m-2}S\left(x_i,x_i,x_{i+1}\right)+S\left(x_{m-1},x_{m-1},x_m\right)\\&\leq 2\sum_{i=n}^{m-2}k^iS\left(x_0,x_0,x_1\right)+k^{m-1}S\left(x_0,x_0,x_1\right)\\&\leq 2[k^n+k^{n+1}+...k^{m-1}]S\left(x_0,x_0,x_1\right)\\&\leq \frac{2k^n}{1-k}S\left(x_0,x_0,x_1\right).\end{aligned}$$

Taking limit as $m,n\to\infty$ we get $S\left(x_n,x_n,x_m\right)\to 0$. This proves that the sequence $\{x_n\}$ is Cauchy and by completeness of X, $x_n\to z$ for some $z\in X$ as $n\to\infty$. Now,

$$\begin{aligned}S(z,z,Tz)&\leq 2S(z,z,x_{n+1})+S(Tz,Tz,x_{n+1})\\&=2S(x_{n+1},x_{n+1},z)+S(Tx_n,Tx_n,Tz)\\&\leq 2S(x_{n+1},x_{n+1},z)+\alpha(S(x_n,x_n,z),S(x_n,x_n,Tx_n),S(z,z,Tz))\\&=2S(x_{n+1},x_{n+1},z)+\alpha(S(x_n,x_n,z),S(x_n,x_n,x_{n+1}),S(z,z,Tz)).\end{aligned}$$

Therefore by taking limit as $n\to\infty$ we get $S(z,z,Tz)\leq\alpha(0,0,S(z,z,Tz))$, which implies that $S(z,z,Tz)=0$. So z is a fixed point of T. For uniqueness, let $u,v\in X$ be two distinct fixed points of T. So by definition of A-contraction,

$$\begin{aligned}S(u,u,v)=S(Tu,Tu,Tv)&\leq\alpha\left(S\left(u,u,v\right),S\left(u,u,Tu\right),S\left(v,v,Tv\right)\right)\\&=\alpha\left(S\left(u,u,v\right),S\left(u,u,u\right),S\left(v,v,v\right)\right)\\&=\alpha\left(S\left(u,u,v\right),0,0\right).\end{aligned}$$

Then by axiom A_2 of α we have $u=v$ and so the fixed point is unique. □

Now we give an example in support of the Theorem 2.3.

Example 2.4. First we take a function $\alpha : \mathbb{R}^3_+ \to \mathbb{R}_+$ as $\alpha(x, y, z) = \beta.(y + z)$, where $0 < \beta < \frac{1}{2}$, which satisfies the property (A_1) obviously. Now

$$a \leq \alpha(a, b, b) = \beta.(b + b) = 2\beta.b \text{ implies } a \leq k.b \text{ where } k = 2\beta < 1,$$

$$a \leq \alpha(b, a, b) = \beta.(a + b) \text{ implies } a \leq k.b \text{ where } k = \frac{\beta}{1-\beta} < 1 \text{ and also}$$

$$a \leq \alpha(b, b, a) = \beta.(b + a) = \text{ implies } a \leq k.b \text{ where } k = \frac{\beta}{1-\beta} < 1.$$

So α satisfies the property (A_2). Now Let $X = [0, 1]$ and $S(x, y, z) = |x - z| + |y - z|$. Clearly (X, S) is a complete S metric space. Let $T : X \to X$ be given by

$$T(x) = \begin{cases} \frac{x}{4}, & \text{for } x \in [0, \frac{1}{2}) \\ \frac{x}{5}, & \text{for } x \in (\frac{1}{2}, 1]. \end{cases}$$

One can check that T is an A-contraction on $X = [0, 1]$ and satisfies all the conditions of the Theorem 2.3. Also T has a unique fixed point at $x = 0$.

Now we show that the above Theorem 2.3 holds for A-contraction mapping, in absence of which, the map T fails to produce any fixed point in the underlying space though other conditions remain invariant.

Example 2.5. Let $X = [0, 1] \subset \mathbb{R}$ and $S(x, y, z) = |x - z| + |y - z|$. Then, (X, S) is a complete S metric space. Take a function α as defined in the previous Example 2.4. Then, α satisfies the properties (A_1) and (A_2). If we assume $T : X \to X$ as

$$T(x) = \begin{cases} 1, & \text{for } x \in [0, 1) \\ \frac{1}{3}, & \text{for } x = 1. \end{cases}$$

Then T is a self mapping on a complete S-metric space $[0, 1]$. Next let $x = \frac{1}{2}$ and $y = 1$, then it is easy to check that $\beta > \frac{1}{2}$, which leads to the conclusion, that T is not an A-contraction mapping. Also, T has no fixed point in X though other conditions of the Theorem 2.3 are being satisfied.

Theorem 2.6. Let (X, S) be a complete S-metric space and let T_1 and T_2 satisfy

$$S(T_1x, T_1x, T_2y) \leq \alpha\left(S(x, x, y), S(x, x, T_1x), S(y, y, T_2y)\right)$$

for all $x, y \in X$ and for some α in A. Then T_1 and T_2 have a unique common fixed point in X.

PROOF. Let us construct the following sequence in X.

$$x_n = \begin{cases} T_1x_{n-1}, & \text{whenever } n \in \mathbb{N} \text{ is odd and} \\ T_2x_{n-1}, & \text{whenever } n \in \mathbb{N} \text{ is even} \end{cases}$$

Then

$$\begin{aligned} S(x_1, x_1, x_2) &= S(T_1x_0, T_1x_0, T_2x_1) \\ &\leq \alpha(S(x_0, x_0, x_1), S(x_0, x_0, T_1x_0), S(x_1, x_1, T_2x_1)) \\ &= \alpha(S(x_0, x_0, x_1), S(x_0, x_0, x_1), S(x_1, x_1, x_2)) \end{aligned} \tag{3}$$

and therefore from the property of α we have $S(x_1, x_1, x_2) \leq kS(x_0, x_0, x_1)$. Also, we see that

$$\begin{aligned} S(x_2, x_2, x_3) = S(x_3, x_3, x_2) &= S(T_1x_2, T_1x_2, T_2x_1) \\ &\leq \alpha(S(x_2, x_2, x_1), S(x_2, x_2, T_1x_2), S(x_1, x_1, T_2x_1)) \\ &= \alpha(S(x_1, x_1, x_2), S(x_2, x_2, x_3), S(x_1, x_1, x_2)) \end{aligned} \tag{4}$$

and we get from the property of α that $S(x_2,x_2,x_3) \leq kS(x_1,x_1,x_2) \leq k^2 S(x_0,x_0,x_1)$. Proceeding in a similar fashion, we see that $S(x_n,x_n,x_{n+1}) \leq kS(x_{n-1},x_{n-1},x_n) \leq k^n S(x_0,x_0,x_1)$ for all $n \in \mathbb{N}$. Then it is a routine calculation to check that $\{x_n\}$ is Cauchy and since X is complete, there exists some $z \in X$ such that $x_n \to z$ as $n \to \infty$. Now,

$$\begin{aligned} S(z,z,T_1 z) &\leq 2S(z,z,x_{2n}) + S(T_1 z, T_1 z, x_{2n}) \\ &= 2S(z,z,x_{2n}) + S(T_1 z, T_1 z, T_2 x_{2n-1}) \\ &\leq 2S(z,z,x_{2n}) + \alpha(S(x_{2n-1},x_{2n-1},z), S(z,z,T_1 z), \\ &\quad S(x_{2n-1},x_{2n-1},x_{2n})). \end{aligned} \tag{5}$$

Since α is continuous, taking n tending to infinity we get $S(z,z,T_1 z) \leq \alpha(0, S(z,z,T_1 z), 0)$ implying that $S(z,z,T_1 z) = 0$ i.e. $T_1 z = z$. In a similar way we can show that $T_2 z = z$ and therefore z is a common fixed point of T_1 and T_2. Uniqueness of fixed point is obvious.

□

3. Result in integral setting

In 2002, Branciari [13] first analyzed the existence of fixed point of a contractive mapping of integral type defined over a complete meric space (X,d).

Theorem 3.1. [13] Let (X,d) be a complete metric space, $c \in (0,1)$ and let $f : X \to X$ be a mapping such that for each $x, y \in X$,

$$\int_0^{d(fx,fy)} \varphi(t)dt \leq c \int_0^{d(x,y)} \varphi(t)dt \tag{6}$$

where $\varphi : [0,+\infty) \to [0,+\infty)$ is a Lesbesgue-integrable mapping which is summable (i.e. with finite integral) on each compact subset of $[0,+\infty)$, nonnegative, and such that for each $\epsilon > 0$, $\int_0^\epsilon \varphi(t)dt > 0$, then f has a unique fixed point $a \in X$ such that for each $x \in X$, $\lim_{n\to\infty} f^n x = a$.

Rhoades [15] extended the result of Branciari by replacing the condition (6) by the following

$$\int_0^{d(fx,fy)} \varphi(t)dt \leq c \int_0^{\max\{d(x,y),d(x,fx),d(y,fy),\frac{[d(x,fy)+d(y,fx)]}{2}\}} \varphi(t)dt. \tag{7}$$

Since then numerous generalizations have been made in this direction (see [15], [14] for details). Motivated by these results we apply and prove the analogue of A-contraction mapping over a complete S-metric space.

An important definition is needed to state our theorem in this section.

Definition 1.2. (Sub additivity)
$u : [0,+\infty) \to [0,+\infty)$ is sub additive on each $[a,b] \subset [0,+\infty)$ if

$$\int_0^{a+b} u(t)dt \leq \int_0^a u(t)dt + \int_0^b u(t)dt. \tag{8}$$

Now we state our result as following.

Theorem 3.2. Let T be a self-mapping of a complete S-metric space (X,S) satisfying the following condition:

$$\int_0^{S(Tx,Tx,Ty)} \varphi(t)dt \leq \alpha\left(\int_0^{S(x,x,y)} \varphi(t)dt, \int_0^{S(x,x,Tx)} \varphi(t)dt, \int_0^{S(y,y,Ty)} \varphi(t)dt\right) \tag{9}$$

for each $x, y \in X$ with some $\alpha \in A$, where $\varphi : [0,+\infty) \to [0,+\infty)$ is a Lesbesgue-integrable mapping which is summable (i.e. with finite integral), sub additive on each $[a,b] \subset [0,+\infty)$, nonnegative, and such that

$$\text{for each } \epsilon > 0, \quad \int_0^\epsilon \varphi(t)dt > 0. \tag{10}$$

Then T has a unique fixed point $z \in X$ and for each $x \in X$, $\lim_n T^n x = z$.

PROOF. Let x_0 be an arbitrary element of X and, for brevity, consider $x_{n+1} = Tx_n$. then for each integer $n \geq 1$, from (9) we get,

$$\begin{aligned}
&\int_0^{S(x_n,x_n,x_{n+1})} \varphi(t)dt \\
&= \int_0^{S(Tx_{n-1},Tx_{n-1},Tx_n)} \varphi(t)dt \\
&\leq \alpha\left(\int_0^{S(x_{n-1},x_{n-1},x_n)} \varphi(t)dt, \int_0^{S(x_{n-1},x_{n-1},Tx_{n-1})} \varphi(t)dt, \int_0^{S(x_n,x_n,Tx_n)} \varphi(t)dt\right) \\
&\leq \alpha\left(\int_0^{S(x_{n-1},x_{n-1},x_n)} \varphi(t)dt, \int_0^{S(x_{n-1},x_{n-1},x_n)} \varphi(t)dt, \int_0^{S(x_n,x_n,x_{n+1})} \varphi(t)dt\right).
\end{aligned}$$

Then by the axiom A_2 of function α,

$$\int_0^{S(x_n,x_n,x_{n+1})} \varphi(t)dt \leq k \int_0^{S(x_{n-1},x_{n-1},x_n)} \varphi(t)dt \tag{11}$$

for some $k \in [0, 1)$ as $\alpha \in A$.
In similar fashion, one can obtain

$$\begin{aligned}
\int_0^{S(x_n,x_n,x_{n+1})} \varphi(t)dt &\leq k \int_0^{S(x_{n-1},x_{n-1},x_n)} \varphi(t)dt \\
&\leq k^2 \int_0^{S(x_{n-2},x_{n-2},x_{n-1})} \varphi(t)dt \\
&\leq ... \\
&\leq k^n \int_0^{S(x_0,x_0,x_1)} \varphi(t)dt.
\end{aligned} \tag{12}$$

Now for $m > n$,

$$\begin{aligned}
S(x_n, x_n, x_m) &\leq 2\sum_{i=n}^{m-2} S(x_i, x_i, x_{i+1}) + S(x_{m-1}, x_{m-1}, x_m) \\
&\leq 2\sum_{i=n}^{m-1} S(x_i, x_i, x_{i+1}).
\end{aligned}$$

Now applying subadditivity of $\varphi(t)$

$$\begin{aligned}
\int_0^{S(x_n,x_n,x_m)} \varphi(t)dt &\leq \int_0^{2S(x_n,x_n,x_{n+1})} \varphi(t)dt + \int_0^{2S(x_{n+1},x_{n+1},x_{n+2})} \varphi(t)dt + ... \\
&\quad + \int_0^{2S(x_{m-2},x_{m-2},x_{m-1})} \varphi(t)dt + \int_0^{2S(x_{m-1},x_{m-1},x_m)} \varphi(t)dt \\
&\leq [k^n + k^{n+1} + ... + k^{m-2} + k^{m-1}] \int_0^{2S(x_0,x_0,x_1)} \varphi(t)dt \\
&= k^n[1 + k + ... + k^{m-n-2} + k^{m-n-1}] \int_0^{2S(x_0,x_0,x_1)} \varphi(t)dt \\
&\leq \frac{k^n}{1-k} \int_0^{2S(x_0,x_0,x_1)} \varphi(t)dt.
\end{aligned}$$

Now taking limit as $m, n \to \infty$, we get $\lim_{m,n\to\infty} \int_0^{S(x_n,x_n,x_m)} \varphi(t)dt = 0$ which, from (10) implies that

$$\lim_{m,n} S(x_n, x_n, x_m) = 0.$$

Therefore, $\{x_n\}$ is Cauchy, hence convergent. Call the limit z.
From (9) we get

$$\begin{aligned}\int_0^{S(Tz,Tz,x_{n+1})}\varphi(t)dt &= \int_0^{S(Tz,Tz,Tx_n)}\varphi(t)dt\\ &\leq \alpha\left(\int_0^{S(z,z,x_n)}\varphi(t)dt, \int_0^{S(z,z,Tz)}\varphi(t)dt, \int_0^{S(x_n,x_n,x_{n+1})}\varphi(t)dt\right).\end{aligned}$$

Taking limit as $n \to \infty$, we get

$$\int_0^{S(Tz,Tz,z)}\varphi(t)dt \leq \alpha\left(0, \int_0^{S(z,z,Tz)}\varphi(t)dt, 0\right).$$

So by the axiom A_2 of function α,

$$\int_0^{S(Tz,Tz,z)}\varphi(t)dt = k.0 = 0$$

which, from (10), implies that $S(Tz, Tz, z) = 0$ or, $Tz = z$.
Next suppose that $w(\neq z)$ be another fixed point of T. Then from (9) we have

$$\begin{aligned}\int_0^{S(z,z,w)}\varphi(t)dt &= \int_0^{S(Tz,Tz,Tw)}\varphi(t)dt\\ &\leq \alpha\left(\int_0^{S(z,z,w)}\varphi(t)dt, \int_0^{S(z,z,Tz)}\varphi(t)dt, \int_0^{S(w,w,Tw)}\varphi(t)dt\right)\\ &= \alpha\left(\int_0^{S(z,z,w)}\varphi(t)dt, \int_0^{S(z,z,z)}\varphi(t)dt, \int_0^{S(w,w,w)}\varphi(t)dt\right)\\ &= \alpha\left(\int_0^{S(z,z,w)}\varphi(t)dt, 0, 0\right).\end{aligned}$$

So by the axiom A_2 of function α,

$$\int_0^{S(z,z,w)}\varphi(t)dt = 0$$

which, from (10), implies that $S(z, z, w) = 0$ or, $z = w$ and so the fixed point is unique. □

Remark 3.3. On setting $\varphi(t) = 1$ over $\mathbb{R}^+$, the contractive condition of integral type transforms into a general contractive condition not involving integrals.

4. An application to homotopy

In this section, we obtain a homotopy result as an application of Theorem 2.3. For this purpose first we give the definition of homotopy between two functions.

Definition 4.1. [16] Let X, Y be two topological spaces, and let $G, S : X \to Y$ be two continuous mappings. Then, a homotopy from G to S is a continuous function $H : X \times [0, 1] \to Y$ such that $H(x, 0) = Gx$ and $H(x, 1) = Sx$, for all $x \in X$. Also, G and S are called homotopic mappings.

Theorem 4.2. Let X be a complete $S-$metric space and U be an open and V be a closed subset of X with $U \subset V$. Let the operator $F : V \times [0, 1] \to X$ satisfies the following conditions:
1) $x \neq F(x, t)$ for every $x \in V \backslash U$ and for any $t \in [0, 1]$,
2) There exists some $\alpha \in A$ such that

$$S(F(x, t), F(x, t), F(y, t)) \leq \alpha(S(x, x, y), S(x, x, F(x, t)), S(y, y, F(y, t))) \tag{13}$$

for all $t \in [0,1]$ and $x, y \in V$,
3) There exists a continuous function $f : [0,1] \to \mathbb{R}$ such that

$$S(F(x,t), F(x,t), F(x,s))) \leq |f(t) - f(s)| \tag{14}$$

$\forall\, t, s \in [0,1]$ and for every $x \in V$,
4) For any $r > 0$ we have $\alpha(a, b, 0) \leq \delta r < r$ whenever $a \leq r$ or $b \leq r$, where $0 < \delta < 1$.
Then $F(., 0)$ has a fixed point if and only if $F(., 1)$ has a fixed point.

Proof. Let us define $G = \{t \in [0,1] : F(x,t) = x \text{ for some } x \in U\}$.
First let us assume that $F(., 0)$ has a fixed point. Then $F(x, 0) = x$ for some $x \in U$ since (1) holds. Then $0 \in G$ and thus G is non-empty. We will show that G is a clopen subset of $[0,1]$, then from connectedness of $[0,1]$ we can easily say that $G = [0,1]$.
First we prove that G is open. let $t_0 \in G$ then there exists $x_0 \in U$ such that $F(x_0, t_0) = x_0$ [as (1) holds]. Therefore there exists $r > 0$ such that $B(x_0, r) \subset U$, where $B(x_0, r) = \{x \in X : S(x, x, x_0) < r\}$.
Now let, $x \in \overline{B(x_0, r)} = \{x \in X : S(x, x, x_0) \leq r\}$ and we choose

$$\epsilon = \frac{1}{2}\left[r - \sup_{x \in \overline{B(x_0,r)}} \alpha\left(S(x, x, x_0), S(x, x, F(x, t_0)), 0\right)\right].$$

Therefore $\epsilon > 0$ by condition (4). Since f is continuous on $[0,1]$, there exists $\eta(\epsilon) > 0$ such that $|f(t) - f(t_0)| < \epsilon$ whenever $t \in (t_0 - \eta(\epsilon), t_0 + \eta(\epsilon)) \subset [0,1]$. Now,

$$\begin{aligned}
S(F(x,t), F(x,t), x_0) &= S(F(x,t), F(x,t), F(x_0, t_0)) \\
&\leq 2S(F(x,t), F(x,t), F(x,t_0)) + S(F(x_0,t_0), F(x_0,t_0), \\
&\quad F(x,t_0)) \\
&= 2S(F(x,t), F(x,t), F(x,t_0)) + S(F(x,t_0), F(x,t_0), \\
&\quad F(x_0,t_0)) \\
&\leq 2|f(t) - f(t_0)| + \alpha(S(x,x,x_0), S(x,x,F(x,t_0)), \\
&\quad S(x_0,x_0,F(x_0,t_0))) \\
&= 2|f(t) - f(t_0)| + \alpha(S(x,x,x_0), S(x,x,F(x,t_0)), 0).
\end{aligned} \tag{15}$$

Therefore, whenever $t \in (t_0 - \eta(\epsilon), t_0 + \eta(\epsilon)) \subset [0,1]$, we get $S(F(x,t), F(x,t), x_0) \leq r$ implying that $F(x,t) \in \overline{B(x_0,r)}$. Therefore $F(.,t) : \overline{B(x_0,r)} \to \overline{B(x_0,r)}$ for every fixed $t \in (t_0 - \eta(\epsilon), t_0 + \eta(\epsilon))$. Now since $F(.,t)$ satisfies all the conditions of Theorem 2.3 we have, $F(.,t)$ has a fixed point in $\overline{B(x_0,r)} \subset V$, but it must be in U as condition (1) holds. Therefore $t \in G$ for every $t \in (t_0 - \eta(\epsilon), t_0 + \eta(\epsilon))$. Hence $(t_0 - \eta(\epsilon), t_0 + \eta(\epsilon)) \subset G$. So G is open in $[0,1]$.
Now we show that G is closed also. Let $\{t_n\} \subset G$ such that $t_n \to t^* \in [0,1]$ as $n \to \infty$. Then there exists $x_n \in U$ such that $x_n = F(x_n, t_n)$ for all $n \in \mathbb{N}$. Moreover we have,

$$\begin{aligned}
S(x_n, x_n, x_m) &= S(F(x_n,t_n), F(x_n,t_n), F(x_m,t_m)) \\
&\leq 2S(F(x_n,t_n), F(x_n,t_n), F(x_n,t_m)) + S(F(x_n,t_m), F(x_n,t_m), \\
&\quad F(x_m,t_m)) \\
&\leq 2|f(t_n) - f(t_m)| + \alpha(S(x_n,x_n,x_m), S(x_n,x_n,F(x_n,t_m)), \\
&\quad S(x_m,x_m,F(x_m,t_m)) \\
&= 2|f(t_n) - f(t_m)| + \alpha(S(x_n,x_n,x_m), S(x_n,x_n,F(x_n,t_m)), 0) \\
&\leq 2|f(t_n) - f(t_m)| + \delta S(x_n,x_n,x_m)
\end{aligned} \tag{16}$$

which implies $S(x_n, x_n, x_m) \leq \frac{2}{1-\delta}|f(t_n) - f(t_m)| \to 0$ as $n, m \to \infty$. Therefore $\{x_n\}$ is Cauchy in X and since X is complete thus it converges to some $x^* \in V$. Now we show that $F(x^*, t^*) = x^*$. Here we see that,

$$\begin{aligned}
S(x_n, x_n, F(x^*,t^*)) &= S(F(x_n,t_n), F(x_n,t_n), F(x^*,t^*)) \\
&\leq 2S(F(x_n,t_n), F(x_n,t_n), F(x_n,t^*)) \\
&\quad + S(F(x_n,t^*), F(x_n,t^*), F(x^*,t^*)) \\
&\leq 2|f(t_n) - f(t^*)| + \alpha(S(x_n,x_n,x^*), S(x_n,x_n,F(x_n,t^*)), \\
&\quad S(x^*,x^*,F(x^*,t^*))).
\end{aligned}$$

Now $S(x_n, x_n, F(x_n, t^*)) \leq |f(t_n) - f(t^*)| \to 0$ as $n \to \infty$. Thus using continuity of α we get,

$$S(x^*, x^*, F(x^*, t^*)) \leq \alpha(0, 0, S(x^*, x^*, F(x^*, t^*))) \tag{17}$$

and therefore by the property of α we have, $S(x^*, x^*, F(x^*, t^*)) \leq k.0 = 0$ implying that $S(x^*, x^*, F(x^*, t^*)) = 0$ that is $F(x^*, t^*) = x^*$. Therefore by condition (1) we get $x^* \in U$ and so $t^* \in G$. Hence G is closed also and so $G = [0, 1]$ that is $F(., 1)$ has also a fixed point. The converse part can be shown in a similar way. □

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References

[1] S.Gähler, *2-metric Raume and ihre topologische strucktur,* Math. Nachr. 26 (1963) 115-148.

[2] S.Gähler, *Lineare 2-normietre Raume*, Math. Nachr. 28 (1965) 1-43.

[3] K. S. Ha, Y. J. Cho, and A. White, *Strictly convex and strictly 2-convex 2-normed spaces*, Mathematica Japonica 33(3) 1988 375-384.

[4] B.C. Dhage, *Generalized metric spaces mappings with fixed point*, Bull. Calcutta Math. Soc. 84 (1992) 329-336.

[5] Z. Mustafa, B. Sims, *A new approach to generalized metric spaces,* J. Nonlinear Convex Anal. 7 (2006) 289-297.

[6] S. Sedghi, K.P.R. Rao, N. Shobe, *Common fixed point theorems for six weakly compatible mappings in D^*-metric spaces,* Internat J. Math. Math. Sci. 6 (2007) 225-237.

[7] S. Sedghi, N. Shobe, H. Zhou, *A common fixed point theorem in D^*-metric spaces, Fixed Point Theory Appl.* (2007) Article ID 27906 13 pages.

[8] S. Sedghi, N. Shobe and A. Aliouche, *A generalization of fixed point theorems in S-metric spaces,* Mat. Vesnik 64(3) (2012) 258-266.

[9] S. Sedghi and N. V. Dung, *Fixed point theorems on S-metric spaces,* Mat. Vesnik 66(1) (2014) 113-124.

[10] M. Akram, A. A. Siddiqui, *A fixed point theorem for A-contractions on a class of generalised metric spaces,* Korean J. Math. Sciences 10(2) (2003) 1-5.

[11] M. Akram, A. A. Zafar, A. A. Siddiqui, *A general class of contractions: A- contractions,* Novi Sad J. Math. 38(1) (2008) 25-33.

[12] M. Saha, D. Dey, *Fixed point theorems for a class of A-contractions on a 2-metric space*, Novi Sad J. Math. 40(1) (2010) 3-8.

[13] A. Branciari, *A fixed point theorem for mappings satisfying a general contractive condition of integral type,* Int. J. Math. Math. Sci 29(9) (2002) 531-536.

[14] D. Dey, A. Ganguly, M. Saha, *Fixed point theorems for mappings under general contractive condition of integral type*, Bull. Math. Anal. Appl. 3(1) (2011) 27-34.

[15] B.E . Rhoades, *Two fixed point theorems for mappings satisfying a general contractive condition of integral type,* International Journal of Mathematics and Mathematical Sciences 63 (2003) 4007-4013.

[16] C. Vetro, F. Vetro, *A Homotopy Fixed Point Theorem in 0-Complete Partial Metric Space,* Filomat 29(9) (2015) 2037-2048.