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ON THE CURVATURE THEORY OF NON-NULL CYLINDRICAL SURFACES IN MINKOWSKI 3-SPACE

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Abstract. This paper presents the curvature theory of non-null cylindrical surfaces in Minkowski 3-space. The definition of the line of striction and generator trihedron for cylindrical surfaces in Minkowski 3-space are given. The derivation formulae and Darboux instantaneous rotation vectors of generator trihedrons which play important role in robot kinematics are found. Moreover, curvature theory of a Lorentzian circular cylinder is given as an example.

Keywords: Curvature theory, Cylindrical surface, Darboux instantaneous rotation vector, Generator trihedron, Minkowski space.

AMS Subject Classification: 53A35

1. Introduction

Ruled surfaces which are generated by the motion of straight lines are widely used in computer aided geometric design (CAGD), industrial areas, spatial mechanisms, physics, kinematics and many other areas [4, 5, 9, 10]. Thus, ruled surfaces become an important issue in differential geometry and engineering mathematics.

Curvature theory which is one of the most popular research areas of ruled surfaces investigates the intrinsic geometric properties of the trajectory of points, lines, and planes embedded in a moving rigid body [9]. The results obtained from curvature theory can be applied to the analysis and synthesis of planar, spherical and spatial mechanisms. Curvature theory of a ruled surface has been studied by several researchers. Freudenstein [3], studied on analysis of higher order path curvature in plane kinematics and used characteristic numbers to characterize plane curves at a point. Kirson [4] examined higher-order curvature theory of a ruled surface in his thesis. Schaff [10] applied the curvature theory of a ruled surface to solve the kinematic problems associated with spatial closed-loop mechanisms. McCarthy [5] derived both scalar and dual formulations to study the curvature theory of a ruled surface and presented relations between two formulations. Ryuh [9] presented the curvature theory of a general ruled surface and special types of ruled surfaces referred to as developables. He gave separate formulations for a cylindrical surface, a cone and a tangent surface.

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The research area of the curvature theory of a ruled surface has also been studied in Minkowski space. Ayyıldız and Yücesan [1] derived the scalar and dual Lorentzian formulations of the curvature theory of line trajectories for non-null curves in the Lorentzian space. Ersoy and Tosun [2] gave the trajectory null scroll in 3-Minkowski space. Önder and Uğurlu [7] gave the Frenet frames and invariants of timelike ruled surfaces and of its directing cone.

Cylindrical surfaces which are developable surfaces are used in many applications in the research areas of robotics and engineering. A robot end-effector, for example, can move on a cylindrical surface. In order to determine the differential properties of the motion, curvature theory of cylindrical surface can be used. Since a cylindrical surface is an exception to a general ruled surface, the curvature theory of a cylindrical surface must be developed separately. In this study, we give the curvature theory of a non-null cylindrical surface in Minkowski 3-space using the technique in [9]. The line of striction are defined, and generator trihedrons, derivation formulae and Darboux instantaneous rotation vectors of the trihedrons are given in detail.

2. Preliminaries

In this section we give a brief summary of basic concepts in Minkowski 3-space.

Minkowski 3-space denoted by IR_1^3 is vector space with the Lorentzian inner product

$$\langle x, y\rangle = x_1y_1 + x_2y_2 - x_3y_3$$

where $x, y \in IR^3$ [6].

Let $x = (x_1,\ x_2,\ x_3)$ be an arbitrary vector in IR_1^3. If $\langle x, x\rangle > 0$ or $x = 0$, then x is called spacelike, if $\langle x, x\rangle < 0$, then x is called timelike, if $\langle x, x\rangle = 0$ and $x \neq 0$, then x is called null (lightlike) vector [6].

Let $x = (x_1,\ x_2,\ x_3)$ and $y = (y_1,\ y_2,\ y_3)$ be two vectors in IR_1^3. The Lorentzian vector product of x and y can be defined by [12]

$$x \times y = (-x_2y_3 + x_3y_2,\ x_1y_3 - x_3y_1,\ x_1y_2 - x_2y_1).$$

Let x and y be timelike vectors in IR_1^3, then there is a real number $\theta \geq 0$ such that $\langle x, y\rangle = -\|x\|\,\|y\|\cosh\theta$ and this number is called the hyperbolic angle between the vectors x and y [8].

Let x and y be spacelike vectors in IR_1^3 and they span a timelike vector subspace, then there is a real number $\theta \geq 0$ such that $|\langle x, y\rangle| = \|x\|\,\|y\|\cosh\theta$ and this number is called the central angle between the vectors x and y [8].

Let x and y be spacelike vectors in IR_1^3 and they span a spacelike vector subspace, then there is a real number $\theta \geq 0$ such that $\langle x, y\rangle = \|x\|\,\|y\|\cos\theta$ and this number is called the spacelike angle between the vectors x and y [8].

Let x be a spacelike vector and y be a timelike vector in IR_1^3, then there is a real number $\theta \geq 0$ such that $|\langle x, y\rangle| = \|x\|\,\|y\|\sinh\theta$ and this number is called the Lorentzian timelike angle between the vectors x and y [8].

A curve $\alpha = \alpha(s)$ in Minkowski 3-space can be spacelike, timelike or null (lightlike) if its velocity vector $\alpha'(s)$ is spacelike, timelike or null (lightlike), respectively [6].

A surface in Lorentzian space IR_1^3 is called a timelike surface if the induced metric on the surface is a Lorentzian metric, i.e., normal vector on the surface is a spacelike vector [13].

3. Curvature theory of a cylindrical surface in Minkowski 3-space

In this section, we give the curvature theory of a cylindrical surface which includes the definitions of line of striction, generator trihedrons and their derivative formulae, the Darboux instantaneous rotation vectors.

A cylindrical surface is a special ruled surface and can be represented by

$$X(t,v)=\alpha(t)+v\ R$$

where α is the directrix, R is a constant vector called ruling, t and v are real-valued parameters [11]. Since the ruling is a constant vector, normalization based on differentiation of a ruling can not be used for a cylindrical surface. Therefore, normalization based on differentiation of the directrix will be used [9]. This normalization can be achieved by using the equation

$$s(t)=\int_0^t \left\| \frac{d\alpha(t)}{dt} \right\|\ dt \tag{1}$$

where s is the arc-length parameter of the directrix.

The definition of line of striction used for a general ruled surface is not valid for a cylindrical surface. Thus, line of striction of a cylindrical surface can be defined as the locus of points on the surface which are at the shortest distance from the origin of the Cartesian reference frame along rulings [9]. From this definition, line of striction of a cylindrical surface is unique and it is perpendicular to the ruling, i.e.,

$$\langle c(s), R\rangle = 0 \tag{2}$$

where $c(s)$ represents line of striction of the cylindrical surface. By differentiating equation (2), we have

$$\left\langle c'(s), R\right\rangle = 0 \tag{3}$$

Equation (3) means that tangent vector of line of striction is perpendicular to the ruling. The line of striction relative to the directrix can be given as

$$c(s)=\alpha(s)-\mu(s)R(s) \tag{4}$$

where μ is a real-valued parameter [9]. The distance from line of striction to directrix along the ruling is $\mu\|R\|$, where $\|R\|$ represents the magnitude of ruling.

By substituting equation (4) into equation (2), we have

$$\langle \alpha(s)-\mu(s)\ R(s), R(s)\rangle = 0 \tag{5}$$

where μ can be obtained from equation (5) as

$$\mu=\frac{\langle \alpha(s), R(s)\rangle}{\langle R(s), R(s)\rangle}.$$

Since all rulings of a cylindrical surface pass through line of striction, line of striction can be thought as a directrix of the cylindrical surface. Thus, a cylindrical surface can be expressed as

$$X(s,v)=c(s)+v\ R.$$

A cylindrical surface in Minkowski 3-space can be classified according to the Lorentzian character of its ruling and its line of striction. Thus, the curvature theory of a cylindrical surface should be studied in two cases: curvature theory of a spacelike cylindrical surface and curvature theory of a timelike cylindrical surface.

3.1. **Curvature theory of a spacelike cylindrical surface.** A spacelike cylindrical surface can be represented by the equation

$$X(s,v) = c(s) + v\ R$$

where, c is a spacelike directrix and also the line of striction of cylindrical surface, ruling R is a constant spacelike vector, s is the arc-length parameter of directrix and v is a real-valued parameter.

The generator trihedron of a spacelike cylindrical surface which is defined on line of striction consists of three orthonormal vectors: the generator vector q, the central normal vector h, and the central tangent vector a. Generator vector can be defined as

$$q = \frac{R}{\|R\|}$$

where $\|R\|$ is the magnitude of ruling. Since ruling R is a spacelike vector, generator vector q is a spacelike vector. From equation (3), it is known that c' is perpendicular to the ruling, thus central tangent vector can be defined as

$$a = c' \tag{6}$$

Note that central tangent vector is tangent to the line of striction. Since line of striction is a spacelike curve, central tangent vector is a spacelike vector. Central normal vector can also be defined as

$$h = a \times q. \tag{7}$$

Since generator vector and central tangent vector are spacelike vectors, central normal vector should be a timelike vector.

From equation (6), the first-order positional variation of line of striction is

$$c' = a. \tag{8}$$

From the definition of a cylindrical surface, we have

$$R' = q' = 0. \tag{9}$$

In order to determine the first-order derivative of central tangent vector a, second-order positional variation of line of striction should be determined. Differentiating equation (8) gives

$$c'' = a'. \tag{10}$$

The line of striction is always perpendicular to constant ruling vector of cylindrical surface and it is a planar curve. From curvature theory of a planar curve, we know that second-order positional variation of line of striction is curvature of line of striction. The center of curvature is on direction line of central normal vector. Therefore, the second order positional variation of line of striction can be expressed by the equation

$$c'' = \kappa_c\ h \tag{11}$$

where $\kappa_c = \|\ c''\|$ is curvature of line of striction. By comparing equation (10) with equation (11), first-order derivative of central tangent vector can be given as

$$a' = \gamma\ h$$

where $\gamma = \kappa_c$ called curvature of a spacelike cylindrical surface characterizes the shape of cylindrical surface. For example, if $\gamma = 0$ then cylindrical surface is a plane or if $\gamma =$ *constant* then cylindrical surface is a right circular cylinder. By differentiating equation (7) and using equation (9), the first-order derivative of central normal vector can also be found as

$$h' = \gamma\ a.$$

Thus, the first-order angular variation of generator trihedron can be expressed in matrix form as

$$\frac{d}{ds}\begin{bmatrix} q \\ h \\ a \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \gamma \\ 0 & \gamma & 0 \end{bmatrix}\begin{bmatrix} q \\ h \\ a \end{bmatrix}$$

The Darboux instantaneous rotation vector of generator trihedron of a spacelike cylindrical surface can be obtained as

$$w = -\gamma\ q.$$

3.2. **Curvature theory of a timelike cylindrical surface.** The central normal vector of a timelike cylindrical surface is a spacelike vector. There are two possible cases in the study of curvature theory of a timelike cylindrical surface. The ruling may be a spacelike vector or a timelike vector. In this section, we study the curvature theory of a timelike cylindrical surface by considering these two cases.

A timelike cylindrical surface with spacelike (resp. timelike) ruling can be expressed by

$$X(s,v) = c(s) + v\ R$$

where, c is a timelike (resp. spacelike) directrix and also line of striction, the ruling R is a constant spacelike (resp. timelike) vector, s is the arc-length parameter of directrix and v is a real-valued parameter.

Generator trihedron of a timelike cylindrical surface can be defined as follows:

$$q = \frac{R}{\|R\|}, a = c', h = -a \times q$$

where q is the generator vector, a is the central tangent vector, and h is the central normal vector. Note that q is a spacelike and a is a timelike vector in the case of timelike cylindrical surface with spacelike ruling and q is a timelike and a is a spacelike vector in the case of timelike cylindrical surface with timelike ruling.

In order to determine the derivation formulae of generator trihedron, we follow a way similar to the case of a spacelike cylindrical surface. The first-order angular variation of generator trihedron of timelike cylindrical surface can be expressed in matrix form as

$$\frac{d}{ds}\begin{bmatrix} q \\ h \\ a \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\varepsilon\gamma \\ 0 & -\gamma & 0 \end{bmatrix}\begin{bmatrix} q \\ h \\ a \end{bmatrix}$$

where γ is the curvature of timelike cylindrical surface and $\varepsilon = \langle q, q\rangle$. The Darboux instantaneous rotation vector of generator trihedron of a timelike cylindrical surface can be obtained as

$$w = -\gamma\ q.$$

Example 3.1. *Let us consider the Lorentzian circular cylinder* $x_1^2 - x_3^2 = r^2$, $r > 0$, *as seen in Figure 1. The parametric form of this surface can be written as* $X(t,v) = (r\cosh t\ ,\ v,\ r\sinh t)$. *The directrix and the ruling of the Lorentzian circular cylinder are* $\alpha(t) = (r\cosh t,\ 0,\ r\sinh t)$ *and* $R = (0,\ 1,\ 0)$, *respectively. From the definition of line of striction of a timelike cylindrical surface, we obtain that the directrix of the Lorentzian circular cylinder is also line of striction. Note that line of striction* c *is a timelike curve* *and the ruling R is a spacelike vector. By using equation (1), normalized parameter can be found as $s = r\,t$.*

The vectors of generator trihedron of the Lorentzian circular cylinder can be found as

$$q = R = (0,\ 1,\ 0),$$
$$a = c' = \left(\sinh\frac{s}{r}\ ,\ 0\ ,\ \cosh\frac{s}{r}\right),$$
$$h = -a \times q = \left(-\cosh\frac{s}{r}\ ,\ 0\ ,\ -\sinh\frac{s}{r}\right),$$

where q is the spacelike generator vector, a is the timelike central tangent vector and h is the spacelike central normal vector. The derivatives of vectors of generator trihedron of the Lorentzian circular cylinder can be found as

$$q' = (0,\ 0,\ 0)\,,$$
$$a' = \left(\frac{1}{r}\cosh\frac{s}{r}\ ,\ 0\ ,\ \frac{1}{r}\sinh\frac{s}{r}\right),$$
$$h' = \left(-\frac{1}{r}\sinh\frac{s}{r}\ ,\ 0\ ,\ -\frac{1}{r}\cosh\frac{s}{r}\right).$$

The first-order angular variation of generator trihedron of the Lorentzian cylinder can also be expressed in matrix form as

$$\frac{d}{ds}\begin{bmatrix} q \\ h \\ a \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1/r \\ 0 & -1/r & 0 \end{bmatrix}\begin{bmatrix} q \\ h \\ a \end{bmatrix}.$$

The curvature of the Lorentzian cylinder is $\gamma = 1/r$. Thus, Darboux instantaneous rotation vector of the Lorentzian circular cylinder can be obtained as $w = -\frac{1}{r}q$.

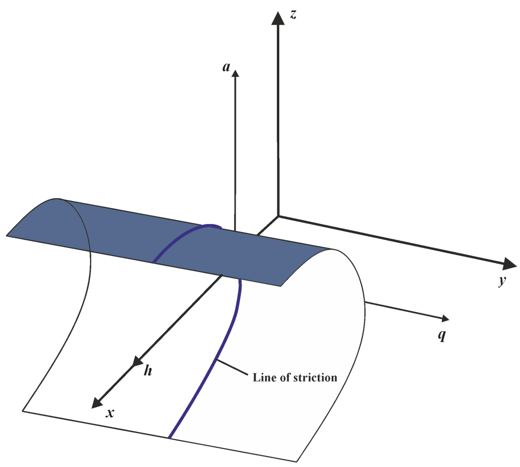

Figure 1 *A Lorentzian circular cylinder.*

4. Conclusions

A cylindrical surface which is a special ruled surface has widely used in robotic science and engineering. Especially, in order to find the differential properties of the motion of a robot end-effector which moves on the cylindrical surface in Minkowski 3-space, the curvature theory of cylindrical surface can be important tool. Since a cylindrical surface is not a general ruled surface, its curvature theory should be developed separately. In this paper, the curvature theory of a cylindrical surface which can be used as a tool of the motion of a robot end-effector in Minkowski 3-space is given in detail.

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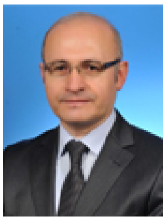

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